Discrete-Time Langevin Motion in a Gibbs Potential

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ABSTRACT

We consider a multivariate Langevin equation in discrete time, driven by a force induced by certain Gibbs’ states. The main goal of the paper is to study the asymptotic behavior of a random walk with stationary increments (which are interpreted as discrete-time speed terms) satisfying the Langevin equation. We observe that (stable) functional limit theorems and laws of iterated logarithm for regular random walks with i.i.d. heavy-tailed increments can be carried over to the motion of the Langevin particle.

Keywords: Langevin Equation; Dynamics of a Moving Particle; Multivariate Regular Variation; Chains with Complete Connections

1. Introduction

We start with the following equation describing a discrete-time motion in \( \mathbb{R}^d, d \geq 1 \), of a particle with mass \( m \) in the presence of a random potential and a viscosity force proportional to velocity:

\[
m(V_n - V_{n-1}) = -\Gamma V_{n-1} + F_n, n \in \mathbb{N}.
\]

Here d-vector \( V_n \) is the velocity at time \( n \), \( d \times d \) matrix \( \Gamma \) represents an anisotropic damping coefficient, and d-vector \( F_n \) is a random force applied at time \( n \). The above equation is a discrete-time counterpart of the Langevin SDE \( dF_t = -\Gamma V_t + dW_t \) [1,2]. Applications of the Langevin equation with a random non-Gaussian term \( W_t \) are addressed, for instance, in [3,4]. Setting \( M = I - m^{-1}\Gamma \) and \( Q_n = m^{-1}F_n \), we obtain:

\[
V_n = MV_{n-1} + Q_n, n \in \mathbb{N}.
\]  \( (1) \)

The random walk \( (X_n)_{n \geq 0} \) associated with this equation is given by

\[
X_n = X_0 + \sum_{k=1}^{n} V_k, n \in \mathbb{N}.
\]  \( (2) \)

Similar models of random motion in dimension one, with i.i.d. forces \( (Q_n)_{n \in \mathbb{Z}} \) were considered in [5-8], see also [9,10] and references therein. See, for instance, [11-14] for interesting examples of applications of Equation (1) with i.i.d. coefficients in various areas.

In this paper we will assume that the coefficients \( (Q_n)_{n \in \mathbb{Z}} \) are induced (in the sense of the following definition) by certain Gibbs’s states.

Definition 1. Coefficients \( (Q_n)_{n \in \mathbb{Z}} \) are said to be induced by random variables \( (Z_n)_{n \in \mathbb{Z}}, \) each valued in a finite set \( D, \) if there exists a sequence of independent random d-vectors \( Q_n = (Q_{n,j})_{n,j \in D} \) which is independent of \( (Z_n)_{n \in \mathbb{Z}} \) and is such that for a fixed \( i \in D, Q_n = (Q_{n,j})_{j \in D} \) are i.i.d. and \( Q_n = Q_{n,Z_n}. \)

The randomness of \( (Q_n)_{n \in \mathbb{Z}} \) is due to two factors:

1) Random environment \( (Z_n)_{n \in \mathbb{Z}} \) which describes a “state of Nature”; and, given the realization of \( (Z_n)_{n \in \mathbb{Z}}, \)

2) The “intrinsic” randomness of systems’ characteristics which is captured by the random variables \( Q_n = Q_{n,Z_n}. \)

Note that when \( (Z_n)_{n \in \mathbb{Z}} \) is a finite Markov chain, \( Q_n = Q_{n,Z_n} \) is a Hidden Markov Model. See, for instance, [15] for a survey of HMM and their applications. Heavy tailed HMM as random coefficients of multivariate linear time-series models have been considered, for instance, in [16,17]. In the context of financial time series, \( Z_n \) can be interpreted as an exogenous factor determined by the current state of the underlying economy. The environment changes due to seasonal effects, response to the news, dynamics of the market, etc. When \( Z_n \) is a function of the state of a Markov chain, stochastic difference Equation (1) is a formal analogue of the Langevin equation with regime switches, which was studied in [18]. The notion of regime shifts or regime switches traces...
back to [19,20], where it was proposed in order to explain the cyclical feature of certain macroeconomic variables.

In this paper we consider \((Z_n)_{n \in \mathbb{Z}}\) that belong to the following class of random processes:

**Definition 2 (21)).** A C-chain is a stationary random process \((Z_n)_{n \in \mathbb{Z}}\) taking values in a finite set (alphabet) \(D\), such that the following holds:

i) For any \(i_1, i_2, \ldots, i_n \in D\),
\[
P(Z_1 = i_1, Z_2 = i_2, \ldots, Z_n = i_n) > 0.
\]

ii) For any \(i_0 \in D\) and any sequence \((i_n)_{n \in \mathbb{N}} \in D^\mathbb{N}\), the following limit exists:
\[
\lim_{n \to \infty} P(Z_0 = i_0 | Z_k = i_k, 1 \leq k \leq n) = P(Z_0 = i_0 | Z_k = i_k, k \geq 1)
\]
where the right-hand side is a regular version of the conditional probabilities.

iii) Let
\[
\gamma_n = \sup \left\{ \frac{P(Z_0 = i_0, Z_k = i_k, 1 \leq k \leq n)}{P(Z_0 = i_0 | Z_k = i_k, k \geq 1)} : i_k = j_k, k \leq n \right\}.
\]
Then, \(\limsup_{n \to \infty} \frac{\gamma_n}{n} < 0\).

C-chains form an important subclass of chains with complete connections/chains of infinite order [22-24]. They can be described as exponentially mixing full shifts, and alternatively defined as an essentially unique random process with a given transition function (g-measure) \(P(X_0 = i_0, X_1 = i_1, k < 0)\) [25]. Stationary distributions of these processes are Gibbs states in the sense of Bowen [21,26]. For any C-chain \((Z_n)_{n \in \mathbb{Z}}\), there exists a Markovian representation [21,25], that is a stationary irreducible Markov chain \((Y_n)_{n \in \mathbb{Z}}\) in a countable state space and a function \(\zeta : S \to D\) such that \((Z_n)_{n \in \mathbb{Z}} = \delta ((\zeta (Y_n))_{n \in \mathbb{Z}})\), where \(\zeta\) means equivalence of distributions. Chains of infinite order are well-suited for modeling of long range-dependence with fading memory, and in this sense constitute a natural generalization of finite-state Markov chains [24,27-30].

We will further assume that the vectors \(Q_{a,d}\) are multivariate regularly varying. Recall that, for \(a \in \mathbb{R}\), a function \(f : \mathbb{R} \to \mathbb{R}\) is said to be regularly varying of index \(\alpha\) if \(f(t) = t^\alpha L(t)\) for some function \(L : \mathbb{R} \to \mathbb{R}\) such that \(L(\lambda t) \sim L(t)\) for any positive \(\lambda > 0\) (i.e., \(L\) is a slowly varying function). Let \(\mathbb{R}_d^+ := [0, \infty)^d\).

**Definition 3 (31)).** A random vector \(Q \in \mathbb{R}^d\) is regularly varying with index \(\alpha > 0\) if there exist a function \(b : \mathbb{R} \to \mathbb{R}\) regularly varying with index \(1/\alpha\) and a Radon measure \(\mu\) in the space \(\mathbb{R}_d^+\) such that \(nP(b^\alpha_Q \in \cdot) \to \mu(\cdot)\), as \(n \to \infty\), where \(\to\) denotes the vague convergence and \(b_\alpha := b(n)\).

We denote by \(\mathfrak{R}_{d,a,b}\) the set of all \(d\)-vectors regularly varying with index \(\alpha\), associated with function \(b\).

The corresponding limiting measure \(\nu\) is called the measure of regular variation associated with \(Q\).

Next, we summarize our assumptions on the coefficients \(Q_{a,d}\) and \(M\). Let \(\|Q\| := \max_{i \in [d]} |Q(i)|\) and \(\|M\| := \sup_{i \in \mathbb{R}^d} \left\{ \|M_Q : \|y\| = 1 \right\} \) for, respectively, a vector \(Q = Q(1), Q(2), \ldots, Q(d)\) \(\in \mathbb{R}^d\) and a \(d \times d\) matrix \(M\).

**Assumption 1.** Let \((Z_n)_{n \in \mathbb{Z}}\) be a stationary C-chain defined on a finite state space \(D\) and suppose that \((Q_n)_{n \in \mathbb{Z}}\) is induced by \((Z_n)_{n \in \mathbb{Z}}\). Assume in addition that:

A1) \(E[\log^+ \|Q_0\|] < +\infty\), where \(x^+ := \max\{x, 0\}\) for \(x \in \mathbb{R}\).

A2) The spectral radius \(\lim_{n \to \infty} \sqrt[n]{\|M^n\|}\) is strictly between zero and one.

A3) There exist a constant \(\alpha > 0\) and a regularly varying function \(b\) with index \(\alpha^{-1}\) such that for all \(i \in D\), \(Q_{a,i} \in \mathfrak{R}_{d,a,b}\) with associated measure of regular variation \(\mu_i\).

### 2. Statement of Results

For any (random) initial vector \(V_0\), the series \(V_n\) converges in distribution, as \(n \to \infty\), to
\[
V = \sum_{i=0}^\infty M^i Q_i,
\]
which is the unique initial value making \((V_n)_{n \in \mathbb{Z}}\) into a stationary sequence [32]. The following result, whose proof is omitted, is a “Gibbsian” version of a “Markovian” [16, Theorem 1]. The claim can be established following the line of argument in [16] nearly verbatim, exploiting the Markovian representation of C-chains obtained in [21].

**Theorem 1.** Let Assumption 1 hold. Then \(V \in \mathfrak{R}_{d,a,b}\) with associated measure of regular variation
\[
\mu_v(\cdot) := \sum_{i=0}^\infty E\left[ \mu_{Q_0} \circ (M^i)^{-1}(\cdot) \right],
\]
where \(\mu \circ \Pi^{-1}(\cdot)\) stands for \(\mu\left(\{x \in \mathbb{R}^d : \Pi x \in \cdot\}\right)\) and \(\mu_{Q_0}(\cdot) := \sum_{i \in D} P(Z_0 = i) \mu_i(\cdot)\).

In a slightly more general setting, the existence of the limiting velocity suggests the following law of large numbers, whose short proof is included in Section 3.1.
Theorem 2. Let Assumption 1 hold with A3) being replaced by the condition \( \mathbb{E} \left[ \| Q_n \| \right] < \infty \). Then
\[
\lim_{n \to \infty} X_n = \mathbb{E} [V] = (I - M)^{-1} \mathbb{E} [Q_0], \text{ a.s.}
\]

Let \( \tilde{Q}_{n} \) denote independent copies of \( Q_0 \), and let \( (A_n)_{n \in \mathbb{Z}} \) a sequence of vectors such that the sequence of processes
\[
\tilde{S}^{(i)} = b_n^{-1} \left( \sum_{k=i}^{n} \tilde{Q}_k - A_{[n]} \right), i > 0,
\]
converges in law as \( n \to \infty \) in the Skorokhod space \( D \left( \mathbb{R}, \mathbb{R}^d \right) \) to a Lévy process \( \xi_{i}^{(a, \mu)}, i > 0, \mu = (\mu)_{i \in \mathbb{D}}, \)
where \( \mu_i \) are introduced in A3) with stationary independent increments, \( \xi_0^{(a, \mu)} = 0 \), and \( \xi_i^{(a, \mu)} \) being distributed according to a stable law of index \( \alpha \) whose domain of attraction includes \( Q_0 \). For an explicit form of the centering sequence \( A_i \) and the characteristic function of \( \xi_{i}^{(a, \mu)} \) see, for instance, [33] or [34]. Remark that one can set \( A_i = 0 \) if \( \alpha < 1 \) and \( A_i = n \mathbb{E} [Q_0] \) if \( \alpha > 1 \). For each \( n \in \mathbb{N} \), define a process \( S^{(n)} \) in \( D \left( \mathbb{R}, \mathbb{R}^d \right) \) by setting
\[
S^{(n)} = b_n^{-1} \left( X_{[n]} - (I - M)^{-1} A_{[n]} \right), i > 0.
\]

Theorem 3. Let Assumption 1 hold with \( \alpha \in (0, 2) \). Then the sequence of processes \( S^{(n)} \) converges weakly in \( D \left( \mathbb{R}, \mathbb{R}^d \right) \), as \( n \to \infty \), to \( (I - M)^{-1} \tilde{Q} \).

It follows from Definition 3 (see, for instance, [31]) that if \( Q \in \mathbb{B}_{d,a,b} \) then the following limit exists for any vector \( x \in \mathbb{R}^d \):
\[
w(x) = \lim_{n \to \infty} n P(Q \cdot x \geq b_n).
\]

Theorem 4. Assume that the conditions of Theorem 3 hold. If \( \alpha = 1 \) assume that in addition the law of \( Q_{0j} \) is symmetric for any \( j \in D \). Let \( w(x) \) be defined by Equation (4) with \( Q = (I - M)^{-1} Q_0 \). Then, for any \( x \in \mathbb{R}^d \) such that \( w(x) > 0 \), we have
\[
\limsup_{n \to \infty} \frac{X_n - A_j}{b_n (\ln n)^{1/\alpha}} \cdot x = \begin{cases} 0 & \text{if } \varepsilon > 0, \\ \infty & \text{if } \varepsilon < 0, \end{cases} \text{ a.s.}
\]

In particular,
\[
\limsup_{n \to \infty} \frac{X_n - A_j}{b_n (\ln n)^{1/\alpha}} \cdot x = e^{b \varepsilon}, \text{ a.s.}
\]

If either Assumption 1 holds with \( \alpha = 2 \) or \( \mathbb{E} \left[ \| Q_0 \| ^2 \right] < \infty \) is assumed instead of A3), then, in view of Equation (6), a Gaussian counterpart of Theorem 3 can be obtained as a direct consequence of general CLTs for uniformly mixing sequences (see, for instance, [35, Theorem 20.1] and [36, Corollary 2]) applied to the sequence \( \left( Q_{n} \right)_{n \in \mathbb{Z}} \). If \( \mathbb{E} \left[ \| Q_0 \| ^2 \right] < \infty \), then a law of iterated logarithm in the usual form follows from Equation (5) and, for instance, [37, Theorem 5] applied to the sequence \( \left( Q_{n} \right)_{n \in \mathbb{Z}} \).

We remark that in the case of i.i.d. additive component \( Q_n \), similar to our results are obtained in [7] for a more general than Equation (1) mapping \( V_{n+1} = \Phi (V_n) \).

3. Proofs

3.1. Proof of Theorem 3

It follows from the definition of the random walk \( X_n \) and Equation (1) that
\[
X_n = \sum_{i=1}^{n} \left( M^i V_0 + \sum_{t=i}^{n} M^{i-t} V_t \right) = \sum_{i=1}^{n} M^i V_0 + \sum_{i=1}^{n} \left( \sum_{t=i}^{n} M^{i-t} - \sum_{t=n+1}^{n} M^{i-t} \right) Q_i.
\]

Note that \( \mathbb{E} \left[ \| Q_0 \| \right] < \infty \) implies
\[
\sum_{n=1}^{\infty} P \left( \| Q_0 \| > n \varepsilon \right) < \infty.
\]

It follows then from the Borel-Cantelli lemma that
\[
\lim_{n \to \infty} \frac{1}{n} \left\| \sum_{i=1}^{n} M^{i+1} V_i \right\| = 0, \text{ a.s.}
\]

Furthermore, we have
\[
P \left( \sum_{i=1}^{n} \left| M^{i+1} V_i \right| < \infty \right) = 1.
\]

Thus the law of large numbers for \( X_n \) follows from the ergodic theorem applied to the sequence \( \left( Q_{n} \right)_{n \in \mathbb{Z}} \) .

3.2. Proof of Theorem 3

Only the second term in the right-most side of Equation (5) contributes to the asymptotic behavior of \( X_n \). The proof rests on the application of Corollary 5.9 in [34] to the partial sums \( \sum_{i=1}^{n} Q_i \). In view of condition iii) in Definition 2 and the decomposition shown in Equation (6), we only need to verify that the following “local dependence” condition (which is condition (5.13) in [34]) holds for the sequence \( \left( Q_{n} \right)_{n \in \mathbb{Z}} \).

\[
\mathbb{E} \left[ \sum_{i=1}^{n} \left| M^{i+1} V_0 \right| > b_{n, \varepsilon} \right] \to n \to \infty, \forall \varepsilon > 0, \forall j \in \mathbb{N}
\]

The above convergence to zero follows from the mix-
ing condition iii) in Definition 2 and the regular variation, as \( t \) goes to infinity, of the marginal distribution tail \( P(\|Q_0\| > t) = \sum P(Z_0 = i) \cdot P(\|Q_{ij}\| > t) \). □

### 3.3. Proof of Theorem 4

For \( i \in D \), let \( k_n(i) \) be the number of occurrences of \( i \) in the set \( \{Z_1, Z_2, \ldots, Z_n\} \). That is,

\[
k_n(i) = \sum_{j=1}^n 1_{\{Z_j = i\}}.
\]

Define recursively \( T_i(0) = 0 \) and

\[
T_i(j) = \inf \{k > T_i(j-1) : Z_k = i\}
\]

(with the usual convention that the greatest lower bound over an empty set is equal to infinity). For \( i \in D, n \in \mathbb{N} \), let

\[
S_{n,i} = \sum_{j=1}^n (I - M)^{-1} Q_{i,j} = C_{n,i},
\]

where

\[
C_{n,i} := \begin{cases} 
0 & \text{if } \alpha \leq 1 \\
n \cdot E[(I - M)^{-1} Q_{0,i}] & \text{if } \alpha \in (1,2).
\end{cases}
\]

Denote

\[
\tilde{X}_n := \sum_{i \in D} S_{n,i}.
\]

Further, for each \( n \in \mathbb{N} \) let \( R_n = 0 \) if \( \alpha \leq 1 \), whereas if \( \alpha > 1 \) let

\[
R_n = \sum_{i \in D} E[(I - M)^{-1} Q_{0,i}] \cdot \{k_n(i) - nP(Z_0 = i)\}.
\]

Then

\[
\tilde{X}_n - R_n = (I - M)^{-1} \sum_{j=1}^n Q_j - A_n
\]

and hence

\[
X_n = (I - M)^{-1} \sum_{j=1}^n Q_j = (X_n - A_n) - (\tilde{X}_n - R_n).
\]

It follows from the decomposition given by Equation (6) along with the Borel-Cantelli lemma that for any \( x \in \mathbb{R}^d \),

\[
\lim_{n \to \infty} \frac{X_n - A_n}{b_n(\ln n)^{1/\alpha}} = 0, \quad \text{a.s.}
\]

Let \( \mathcal{A}_n(i) = 1_{\{Z_n = i\}} - P(Z_0 = i) \). Then

\[
k_n(i) - nP(Z_0 = i) = \sum_{j=1}^n \mathcal{A}_j(i).
\]

It follows, for instance, from Theorem 5 in [37] that if \( \alpha > 1 \) then for any \( x \in \mathbb{R}^d \), the following limit exists and the identity holds with probability one:

\[
\limsup_{n \to \infty} \frac{R_n}{\sqrt{2n \ln \ln n}} = \sum_{i \in D} E[(I - M)^{-1} Q_{0,i}] \cdot X
\]

\[
\times \limsup_{n \to \infty} \frac{\sum_{j=1}^n \mathcal{A}_j(i) \cdot x}{\sqrt{2n \ln \ln n}} < \infty.
\]

Therefore (since \( b_n \) is regularly varying with index \( \alpha < 2 \), in order to complete the proof Theorem 4 it suffices to show that for any \( x \in \mathbb{R}^d \) that satisfies the condition \( w(x) > 0 \) of the theorem, we have

\[
\limsup_{n \to \infty} \frac{\tilde{X}_n \cdot x}{b_n(\ln n)^{1/\alpha}} = \begin{cases} 0 & \text{if } \varepsilon > 0, \\
\infty & \text{if } \varepsilon < 0, \quad \text{a.s.}
\end{cases}
\]

We first observe that by the law of iterated logarithm for heavy-tailed i.i.d. sequences (see Theorems 1.6.6 and 3.9.1 in [33]),

\[
\limsup_{n \to \infty} \frac{S_{n,i} \cdot x}{b_n(\ln n)^{1/\alpha}} = 0, \quad \text{a.s.}
\]

for any \( i \in D, x \in \mathbb{R}^d \), and \( \varepsilon > 0 \). Since by the ergodic theorem,

\[
\lim_{n \to \infty} k_n(i)/n = P(Z_0 = i) \in (0, \infty), \quad \text{a.s.,}
\]

this yields

\[
\limsup_{n \to \infty} \frac{S_{n,i} \cdot x}{b_n(\ln n)^{1/\alpha}} = 0, \quad \text{a.s.}
\]

and hence

\[
\limsup_{n \to \infty} \frac{\tilde{X}_n \cdot x}{b_n(\ln n)^{1/\alpha}} = 0, \quad \text{a.s.}
\]

On the other hand, if \( \varepsilon < 0 \), Theorem 3.9.1 in [33] implies that for any \( i \in D \) and any \( x \in \mathbb{R}^d \) such that \( w(x) > 0 \), we have

\[
\limsup_{n \to \infty} \frac{S_{n,i} \cdot x}{b_n(\ln n)^{1/\alpha}} = \infty, \quad \text{a.s.}
\]

To conclude the proof of the theorem it thus remains to show that for any \( i, j \in D \), any \( x \in \mathbb{R}^d \), and all \( \delta \in (1/(2\alpha), 1/\alpha) \),

\[
P(E_{i,j}(n) \cap E_{j,i}(n) \cap \mathcal{A}) = 0, \quad (7)
\]

where, for \( i \in D \), the events \( E_{i,j}(n) \) are defined as follows:

\[
E_{i,j}(n) := \left\{ \left| S_{n,i,j} \cdot x \right| > b_n(\ln n)^{\delta} \right\}.
\]

For \( i \in D \), let \( \xi_i = 2 \cdot P(Z_0 = i) \) and define

\[
G_{n,i} := \max_{1 \leq k \leq n} \left| S_{n,k} \cdot x \right| > b_n(\ln n)^{\delta} \right\}.
\]

Then

\[
P(E_{i,j}(n) \cap E_{j,i}(n) \cap \mathcal{A}) \leq P(G_{n,i} \cap G_{n,j}) + P(k_n(i) > n\xi_i)
\]

\[
+ P(k_n(j) > n\xi_j)
\]

\[
= P(G_{n,i}) \cdot P(G_{n,j}) + P(k_n(i) > n\xi_i)
\]

\[
+ P(k_n(j) > n\xi_j)
\]

\[
= 0.
\]
The Ruelle-Perron-Frobenius theorem (see [26]) implies that the sequence \( k_n \) satisfies the large deviation principle (by the Gärtner-Ellis theorem), and hence

\[
P \left( k_n(i) > n \xi \right) < C e^{-\lambda_i n}
\]

for some constants \( C > 0 \) and \( \lambda_i > 0 \). Furthermore, for any \( A > 0 \), \( n_k = \left[ A^k \right] \), and \( \beta \in (0, \alpha) \) there exists a constant \( C = C(\alpha, \beta) \) such that (see [33, p. 177]),

\[
P \left( E_{n_k} \right) \leq C k^{-\beta}
\]

Therefore, since \( \delta > 2/\alpha > 0 \), we can choose \( \beta \in (0, \alpha) \) such that

\[
P \left( E_{n_k}^\beta \right) \cdot P \left( E_{n_k}^\gamma \right) \leq C_0 k^{-\gamma}, \text{ with suitable } C_0 > 0 \]

and \( \gamma > 1 \). A standard argument using the Borel-Cantelli lemma imply then the identity in Equation (7).

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