Application of $\alpha\delta$-Closed Sets

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ABSTRACT

In this paper, we introduce the notion of $\alpha\delta$-US spaces. Also we study the concepts of $\alpha\delta$-convergence, sequentially $\alpha\delta$-continuity and sequentially $\alpha\delta$-sub-continuity and derive some of their properties.

Keywords: $\alpha\delta$-US Spaces; $\alpha\delta$-Convergence; Sequentially $\alpha\delta$-Compactness; Sequentially $\alpha\delta$-Continuity; Sequentially $\alpha\delta$-Sub-Continuity

1. Introduction

In 1967, A. Wilansky [1] introduced and studied the concept of US spaces. Also, the notion of $\alpha\delta$-closed sets of a topological space is discussed by R. Devi, V. Kokilavani and P. Basker [2,3]. The concept of slightly continuous functions is introduced and investigated by Erdal Ekici et al. [4]. In this paper, we define that a sequence $\{x_n\}$ in a space $X$ is $\alpha\delta$-converges to a point $x \in X$ if $\{x_n\}$ is eventually in every $\alpha\delta$-open set containing $x$. Using this concept, we define the $\alpha\delta$-US space, Sequentially-$\alpha\delta$-continuous, Sequentially-Nearly-$\alpha\delta$-continuous, Sequentially-Sub-$\alpha\delta$-continuous and Sequentially-$\alpha\delta$O-compact of a topological space $(X, \tau)$.

2. Preliminaries

Throughout this paper, spaces $X$ and $Y$ always mean topological spaces. Let $X$ be a topological space and $A$, a subset of $X$. The closure of $A$ and the interior of $A$ are denoted by $cl(A)$ and $int(A)$, respectively. A subset $A$ is said to be regular open (resp. regular closed) if $A = int(cl(A))$ (resp. $A = cl(int(A))$, the $\delta$-interior [5] of a subset $A$ of $X$ is the union of all regular open sets of $X$ contained in $A$ and is denoted by $Int_\delta(A)$. The subset $A$ is called $\delta$-open if $A = Int_\delta(A)$, i.e., a set is $\delta$-open if it is the union of regular open sets. The complement of a $\delta$-open set is called $\delta$-closed.

Alternatively, a set $A \subset (X, \tau)$ is called $\delta$-closed if $A = cl_\delta(A)$, where $cl_\delta(A) = \{x \mid x \in U \in \tau \Rightarrow \text{int}(cl(A)) \cap A \neq \emptyset\}$. The family of all $\delta$-open (resp. $\delta$-closed) sets in $X$ is denoted by $\deltaO(X)$ (resp. $\deltaC(X)$). A subset $A$ of $X$ is called $\alpha$-open [6] if $A \subseteq \text{int}(cl(int(A)))$ and the complement of an $\alpha$-open are called $\alpha$-closed. The intersection of all $\alpha$-closed sets containing $A$ is called the $\alpha$-closure of $A$ and is denoted by $\alphacl(A)$, Dually, $\alpha$-interior of $A$ is defined to be the union of all $\alpha$-open sets contained in $A$ and is denoted by $\alphaInt(A)$.

We recall the following definition used in sequel.

Definition 2.1. A subset $A$ of a space $X$ is said to be
(a) An $\alpha$-generalized closed [7] ($ag$-closed) set if $\alphacl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open in $(X, \tau)$.
(b) An $\alpha\delta$-closed [8] set if $\alphacl_\delta(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $ag$-open in $(X, \tau)$.

The complement of a $\alpha\delta$-closed set is said to be $\alpha\delta$-open. The intersection of all $\alpha\delta$-closed sets of $X$ containing $A$ is called $\alpha\delta$-closure of $A$ and is denoted by $\alpha\deltacl(A)$. The union of all $\alpha\delta$-open sets of $X$ contained in $A$ is called $\alpha\delta$-interior of $A$ and is denoted by $\alpha\deltaint(A)$.

3. $\alpha\delta$-US Spaces

Definition 3.1. A sequence $\{x_n\}$ in a space $X$, $\alpha\delta$-converges to a point $x \in X$ if $\{x_n\}$ is eventually in every $\alpha\delta$-open set containing $x$.

Definition 3.2. A space $X$ is said to be $\alpha\delta$-US if every sequence in $X$, $\alpha\delta$-converges to a point of $X$.

Definition 3.3. A space $X$ is said to be
(a) $T_{\alpha\delta}^{\ast}$ if each pair of distinct points $x$ and $y$ in $X$ there exists an $\alpha\delta$-open set $U$ in $X$ such that $x \in U$ and $y \notin U$ and a $\alpha\delta$-open set $V$ in $X$ such that $y \in V$ and $x \notin V$.
(b) $T_{\alpha\delta}$ if for each pair of distinct points $x$ and $y$ in $X$ there exists an $\alpha\delta$-open sets $U$ and $V$ such

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that \( U \cap V = \varnothing \) and \( x \in U \), \( y \in V \).

**Theorem 3.4.** Every \( a\delta \)-US-space is \( T_{1}^{a\delta} \).

**Proof.** Let \( X \) be an \( a\delta \)-US-space and \( x, y \) be two distinct points of \( X \). Consider the sequence \( \{x_n\} \), where \( x_n = x \) for any \( n \in N \). Clearly \( \{x_n\} a\delta \)-converges to \( x \). Since \( x \neq y \) and \( X \) is \( a\delta \)-US, \( \{x_n\} \) does not \( a\delta \)-converges to \( y \), i.e., there exists an \( a\delta \)-open set \( U \) containing \( x \) but not \( y \). Similarly, we obtain an \( a\delta \)-open set \( V \) containing \( y \) but not \( x \). Thus, \( X \) is \( T_{1}^{a\delta} \).

**Theorem 3.5.** Every \( T_{2}^{a\delta} \)-space is \( a\delta \)-US.

**Proof.** Let \( X \) be a \( T_{2}^{a\delta} \) space and \( \{x_n\} \) a sequence in \( X \). Assume that \( \{x_n\} a\delta \)-converges to two distinct points \( x \) and \( y \). Then \( \{x_n\} \) is eventually in every \( T_{2}^{a\delta} \) and \( \{x_n\} \) is eventually in two disjoint \( a\delta \)-open sets. This is a contradiction. Therefore, \( X \) \( a\delta \)-US.

**Definition 3.6.** A subset \( A \) of a space \( X \) is said to be

(a) Sequentially \( a\delta \)-closed if every sequence in \( A \) \( a\delta \)-converges to a point in \( A \),

(b) Sequentially \( a\delta \)O-compact if every sequence in \( A \) has a subsequence which \( a\delta \)-converges to a point in \( A \).

**Theorem 3.7.** A space is \( a\delta \)-US if and only if the diagonal subset \( \Delta \) is sequentially \( a\delta \)-closed.

**Proof.** Suppose that \( X \) is an \( a\delta \)-US space and \( \{(x_n, x_n)\} \) is a sequence in the diagonal \( \Delta \). It follows that \( \{x_n\} \) is a sequence in \( X \). Since \( X \) is \( a\delta \)-US, the sequence \( \{(x_n, x_n)\} \) \( a\delta \)-converges to \( (x, x) \) which clearly belongs to \( \Delta \). Therefore, \( \Delta \) is a sequentially \( a\delta \)-closed subset of \( X \times X \). Conversely, suppose that the diagonal \( \Delta \) is sequentially \( a\delta \)-closed subset of \( X \times X \). Assume that a sequence \( \{x_n\} \) is \( a\delta \)-converging to \( x \) and \( y \). Then it follows that \( \{(x_n, x_n)\} a\delta \)-converges to \( (x, y) \). By hypothesis, since \( \Delta \) is sequentially \( a\delta \)-closed, we have \( (x, y) \in \Delta \). Thus \( x = y \). Therefore, \( X \) \( a\delta \)-US.

**Theorem 3.8.** If a space \( X \) is \( a\delta \)-US and a subset \( M \) of \( X \) is sequentially \( a\delta \)O-compact, then \( M \) is sequentially \( a\delta \)-closed.

**Proof.** Assume that \( \{x_n\} \) is any sequence in \( M \) which \( a\delta \)-converges to a point \( x \in X \). Since \( M \) is sequentially \( a\delta \)O-compact, there exists a subsequence \( \{x_{n_m}\} \) of \( \{x_n\} \) \( a\delta \)-converges to \( m \in M \). Since \( X \) is \( a\delta \)-US, we have \( x = m \). This shows that \( M \) is sequentially \( a\delta \)-closed.

**Theorem 3.9.** The product space of an arbitrary family of \( a\delta \)-US topological space is an \( a\delta \)-US topological space.

**Proof.** Let \( \{x_\lambda : \lambda \in \Delta \} \) be a family of \( a\delta \)-US topological spaces with the index set \( \Delta \). The product space of \( \{x_\lambda : \lambda \in \Delta \} \) is denoted by \( \prod X_\lambda \). Let \( \{x_\lambda (\lambda)\} \) be a sequence in \( \prod X_\lambda \). Suppose that \( \{x_\lambda (\lambda)\} a\delta \)-converges to two distinct points \( x \) and \( y \) in \( \prod X_\lambda \). Then there exists a \( \lambda_0 \in \Delta \) such that \( x(\lambda_0) \neq y(\lambda_0) \). Then \( \{x_\lambda (\lambda_0)\} \) is a sequence in \( X_{\lambda_0} \).

Let \( V_{\lambda_0} \) be any \( a\delta \)-open in \( X_{\lambda_0} \) containing \( x(\lambda_0) \). Then \( V = \bigcap_{\lambda \in \Delta} X_\lambda \) is a \( a\delta \)-open set of \( \prod X_\lambda \) containing \( x \). Therefore, \( \{x_\lambda (\lambda)\} \) is eventually in \( V \). Thus \( \{x_\lambda (\lambda_0)\} \) is eventually in \( V_{\lambda_0} \) and it \( a\delta \)-converges to \( x(\lambda_0) \). Similarly, the sequence \( \{x_\lambda (\lambda_0)\} a\delta \)-converges to \( y(\lambda_0) \). This is a contradiction as \( X_{\lambda_0} \) is an \( a\delta \)-US space.

Therefore, the product space \( \prod X_\lambda \) is \( a\delta \)-US.

4. Sequentially \( a\delta \)O-Compact Preserving Functions

**Definition 4.1.** A function \( f : X \rightarrow Y \) is said to be

(a) Sequentially-\( a\delta \)-continuous at \( x \in X \) if the sequence \( \{f(x_n)\} a\delta \)-converges to \( f(x) \) whenever a sequence \( \{x_n\} a\delta \)-converges to \( x \).

(b) Sequentially-\( a\delta \)O-compact preserving if the image \( f(M) \) of every sequentially \( a\delta \)O-compact set \( M \) of \( X \) is a sequentially \( a\delta \)O-compact subset of \( Y \).

**Theorem 4.2.** Let \( f_1 : X \rightarrow Y \) and \( f_2 : X \rightarrow Y \) be two sequentially \( a\delta \)-continuous functions. If \( Y \) is \( a\delta \)-US, then the set \( E = \{ x \in X : f_1(x) = f_2(x) \} \) is sequentially \( a\delta \)-closed.

**Proof.** Suppose that \( Y \) is \( a\delta \)-US and \( \{x_n\} \) is any sequence in \( E \) that \( f_1 \)-converges to \( x \in X \). Since \( f_1 \) and \( f_2 \) are sequentially \( a\delta \)-continuous functions, the sequence \( \{f_1(x_n)\} \) (respectively, \( \{f_2(x_n)\} \)) converges to \( f_1(x) \) (respectively, \( f_2(x) \)). Since \( x \in E \) for each \( n \in N \) and \( Y \) is \( a\delta \)-US, \( f_1(x) = f_2(x) \) and hence \( x \in E \). This shows that \( E \) is sequentially \( a\delta \)-closed.

**Lemma 4.3.** Every function \( f : X \rightarrow Y \) is sequentially sub \( a\delta \)US \( a\delta \)-US continuous if \( Y \) is sequentially \( a\delta \)O-compact.

**Proof.** Let \( \{x_n\} \) be a sequence in \( X \) that \( a\delta \)-US converges to \( x \in X \). It follows that \( \{f(x_n)\} \) is a sequence in \( Y \). Since \( Y \) is sequentially \( a\delta \)-compact,
there exists a subsequence \(\{f(x_n)\}\) of \(\{f(x)\}\) that \(a_\delta\)-converges to a point \(y \in Y\). Therefore \(f: X \to Y\) is sequentially sub \(a_\delta\)-continuous.

**Theorem 4.4.** Every sequentially nearly \(a_\delta\)-continuous function is sequentially \(a_\delta\Omega\)-compact preserving.

**Proof.** Let \(f: X \to Y\) be a sequentially nearly \(a_\delta\)-continuous function and \(M\) be any sequentially \(a_\delta\Omega\)-compact subset of \(X\). We will show that \(f(M)\) is a sequentially \(a_\delta\Omega\)-compact subset of \(Y\). So, assume that \(\{y_n\}\) is any sequence in \(f(M)\). Then for each \(n \in N\), there exists a point \(x_n \in M\) such that \(f(x_n) = y_n\). Now \(M\) is sequentially \(a_\delta\Omega\)-compact, so there exists a subsequence \(\{x_{n_k}\}\) of \(\{x_n\}\) that \(a_\delta\)-converges to a point \(x \in M\). Since \(f\) is sequentially nearly \(a_\delta\)-continuous, there exists a subsequence \(\{x_{n_k}(i)\}\) of \(\{x_n(i)\}\) such that \(f(x_{n_k}(i))\) \(a_\delta\)-converges to \(f(x)\). Therefore, there exists a subsequence \(\{y_{n_k}(i)\}\) of \(\{y_n(i)\}\) that \(a_\delta\)-converges to \(f(x)\). This implies that \(f(M)\) is a sequentially \(a_\delta\Omega\)-compact set of \(Y\).

**Theorem 4.5.** Every sequentially \(a_\delta\Omega\)-compact preserving function is sequentially sub-\(a_\delta\)-continuous.

**Proof.** Suppose that \(f: X \to Y\) is a sequentially \(a_\delta\Omega\)-compact preserving function. Let \(x\) be any point of \(X\) and \(\{x_n\}\) a sequence that \(a_\delta\)-converges to \(x\). We denote the set \(\{x_n: n \in N\}\) by \(A\) and put \(M = A \cup \{x\}\). Since \(\{x_n\}\) \(a_\delta\)-converges to \(x\), \(M\) is sequentially \(a_\delta\Omega\)-compact. By hypothesis, \(f\) is sequentially \(a_\delta\Omega\)-compact subset of \(Y\). Now in \(f(M)\) there exists a subsequence \(\{f(x_{n_k})\}\) of \(\{f(x_n)\}\) that \(a_\delta\)-converges to a point \(y \in f(M)\). This implies that \(f\) sequentially sub-\(a_\delta\)-continuous.

**Theorem 4.6.** A function \(f: X \to Y\) is sequentially \(a_\delta\Omega\)-compact preserving if and only if \(f|_M: M \to f(M)\) is sequentially sub-\(a_\delta\)-continuous for each sequentially \(a_\delta\Omega\)-compact set \(M\) of \(X\).

**Proof.** Necessity: Suppose that \(f: X \to Y\) is a sequentially \(a_\delta\Omega\)-compact preserving function. Then \(f(M)\) is sequentially \(a_\delta\Omega\)-compact in \(Y\) for each sequentially \(a_\delta\Omega\)-compact subset \(M\) of \(X\). Therefore, by Theorem 3.5 \(f|_M: M \to f(M)\) is sequentially sub-\(a_\delta\)-continuous.

Sufficiency: Let \(M\) be any sequentially \(a_\delta\Omega\)-compact set of \(X\). We will show that \(f(M)\) is sequentially \(a_\delta\Omega\)-compact subset of \(Y\). Let \(\{y_n\}\) be any sequence in \(f(M)\). Then for each \(n \in N\), there exists a point \(x_n \in M\) such that \(f(x_n) = y_n\). Since \(\{x_n\}\) is a sequence in the sequentially \(a_\delta\Omega\)-compact set \(M\) there exists a subsequence \(\{x_{n_k}\}\) of \(\{x_n\}\) that \(a_\delta\)-converges to a point in \(M\). By hypothesis \(f|M: M \to f(M)\) is sequentially sub-\(a_\delta\)-continuous, hence there exists a subsequence \(\{y_{n_k}\}\) of \(\{y_n\}\) that \(a_\delta\)-converges to \(y \in f(M)\). This implies that \(f(M)\) is sequentially \(a_\delta\Omega\)-compact in \(Y\).

**Corollary 4.7.** If a function \(f: X \to Y\) is sequentially sub-\(a_\delta\)-continuous and \(f(M)\) is sequentially \(a_\delta\)-closed in \(Y\) for each sequentially \(a_\delta\Omega\)-compact set \(M\) of \(X\), then \(f\) is sequentially \(a_\delta\Omega\)-compact preserving.

**Proof.** It will be sufficient to show that \(f|M: M \to f(M)\) is sequentially sub-\(a_\delta\)-continuous for each sequentially \(a_\delta\Omega\)-compact set \(M\) of \(X\) and by Lemma 3.3. We have already done. So, let \(\{x_n\}\) be any sequence in \(M\) that \(a_\delta\)-converges to a point \(x \in M\). Then, since \(f\) is sequentially sub-\(a_\delta\)-continuous there exists a subsequence \(\{x_{n_k}\}\) of \(\{x_n\}\) and a point \(y \in Y\) such that \(f(x_{n_k})\) \(a_\delta\)-converges to \(y\).

Since \(\{f(x_{n_k})\}\) is a sequence in the sequentially \(a_\delta\)-closed set \(f(M)\) of \(Y\), we obtain \(y \in f(M)\). This implies that \(f|M: M \to f(M)\) is sequentially sub-\(a_\delta\)-continuous.

5. Slightly \(a_\delta\)-Continuous Functions

**Definition 5.1.** A function \(f: X \to Y\) is said to be slightly \(a_\delta\)-continuous if for each \(x \in X\) and for each \(v \in CO(Y, f(x))\), there exists \(U \in a_\delta O(X, x)\) such that \(f(U) \subseteq V\), where \(CO(Y, f(x))\) is the family of clopen sets containing \(f(x)\) in a space \(Y\).

**Definition 5.2.** Let \((D, \leq\) be a directed set \(A\) net \(\{x_\lambda: \lambda \in D\}\) in \(X\) is said to be \(a_\delta\)-convergent to a point \(x \in X\) if \(\{x_\lambda\}_{\lambda \in D}\) is eventually in each \(U \in a_\delta O(X, x)\).

**Theorem 5.3.** For a function \(f: X \to Y\), the following are equivalent:

(a) \(f\) is slightly \(a_\delta\)-continuous.

(b) \(f^{-1}(v) \in a_\delta O(X)\) for each \(V \in CO(Y)\).

(c) \(f^{-1}(v)\) is \(a_\delta\)-cl-open for each \(V \in CO(Y)\).

(d) for each \(x \in X\) and for each net \(\{x_\lambda\}_{\lambda \in D}\) in \(X\).

**Proof.** (a) \(\Rightarrow\) (b). Let \(V \in CO(Y)\) and let \(x \in f^{-1}(V)\), then \(x \in V\). Since \(f\) is slightly \(a_\delta\)-continuous, there is a \(U \in a_\delta O(X, x)\) such that \((U) \subseteq V\). Thus \(f^{-1}(U) = \bigcup \{U: x \in f^{-1}(V)\}\), that is \(f^{-1}(U)\) is a union of \(a_\delta\)-open sets. Hence \(f^{-1}(U) \in a_\delta O(X)\).

(b) \(\Rightarrow\) (c). Let \(V \in CO(Y)\), then \((Y-V) \in CO(X)\).

By hypothesis \(f^{-1}(Y-V) = X-f^{-1}(V) \in a_\delta O(X)\).

Thus \(f^{-1}(U)\) is \(a_\delta\)-closed.

(c) \(\Rightarrow\) (d). Let \(\{x_\lambda\}_{\lambda \in D}\) be a net in \(X a_\delta\)-con-
verging to \( x \) and let \( V \in CO(Y, f(x)) \). There is thus a \( U \in \alpha \delta O(X, x) \) such that \( (U) \subset V \). There is thus a \( \lambda_0 \in D \) such that \( \lambda_0 \leq \lambda \) implies \( x_\lambda \in U \) since \( \{x_\lambda\}_{\lambda \in D} \) is \( \alpha \delta \)-convergent to \( x \). Thus \( f(x_\lambda) \in f(U) \subset V \) for all \( \lambda \). Thus \( \{f(x_\lambda)\}_{\lambda \in D} \) is \( \alpha \delta \)-convergent to \( f(x) \).

\((d) \Rightarrow (a)\) Suppose that \( f \) is not slightly \( \alpha \delta \)-continuous at a point \( x \in X \), then there exists a \( V \in CO(Y, f(x)) \) such that \( f(U) \) does not contained in \( V \) for each \( U \in \alpha \delta O(X, x) \). So

\[
f(U) \cap (Y - V) \neq \emptyset \text{ and thus } U \cap f^{-1}(Y - V) \neq \emptyset \text{ for each } U \in \alpha \delta O(X, x), \text{ since } \alpha \delta O(X, x) \text{ is directed by set inclusion } C, \text{ there exists a selection function } x_\lambda \text{ from } \alpha \delta O(X, x) \text{ into } X \text{ for each } U \in \alpha \delta O(X, x).
\]

Thus \( \{x_\lambda\}_{U} \in \alpha \delta O(X, x) \) is a net in \( \alpha \delta \)-converging to \( x \). Since \( X_U \in U \cap f^{-1}(Y - V) = U - f^{-1}(V) \) and so \( f(x_\lambda) \notin V \) for each \( U \),

\[
\{f(x_\lambda)\}_{U} \in \alpha \delta O(X, x) \text{ is not eventually in } V \subset CO(Y, f(x)), \text{ which is a contradiction. Hence } (a) \text{ holds.}
\]

**Theorem 5.4.** If \( f : X \to Y \) is slightly \( \alpha \delta \)-continuous and \( g: Y \to Z \) is slightly continuous, then their composition \( g \circ f \) is slightly \( \alpha \delta \)-continuous.

**Proof.** Let \( V \in CO(Z) \), then \( g^{-1}(V) \subset CO(Y) \).
Since \( f \) is slightly \( \alpha \delta \)-continuous,

\[
f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V) \subset \alpha \delta O(X). \text{ Thus } g \circ f \text{ is slightly } \alpha \delta \text{-continuous.}
\]

**Theorem 5.5.** The following are equivalent for a function \( f : X \to Y \):

\((a)\) \( f \) is slightly \( \alpha \delta \)-continuous,
\((b)\) for each \( x \in X \) and for each \( V \in CO(Y, f(x)) \), there exists a \( \alpha \delta \)-open set \( U \) such that \( f(U) \subset V \),
\((c)\) for each closed set \( F \) of \( Y \), \( f^{-1}(F) \) is \( \alpha \delta \)-closed,
\((d)\) \( f(cl(A)) \subset \alpha \delta cl(f(A)) \) for each \( A \subset X \) and
\((e)\) \( f^{-1}(B) \subset f^{-1}(\alpha \delta cl(B)) \) for each \( B \subset Y \).

**Proof.** \((a) \Rightarrow (b)\) Let \( x \in X \) and \( V \in CO(Y, f(x)) \) by Theorem 4.3. \( f^{-1}(V) \) is clopen. Put \( U = f^{-1}(V) \), then \( x \in U \) and \( f(U) \subset V \).

\((b) \Rightarrow (c)\) is obvious.

\((c) \Rightarrow (d)\) since \( \alpha \delta cl(f(A)) \) is the smallest \( \alpha \delta \)-closed set containing \( f(A) \), hence by \((c)\), we have \((d)\).

\((d) \Rightarrow (e)\) for each \( V \subset Y \),

\[
f(cl(f^{-1}(B))) \subset \alpha \delta cl(f^{-1}(B)) \subset \alpha \delta cl(B). \text{ Hence } f(cl(f^{-1}(B))) \subset \alpha \delta cl(B)
\]

\[
\Rightarrow f^{-1}(B) \subset f^{-1}(\alpha \delta cl(B))
\]

\((e) \Rightarrow (a)\) Let \( V \in CO(Y) \), then \( (Y - V) \subset CO(X) \), by \((e)\), we have

\[
cl(f^{-1}(Y - V)) \subset f^{-1}(\alpha \delta cl(Y - V)) = f^{-1}(Y - V), \text{ since every closed set is } \alpha \delta \text{-closed, thus } f^{-1}(Y - V) = X - f^{-1}(V) \text{ is closed and thus } \alpha \delta \text{-closed, thus } f^{-1}(V) \in \alpha \delta O(X) \text{ and } f \text{ is slightly } \alpha \delta \text{-continuous.}
\]

**Theorem 5.6.** If \( f : X \to Y \) is a slightly \( \alpha \delta \)-continuous injection and \( Y \) is clopen \( T_1 \), then \( X \) is \( T_{1}^{\text{ad}} \).

**Proof.** Suppose that \( Y \) is clopen \( T_1 \). For any distinct points \( x \) and \( y \) in \( X \), there exist \( U, V \in CO(Y) \) such that \( f(x) \in V, f(y) \notin V, f(x) \notin W \) and \( f(y) \in W \). Since \( f \) is slightly \( \alpha \delta \)-continuous, \( f^{-1}(V) \) and \( f^{-1}(W) \) are \( \alpha \delta \)-open subsets of \( X \) such that \( x \in f^{-1}(V), y \notin f^{-1}(V), x \notin f^{-1}(W) \) and \( y \in f^{-1}(W) \). This shows that \( X \) is \( T_{1}^{\text{ad}} \).

**Theorem 5.7.** If \( f : X \to Y \) is a slightly \( \alpha \delta \)-continuous surjection and \( Y \) is clopen \( T_2 \), then \( X \) is \( T_{2}^{\text{ad}} \).

**Proof.** For any pair of distinct points \( x \) and \( y \) in \( X \), there exist disjoint clopen sets \( U \) and \( V \) in \( Y \) such that \( f(x) \in U \) and \( f(y) \in V \). Since \( f \) is slightly \( \alpha \delta \)-continuous, \( f^{-1}(U) \) and \( f^{-1}(V) \) are \( \alpha \delta \)-open in \( X \) containing \( x \) and \( y \) respectively. Therefore \( f^{-1}(U) \cap f^{-1}(V) = \emptyset \) because \( U \cap V = \emptyset \). This shows that \( X \) is \( T_{2}^{\text{ad}} \).

**Definition 5.8.** A space is called \( \alpha \delta \)-regular if for each \( \alpha \delta \)-closed set \( F \) and each point \( x \notin F \), there exist disjoint open sets \( U \) and \( V \) such that \( F \subset U \) and \( x \in V \).

**Definition 5.9.** A space is said to be \( \alpha \delta \)-normal if for every pair of disjoint \( \alpha \delta \)-closed subsets \( F_1 \) and \( F_2 \) of \( X \), there exist disjoint open sets \( U \) and \( V \) such that \( F_1 \subset U \) and \( F_2 \subset V \).

**Theorem 5.10.** If \( f \) is slightly \( \alpha \delta \)-continuous injective open function from an \( \alpha \delta \)-regular space \( X \) onto a space then \( Y \) is clopen regular.

**Proof.** Let \( F \) be clopen set in \( Y \) and be \( y \notin F \), take \( y = f(x) \). Since \( f \) is slightly \( \alpha \delta \)-continuous, \( f^{-1}(F) \) is a \( \alpha \delta \)-closed set, take \( G = f^{-1}(F) \), we have \( x \notin G \). Since \( X \) is \( \alpha \delta \)-regular, there exist disjoint open sets \( U \) and \( V \) such that \( G \subset U \) and \( x \in V \). We obtain that \( F = f(G) \subset f(U) \) and \( y = f(x) \in f(V) \) such that \( f(U) \) and \( f(V) \) are disjoint open sets. This shows that \( Y \) is clopen regular.
Theorem 5.11. If \( f \) is slightly \( \alpha \delta \)-continuous injective open function from a \( \alpha \delta \)-normal space \( X \) onto a space \( Y \), then \( Y \) is \( cl \)-open normal.

Proof. Let \( F_1 \) and \( F_2 \) be disjoint \( cl \)-open subsets of \( Y \). Since \( f \) is slightly \( \alpha \delta \)-continuous, \( f^{-1}(F_1) \) and \( f^{-1}(F_2) \) are \( \alpha \delta \)-closed sets. Take \( U = f^{-1}(F_1) \) and \( V = f^{-1}(F_2) \). We have \( U \cap V = \emptyset \). Since \( X \) is \( \alpha \delta \)-regular, there exist disjoint open sets \( A \) and \( B \) such that \( U \subset A \) and \( V \subset B \). We obtain that \( F_1 = f(U) \subset f(A) \) and \( F_2 = f(V) \subset f(B) \) such that \( f(A) \) and \( f(B) \) are disjoint open sets. Thus, \( Y \) is \( cl \)-open normal.

REFERENCES
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