On Continuous Limiting Behaviour for the $q(n)$-Binomial Distribution with $q(n) \to 1$ as $n \to \infty$

Malvina Vamvakari
Department of Informatics and Telematics, Harokopio University of Athens, Athens, Greece
Email: mvamv@hua.gr

Received September 28, 2012; revised October 28, 2012; accepted November 5, 2012

ABSTRACT

Recently, Kyriakoussis and Vamvakari [1] have established a $q$-analogue of the Stirling type for $q$-constant which have lead them to the proof of the pointwise convergence of the $q$-binomial distribution to a Stieltjes-Wigert continuous distribution. In the present article, assuming $q(n)$ a sequence of $n$ with $q(n) \to 1$ as $n \to \infty$, the study of the affect of this assumption to the $q(n)$-analogue of the Stirling type and to the asymptotic behaviour of the $q(n)$-Binomial distribution is presented. Specifically, a $q(n)$ analogue of the Stirling type is provided which leads to the proof of deformed Gaussian limiting behaviour for the $q(n)$-Binomial distribution. Further, figures using the program MAPLE are presented, indicating the accuracy of the established distribution convergence even for moderate values of $n$.

Keywords: Stirling Formula; $q(n)$-Factorial Number of Order $n$; Saddle Point Method; $q(n)$-Binomial Distribution; Pointwise Convergence; Gauss Distribution

1. Introduction and Preliminaries

In last years, many authors have studied $q$-analogues of the binomial distribution (see among others [2-4]). Specifically, Kemp and Kemp [3] defined a $q$-analogue of the binomial distribution with probability function in the form

$$
\begin{align*}
    f_X(x) &= \binom{n}{x}_q \theta^x (1 + \theta q^{x+1})^{-1}, \\
    x &= 0, 1, \ldots, n,
\end{align*}
$$

where $\theta > 0, 0 < q < 1$, by replacing the loglinear relationship for the Bernoulli probabilities in Poissonian random sampling with loglinear odds relationship. Also, Kemp [4] defined (1) as a steady state distribution of birth-abort-death process.

Furthermore, Charalambides [2] considering a sequence of independent Bernoulli trials and assuming that the odds of success at the $i$th trial given by

$$
\pi_i = \theta q^{i-1}, i = 1, 2, \cdots, 0 < q < 1, 0 < \theta < \infty,
$$

is a geometrically decreasing sequence with rate $q$, derived that the probability function of the number $X$ of successes up to $n$-trail is the $q$-analogue of the binomial distribution with p.f. given by Equation (1).

For $q$ constant, the $q$-binomial distribution has finite mean and variance when $n \to \infty$. Thus, the asymptotic normality in the sense of the DeMoivre-Laplace classical limit theorem did not conclude, as in the case of ordinary hypergeometric series discrete distributions. Also, asymptotic methods—central or/and local limit theorems—are not applied as in Bender [5], Canfield [6], Flajolet and Soria [7], Odlyzko [8] et al.

Recently, Kyriakoussis and Vamvakari [1], for $q$ constant, established a limit theorem for the $q$-binomial distribution by a pointwise convergence in a $q$-analogue sense of the DeMoivre-Laplace classical limit theorem. Specifically, the pointwise convergence of the $q$-binomial distribution to a Stieltjes-Wigert continuous distribution was proved. In detail, transferred from the random variable $X$ of the $q$-binomial distribution (1) to the equal-distributed deformed random variable $Y = [X]_{n/q}$, then, for $n \to \infty$, the $q$-binomial distribution was approximated by a deformed standardized continuous Stieltjes-Wigert distribution as follows.
The derived estimate for the \( q \)-factorial numbers of order \( n \), is based on the analysis of the \( q \)-Exponential function

\[
E_q \left( (1-q)x \right) = \sum_{n=0}^{\infty} \left[ \frac{q^n}{[n]_q !} \right] x^n
\]

rewriting

\[
\exp \left( g(x) \right) = \prod_{j=1}^{\infty} \left( 1 + \left( 1-q \right) x q^{-j-1} \right)^{\nu}
\]

which is the ordinary generating function (g.f.) of the numbers \( \frac{q^n}{[n]_q !} \), \( n = 0, 1, 2, \ldots \).

Rewriting \( E_q \left( (1-q)x \right) \) as follows

\[
E_q (x) = \exp \left( g(x) \right),
\]

where

\[
g(x) = \sum_{j=1}^{\infty} \log \left( 1 + (1-q)x q^{-j-1} \right).
\]

because of the large dominant singularities of the generating function \( E_q (x) \), a well suited method for analyzing this is the saddle point method.

Using an approach of the saddle point method inspired from [9-12] and [1], the following theorem gives an asymptotic for the \( q(n) \)-factorial number of order \( n \).

**Theorem 1.** The \( q(n) \)-factorial numbers of order \( n \), \( [n]_q ! \), where

A) \( q = q(n) \) with \( q(n) \rightarrow 1 \) as \( n \rightarrow \infty \) and \( q(n)^r = \Omega(1) \)

or

B) \( q = q(n) \) with \( q(n)^r = o(1) \) have the following asymptotic expansion for \( n \rightarrow \infty \)

\[
[n]_q ! = (2\pi)^{\frac{1}{2}} \left[ \frac{q^n}{[n]_q !} \right]^{1/2} \exp \left( -g(r) \right) \left[ r g'(r) + r^2 g''(r) \right]^{1/2} r^n
\]

\[
\left[ 1 + \sum_{k=1}^{N} S_k(r) \left( q^n (1-q) \right)^k + O \left( \left( q^n (1-q) \right)^{N+1} \right) \right]^{-1}
\]

where \( N \) is a positive integer, \( r \) is the real solution of the equation

\[
rg'(r) = n
\]

and

\[
[r] = \frac{1-q^r}{1-q}, \text{ the } q\text{-number } t.
\]
The absence of a linear term in $\theta$ indicates a saddle point. The function $\text{e}^{G(\theta)}$ is unimodal with its peak at $\theta = 0$.

An estimation of the $q$-factorial numbers of order $n, [n]_q^{(1)}$, with $q$ defined by conditions (A) or (B) should naturally proceed by isolating separately small portions of the contour (corresponding to $x$ near the real axis) as follows.

A) For $q = q(n)$ with $q(n)^\ast = \Omega(1)$ we set
\[
I_1 = \frac{1}{\sqrt{2\pi}} \int_0^\pi \exp[G(\theta)]d\theta, \\
I_2 = \frac{1}{\sqrt{2\pi}} \int_{\pi}^{\pi+\delta} \exp[G(\theta)]d\theta,
\]
and choose $\delta$ such that the following conditions are true (see [12]):

C1) $n\delta^2 \to \infty$, that is $\delta \gg n^{-1/2}$
C2) $n\delta \to 0$, that is $\delta \ll n^{-3/4}$,
where “$\ll$” means “much smaller than”. A suitable choice for $\delta$ is $n^{-3/8}$.

As $\text{e}^{G(\theta)}$ decreases in $[\delta, \pi]$,
\[
\text{e}^{G(\theta)} \leq \text{e}^{G(\delta)}, \delta \leq \theta \leq 2\pi - \delta. 
\]

We will show in the sequel that from C1) and C2) it follows that $\text{e}^{G(\delta)}$ is exponentially small, being dominated by a term of the form $\text{e}^{-C(\delta)q}$.

Indeed we have
\[
G(\delta) \sim -\frac{1}{2} (rg'(r) + r^2g^*(r))\delta^2
\]
or
\[
G(\delta) \sim -\frac{1}{2} (rg'(r) + r^2g^*(r))n^{-3/4}. 
\]

But
\[
rg'(r) + r^2g^*(r) \sim \frac{1}{\log q^{-1}}(q^{-1} - 1)
\]
or
\[
rg'(r) + r^2g^*(r) \sim \frac{1 - q}{\log q^{-1}}(q^{-1} + q^{-2} + \cdots + 1). 
\]

For $q = q(n)$ with $q(n)^\ast = \Omega(1)$ we get
\[
G(\delta) = O\left(-\frac{1}{2} n^{-3/4}\right). 
\]

From which we find that
\[
|I_2| = O\left(e^{G(\delta)}\right) = O\left(e^{-\frac{3}{4}n^{3/4}}\right). 
\]
Thus, by C1), $\delta$ has been taken large enough so that the central integral $I_1$ "captures" most of the contribution, while the remainder integral $I_2$ is exponentially small by (19).

We now turn to the precise evaluation of the central integral $I_1$ . We have

$$I_1 = \frac{1}{[rg'(r)+r^2 g^*(r)]^{1/2}} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left[ -\phi^2 + \phi^2 \sum_{k=1}^{\infty} \alpha_k (r) \frac{(\psi \phi)^k}{k!} \right] d\phi$$

where

$$\epsilon = \left[ \frac{1}{2} \left( rg'(r) + r^2 g^*(r) \right) \right]^{1/2} \delta.$$  \hspace{1cm} (21)

Note that $\epsilon \to \infty$ as $n \to \infty$, since

$$\epsilon = n^{-3/8} \left[ \frac{1}{2} \left( rg'(r) + r^2 g^*(r) \right) \right]^{1/2} = n^{3/8} \left[ \frac{1}{2} \left( 1 + \frac{r^2 g^*(r)}{g'(r)} \right) \right]^{1/2} > Cn^{3/8},$$

where $C$ is a positive constant.

B) For $q = q(n)$ with $q(n)^n = o(1)$ we set

$$I_5 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r^{-\theta} \exp \left[ G(\theta) \right] d\theta,$$

and choose $\delta$ such that the conditions C1) and C2) are true. We suitably select $\delta = n^{-3/8}$.

As $e^{G(\theta)}$ decreases in $[\delta, \pi]$

$$e^{G(\theta)} \leq e^{G(\delta)}, \delta \leq \theta \leq 2\pi - \delta.$$  \hspace{1cm} (23)

We will now show that $e^{G(\delta)}$ is dominated by a term of the form $O(1)$. Indeed, form C1), C2), 16) and 17) it follows that

$$\exp(G(\delta)) \sim \exp \left( -\frac{1}{2 \log q} (1-q^n) n^{-3/4} \right).$$  \hspace{1cm} (24)

From which we get

$$|I_4| = O\left( r^{-\theta} e^{G(\delta)} \right) = O\left( q e^{\frac{1}{2 \log q} (1-q^n) n^{-3/4}} \right).$$  \hspace{1cm} (25)

Thus, for $q = q(n)$ with $q(n)^n = o(1)$ the integral $I_4$ is negligibly small. We now turn to the precise evaluation of the central integral $I_1$. Since

$$I_5 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r^{-\theta} \exp \left[ G(\theta) \right] d\theta$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\delta} r^{-\theta} \exp \left[ G(\theta) \right] d\theta$$

$$- \frac{1}{\sqrt{2\pi}} \int_{\delta}^{\infty} r^{-\theta} \exp \left[ G(\theta) \right] d\theta$$

we have

$$I_5 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r^{-\theta} \exp \left[ G(\theta) \right] d\theta$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\delta} r^{-\theta} \exp \left[ G(\theta) \right] d\theta$$

$$- \frac{1}{\sqrt{2\pi}} \int_{\delta}^{\infty} r^{-\theta} \exp \left[ G(\theta) \right] d\theta$$

\begin{equation}
\text{we have}
\end{equation}

$$I_5 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\delta} r^{-\theta} \exp \left[ G(\theta) \right] d\theta$$

$$- \frac{1}{\sqrt{2\pi}} \int_{\delta}^{\infty} r^{-\theta} \exp \left[ G(\theta) \right] d\theta$$

and

$$g'(r) + r^2 g^*(r) \geq 1 - \frac{q}{\log q} n_{h/q} q^{-1}.$$  \hspace{1cm} (26)

So, by substituting these to our estimation (7) the following corollary is proved.

**Corollary 1.** The $q$-factorial numbers of order $[n]_q [n]_q !$, where

A) $q = q(n)$ with $q(n) \to 1$ as $n \to \infty$

and $q(n)^n = O(1)$

or

B) $q = q(n)$ with $q(n)^n = o(1)$

have the following asymptotic expansion for $n \to \infty$

$$[n]_q ! = \frac{(2\pi (1-q))^{1/2}}{(q \log q)^{1/2}} q^{n/2} \left[ [n]_q [n]_q ^{1/2} \right] \left( 1 + O(q^n (1-q)) \right).$$  \hspace{1cm} (27)
2.2. Deformed Gaussian Limiting Behaviour for the $q(n)$-Binomial Distributions with $q(n) \to 1$ as $n \to \infty$

Transferred from the random variable $X$ of the $q$-binomial distribution (1) to the equal-distributed deformed random variable $Y = [X]_{\Omega_q}$, the mean value and variance of the random variable $Y$, say $\mu_q$ and $\sigma^2_q$, respectively, are given by the next relations

$$\mu_q = \left[ n \right] \frac{\theta}{1+\theta q^{n-1}}$$

and

$$\sigma^2_q = \frac{\theta^2 \left[ n \right] \left[ n-1 \right]}{q(1+\theta q^{n-1})} + \frac{\theta \left[ n \right]}{(1+\theta q^{n-1})^2}$$

(see Kyriakoussis and Vamvakari [1]).

Using the standardized r.v.

$$Z = \frac{[X]_{\Omega_q} - \mu_q}{\sigma_q}$$

with $\mu_q$ and $\sigma_q$ given in (28) and (29), the $q$-analogue Stirling asymptotic formula (27), and inspired by [1], the following theorem explores the continuous limiting behaviour of the $q(n)$-binomial distribution with $q(n) \to 1$ as $n \to \infty$.

**Theorem 2.** Let the p.f. of the $q$-binomial distribution be of the form

$$f_X(x) = \binom{n}{x} q^x (1-q)^{n-x}, x = 0, 1, \ldots, n,$$

where $\theta = \theta_n$, $n=0,1,2,\ldots$ such that $\theta_n \to \infty$, as $n \to \infty$. Then, for

A) $q = q(n)$ with $q(n)^n = \Omega(1)$

or

B) $q = q(n)$ with $q(n) \to 1$ as $n \to \infty$ and $q(n)^n = o(1)$ and $\theta_n = q^{-a}$ with $0 < a < 1$ constant

the $q(n)$-binomial distribution is approximated, for $n \to \infty$, by a deformed standardized Gauss distribution as follows

$$f_X(x) \approx \frac{\left( \log q^{-1} \right)^{1/2}}{(2\pi)^{1/2}} \cdot \exp \left\{ \frac{1}{2} \left( \frac{[X]_{\Omega_q} - \mu_q}{\sigma_q} \right)^2 \right\}, x \geq 0.$$  

**Proof.** Using the $q$-analogue of Stirling type (27), for $q = q(n)$ with $q(n) \to 1$ and $q(n)^n = \Omega(1)$ or $q(n)^n = o(1)$, the $q$-binomial distribution (1), is approximated by

$$\sum_{j=0}^{\infty} \left( \frac{\left( \log q^{-1} \right)^{1/2}}{(2\pi)^{1/2}} \cdot \exp \left\{ \frac{1}{2} \left( \frac{[X]_{\Omega_q} - \mu_q}{\sigma_q} \right)^2 \right\} \right),$$

$$f_X(x) \approx \frac{\left( \log q^{-1} \right)^{1/2}}{(2\pi)^{1/2}} \cdot \exp \left\{ \frac{1}{2} \left( \frac{[X]_{\Omega_q} - \mu_q}{\sigma_q} \right)^2 \right\}, x \geq 0.$$
where \( Li \) the dilogarithmic function and \( \beta_2 \) the Bernoulli number of order 2.

Moreover, working similarly for the sum appearing in the product
\[
\prod_{j=1}^{n} (1 + \theta_j q^{j-1}) = \exp \left( \sum_{j=1}^{n} \log (1 + \theta_j q^{j-1}) \right)
\]
the next estimation is obtained
\[
\sum_{j=1}^{n} \log (1 + \theta_j q^{j-1}) = \frac{1}{2 \log q} \log^2 (\theta_1) + L_2 \left( \frac{\theta_1}{\theta_1 + 1} \right) + O(\log q),
\]
\[
+ \frac{\beta_2}{2} \log q \frac{\theta_1}{1 + \theta_1} + O(\theta_1^{-1}).
\]

Applying all the previous estimations (32)-(39) to the approximation (31), carrying out all the necessary manipulations and for \( \theta_n \to \infty \), by both conditions A) and B), we derive our final asymptotic (30).

**Remark 2.** A realization of the sequence \( q(n), n = 0, 1, 2, \cdots \) considered in the above theorem 1A) is
\[
q(n) = 1 - \frac{\beta}{n}, 0 < \beta \leq 1
\]
with
\[
q(n)^n = \exp (-\beta).
\]

**Remark 3.** Possible realizations of the sequence \( q(n), n = 0, 1, 2, \cdots \) considered in the above theorem 2B) are among others the next two ones
\[
q(n) = 1 - \frac{1}{\ln(n)} \quad \text{or} \quad q(n) = 1 - \frac{1}{n^c}, 0 < c < 1.
\]

**Corollary 2** Let the random variable \( X \) with p.f. that of the \( q(n) \)-binomial distribution as in Theorem 2. Then for \( n \to \infty \) the following approximation holds
\[
P(a \leq X \leq b) \approx \frac{1}{2} \operatorname{Erf} \left( u_{a,b} \right) - \frac{1}{2} \operatorname{Erf} \left( u_a \right),
\]
\[
0 \leq a \leq b,
\]
where
\[
u_a = \left[ \left( \frac{a-1/2}{\ln q} - \mu_q \right) \sigma_q \right] / \mu_q \left( 2 \log q^{-1} \right)^{1/2}.
\]

with \( \operatorname{Erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t \exp \left( -x^2 \right) dx \) the Gauss error function.

**Proof**. Using the approximation (2) and the classical continuity correction we have that
\[
P(a \leq X \leq b) = \sum_{x=a}^{b} P(X = x)
\]
\[
\approx \int_0^{b-1/2} \left( \log q^{-1} \right)^{1/2} \frac{1}{(2\pi)^{1/2}} \cdot \exp \left( -\frac{1}{2} \left( \frac{\sigma_q \left[ x_{h_q} - \mu_q \right]^2}{\sigma_q} \right) \right) dx.
\]

Setting
\[
z = \left[ \frac{x}{\ln q} - \mu_q \right] / \sigma_q
\]
the approximation (42) becomes
\[
P(a \leq X \leq b) = \sum_{x=a}^{b} P(X = x)
\]
\[
\approx \frac{\sigma_q \mu_q^{-1}}{(\log q^{-1})^{1/2} (2\pi)^{1/2}} \int_{a-1/2}^{b-1/2} \exp \left( -\frac{1}{2} \left( \frac{\sigma_q^2 z^2}{\mu_q (\log q^{-1})^{1/2}} \right) \right) dz.
\]

Carrying out all the necessary manipulations, we get the final approximation (40).

**3. Figures Using Maple**

In this section, we present a computer realization of approximation (30), by providing figures using the computer program MAPLE and the \( q \)-series package developed by F. Garvan [13] which indicate good convergence even
for moderate values of $n$. Analytically, for the random variable $X$, we give the Figures 1 and 2 realizing Theorem 2(A), by demonstrating with diamond blue points the exact probability

$$f_x(x) = P(X = x) = \text{Prob}\left(x - \frac{1}{2} \leq X \leq x + \frac{1}{2}\right), \quad (44)$$

$x = 0, 1, 2, \ldots, n$.

and with diamond green points the continuous probability approximation

$$b_n(x) = \text{Prob}\left(x - \frac{1}{2} \leq X \leq x + \frac{1}{2}\right) \equiv \frac{1}{2} \text{Erf}(u_{x+1}) - \frac{1}{2} \text{Erf}(u_x), \quad (45)$$

$0 \leq x \leq n$

with $u_x$ and $u_{x+1}$ given by Equation (41), for $q = q(n) = 1 - \frac{1}{n}$,

$$\theta = \theta_n = \left(\exp(1) - 1\right) / \left(\exp(1) - 2\exp(1)q^n + q^n\right)$$

and $n = 50, 100$.

Note that similar good convergence even for moderate values of $n$ have been implemented for Theorem 2B).

The procedure in MAPLE which realizes the exact probability (44) and its approximation (45) for given $n, q$ and theta for both Theorem 2A) and 2B), is available under request.

4. Concluding Remarks

In this article, a deformed Gaussian limiting behaviour

for the $q(n)$-Binomial distribution has been established. The proofs have been concentrated on the study of the sequence $q(n)$ and the parameters of the considered distributions as sequences of $n$. Further, figures using the program MAPLE have been presented, indicating the accuracy of the established distribution convergence even for moderate values of $n$.

5. Acknowledgements

The author would like to thank Professor A. Kyriakoussis for his helpful comments and suggestions.

REFERENCES


