On the Set of 2 – Common Consequent of Primitive Digraphs with Exact \( d \) Vertices Having Loop

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ABSTRACT
Let \( d \) and \( n \) are positive integers, \( n \geq 2, 1 \leq d \leq n \). In this paper we obtain that the set of the 2 – common consequent of primitive digraphs of order \( n \) with exact \( d \) vertices having loop is \( \left\{ 1, 2, \ldots, n - \left\lfloor \frac{d}{2} \right\rfloor \right\} \).

Keywords: Boolean Matrix; Common Consequent; Primitive Digraph

1. Introduction
Let \( V = \{a_1, \ldots, a_n\} \) be a finite set of order \( n \), \( G = (V, E) \) be a digraph. Elements of \( V \) are referred as vertices and those of \( E \) as arcs. The arc of \( E \) from vertex \( u \) to vertex \( v \) is denoted by \((u, v)\). Let \( A \) be a \( n \times n \) matrix over the Boolean algebra \( \{0,1\} \). If the adjacency matrix of \( G \) is \( A \), where \( A = (a_{ij}) \), \( m_i = 1 \), if \((a_i, a_j) \in E \) and \( m_i = 0 \) otherwise, then \( A \) is Boolean matrix. \( G \) is called adjoint digraph of \( A \).

The map: \( A \leftrightarrow G \) is isomorphism.
Let \( G' \) be a digraph corresponding to the \( A' \), and
\[
a G' = \left\{ a_j \in V \left| \left( a_i, a_j \right) \in E \left( G' \right) \right. \right\},
\]
where \( l > 0 \) is an integer.
In 1983, Š. Schwarz [1] introduced a concept of the common consequent as follows.

Definition 1.1 Let \( G \) be a digraph. We say that a pair of vertices \((a_i, a_j), a_i \neq a_j\), has a common consequent (c.c.) if there is an integer \( l > 0 \) such that
\[
a G' \cap a G' \neq \phi
\]
If \((a_i, a_j)\) have a c.c. then the least integer \( l > 0 \) for which (1) holds is denoted by \( L_c(a_i, a_j) \).

Definition 1.2 Let \( G \) be a digraph. The generalized vertex exponent of \( G \), denoted by \( \exp_G (1) \), is the least integer \( l > 0 \) such that
\[
\bigcap_{i=1}^{n} a_i G' \neq \phi
\]
In 1996, Bolian Liu [2] extends the common consequent to the \( k \) – common consequent \((k – c.c.)\) as follows.

Definition 1.3 Let \( G \) be a digraph. We say that a group of vertices
\[
\{a_1, \ldots, a_k\} \subseteq V = \{a_1, \ldots, a_n\},
\]
\[
2 \leq k \leq n, a_i \neq a_s, s \neq i,
\]
has a \( k \) – common consequent \((k – c.c.)\), if there is an integer \( l > 0 \) such that
\[
\bigcap_{i=1}^{k} a_i G' \neq \phi
\]
If \((a_1, \ldots, a_k)\) have a \( k – c.c. \), then the least integer \( l > 0 \) for which (3) holds is denoted by \( L_c(a_1, \ldots, a_k) \).
If there is at least one group \( \{a_1, \ldots, a_k\} \) for which \( L_c(a_1, \ldots, a_k) \) exists, we define
\[
L_c(k) = \max L_c(a_1, \ldots, a_k),
\]
where \( \{a_1, \ldots, a_k\} \) runs through all groups with \( k \) elements for which \( L_c(a_1, \ldots, a_k) \) exists. If there is no group \( \{a_1, \ldots, a_k\} \) for which \( L_c(a_1, \ldots, a_k) \) exists, we define \( L_c(k) = 0 \). \( L_c(k) \) is called \( k \) – c.c. of \( G \).
A digraph \( G \) is said to be strongly connected if there exists a path from \( u \) to \( v \) for all \( u, v \in V(G) \). A digraph \( G \) is said to be primitive if there exists a positive integer \( p \) such that there is a walk of length \( p \) from \( u \) to \( v \) for all \( u, v \in V(G) \). The smallest such \( p \) is called the primitive exponent of \( G \).
A digraph \( G \) is primitive iff \( G \) is strongly connected and the greatest common divisor of all cycle lengths of \( G \) is 1.

Let \( V = \{a_1, \ldots, a_n\} \) and \( P_n(d) \) be the set of all primitive digraphs of order \( n \) with exact \( d \) vertices having loop. It is obvious that if \( G \in P_n(d) \), then
Let \( L_G \left( a_1, \ldots, a_n \right) \) exists for any group \( \left\{ a_1, \ldots, a_n \right\} \), \( 2 \leq k \leq n \). We define
\[
L(n, d, k) = \max \left\{ L_G \left( k \right) \mid G \in P_n \left( d \right) \right\}.
\]

The properties of primitive digraphs and its \( k \)-c.c. see [3-5]. In this paper we obtain that the set of the 2 common consequent of primitive digraphs of order \( n \) with exact \( d \) vertices having loop is
\[
\left\{ 1, 2, \ldots, n - \left\lfloor \frac{d}{2} \right\rfloor \right\},
\]
where \( n \) and \( d \) are positive integers, \( n \geq 2, 1 \leq d \leq n \), \( \left\lfloor a \right\rfloor \) is the least integer greater or equal to \( a \).

2. Preliminaries

It is easy to see that \( L(n, d, k) \) exists by [1].

**Lemma 2.1** Let \( G = G(V) \) be a primitive digraph of order \( n \geq 2 \) and \( V_i \) be a nonempty proper subset of \( V \), then \( V_i G \) contains at least one element of \( V \) which is not contained in \( V_i \).

**Proof:** Since vertex \( a \) has a loop, hence \( a \in G \), and \( a_G^{d+i} \geq k \) by lemma 2.1.

The follow lemma is obvious.

**Lemma 2.3** [2] If \( 2 \leq k_1 \leq k_2 \leq n \), \( G \) is a primitive digraph, then \( L_G \left( k_1 \right) \leq L_G \left( k_2 \right) \).

**Lemma 2.4** Let \( V = \left\{ a_1, \ldots, a_n \right\} \),
\[
E = \left\{ \left( a_i, a_j \right), \left( a_i, a_j, a \right), \left( a_j, a_i \right) \mid 1 \leq i < j \leq n \right\},
\]
\[
G_0 = (V, E),
\]
where \( n, d, k \) are integers and \( 1 \leq d \leq n \), \( 2 \leq k \leq n \), then
\[
L_{G_0} (2) = n - \left\lfloor \frac{d}{2} \right\rfloor \text{ and } L_{G_0} (n) = n - 1.
\]

**Proof:** First of all, It is obvious that \( G_0 \) is belong to \( P_n \left( d \right) \).

Let \( V_i = \left\{ a_1, \ldots, a_d \right\}, V_2 = \left\{ a_{d+1}, \ldots, a_n \right\} \), then \( V_i \) is a set in which every vertex have a loop, For all \( u, v \in V, u \neq v \).

**Case 1** \( u, v \in V_1 \).

There exists a walk of length less than or equal to \( n - \left\lfloor \frac{n}{2} \right\rfloor \) form \( u \) to \( v \) (or from \( v \) to \( u \)), and
\[
n - \left\lfloor \frac{n}{2} \right\rfloor \leq n - \left\lfloor \frac{d}{2} \right\rfloor \text{, then } uG_0^{\left\lfloor \frac{d}{2} \right\rfloor} \cap vG_0^{\left\lfloor \frac{d}{2} \right\rfloor} \neq \phi.
\]

**Case 2** \( u, v \in V_2 \).

There exists a walk of length less than or equal to \( n - d \) form \( u \) to \( a_1 \) (and form \( v \) to \( a_1 \)),
\[
n - d \leq n - \left\lfloor \frac{d}{2} \right\rfloor,
\]
then
\[
uG_0^{\left\lfloor \frac{d}{2} \right\rfloor} \cap vG_0^{\left\lfloor \frac{d}{2} \right\rfloor} \neq \phi.
\]

**Case 3** \( u \in V_1, v \in V_2 \).

There exist a walk of length less than or equal to \( n - d \) form \( v \) to \( x \in V \), by Lemma 2.2,
\[
uG_0^{\left\lfloor \frac{d}{2} \right\rfloor} \cap vG_0^{\left\lfloor \frac{d}{2} \right\rfloor} \neq \phi.
\]

So we have \( uG_0^{\left\lfloor \frac{d}{2} \right\rfloor} \cap vG_0^{\left\lfloor \frac{d}{2} \right\rfloor} \neq \phi \) for all \( u, v \in V \).

Note that if \( l < n - \left\lfloor \frac{d}{2} \right\rfloor \), then \( a_{d+1}G_0^{l} \cap a_{d+1}G_0^{l} = \phi \).

Hence \( L_{G_0} (2) = n - \left\lfloor \frac{d}{2} \right\rfloor \) and \( L_{G_0} (n) = n - 1 \). The proof is now completed.

3. The Main Results

**Theorem 3.1** Let \( G \in P_n \left( d \right) \), \( n, d \) be integers,
\[
n \geq 3, 1 \leq d \leq n,
\]
then \( L(n, 2, d) = n - \left\lfloor \frac{d}{2} \right\rfloor \).

**Proof:** Let \( V = \left\{ a_1, \ldots, a_n \right\} \) be set of vertices of \( G \) and \( V_1 \) be subset of \( V \) in which each vertex have a loop, \( V_2 = V - V_1 \), for all \( u, v \in V, u \neq v \).

**Case 1** \( u, v \in V_1 \).

**Case 2** \( u, v \in V_2 \).

**Case 3** \( u \in V_1, v \in V_2 \).

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There exists a walk of length less than or equal to $n - \left[ \frac{n}{2} \right]$ from $u$ to $v$ (or from $v$ to $u$), and $n - \left[ \frac{n}{2} \right] \leq n - \left[ \frac{d}{2} \right]$, then $u^d \in G^{n-\left[\frac{d}{2}\right]} \cap v^d \in G^{n-\left[\frac{d}{2}\right]} \neq \phi$.

**Case 2** $u, v \in V_2$.

Suppose that there be a walk of length equal to $n - \left[ \frac{d}{2} \right]$ of $u : u_1 u_2 \cdots u_{d-1} v x_1 x_2 \cdots x_p$, and there be a walk of length equal to $n - \left[ \frac{d}{2} \right]$ of $v : v_1 v_2 \cdots v_{d-1} u y_1 y_2 \cdots y_q$,

where $s + p = t + q = n - \left[ \frac{d}{2} \right]$.

Let $X = \{x_1 x_2, \ldots, x_p\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$. If there be one vertex of $X$ or $Y$ belong to $V_1$, then $u^d \in G^{n-\left[\frac{d}{2}\right]} \cap v^d \in G^{n-\left[\frac{d}{2}\right]} \neq \phi$.

Otherwise, $V - \{u, u_1, u_2, \ldots, u_{d-1}, v, x_1 x_2, \ldots, x_p\}$ and $V - \{v, v_1, v_2, \ldots, v_{d-1}, u, y_1 y_2, \ldots, y_q\}$ contains at most $\left[ \frac{d}{2} \right]$ elements of $V_1$. In other word, $\{u, u_1, u_2, \ldots, u_{d-1}\}$ contains at least $\left[ \frac{d}{2} \right]$ elements of $V_1$. Note that $G$ is strongly connected, $u \neq v$. There exists a walk of length less than or equal to $n - \left[ \frac{d}{2} \right]$ from $v$ to one vertex of $\{u_1, u_2, \ldots, u_{d-1}\}$ which belong to $V_1$. Therefore $u^d \in G^{n-\left[\frac{d}{2}\right]} \cap v^d \in G^{n-\left[\frac{d}{2}\right]} \neq \phi$.

**Case 3** $u \in V_1, v \in V_2$.

There exist a walk of length less than or equal to $n - d$ form $v$ to $x \in V_1$, by Lamma 2.2

$$u^d \in G^{n-\left[\frac{d}{2}\right]} \cap v^d \in G^{n-\left[\frac{d}{2}\right]} \neq \phi$$

So we have $u^d \in G^{n-\left[\frac{d}{2}\right]} \cap v^d \in G^{n-\left[\frac{d}{2}\right]} \neq \phi$ for all $u, v \in V$.

Hence $L(n, 2, d) \leq n - \left[ \frac{d}{2} \right]$.

Note that $L_{C_0}(2) = n - \left[ \frac{d}{2} \right]$,

then $L(n, 2, d) = n - \left[ \frac{d}{2} \right]$.

The proof is completed.

**Corollary 3.2** Let $G \in P_s(d)$ and $n, d$ be integers, $n \geq 2, 1 \leq d \leq n$, then $L(n, n, d) = n - 1$.

**Proof:** Let $V$ be a set of vertices of $G$ and let $u$ be an arbitrary vertex belong to $V$, then there exists a walk of length $n - 1$ from $u$ to $x$, where $x$ having a loop. Hence

$$\bigcap_{i=1}^n u_i G^{e_i} \neq \phi, L(n, n, d) = n - 1$$

Note that $L_{C_0}(n) = n - 1$ by Lemma 2.4, hence

$$L(n, n, d) = n - 1$$

Applying Lemma 2.3, Theorem 2.1 and Theorem 2.2, we have conclusion.

**Corollary 3.3** Let $G \in P_s(d)$, $n, k, d$ and be integers, $n \geq 2, 2 \leq k \leq n, 1 \leq d \leq n$, then $n - \left[ \frac{d}{2} \right] \leq L(n, k, d) \leq n - 1$.

**Corollary 3.4** Let $G$ be a primitive digraph of order $n$ with girth $s (1 \leq s \leq n - 1)$, then $L_G(2) \leq s \left(n - \left[ \frac{s}{2} \right]\right)$.

**Proof:** Since $G$ is a primitive digraph of order $n$ with girth $s$, then $G^s$ is a primitive digraph of order $n$ with exactly $s$ vertices having loop. By Theorem 3.1, we have

$$L_G(2) \leq s \left(n - \left[ \frac{s}{2} \right]\right)$$

**Theorem 3.5** Let $n$ and $d$ be integers, $1 \leq d \leq n, n \geq 2$, then there exists $Q \in P_s(V, d)$ so that $L_Q(n, 2) = r$ for arbitrary $r \in \left(1, 2, \ldots, n - \left[ \frac{d}{2} \right]\right)$.

**Proof:** Let $V = \{v_1, v_2, \ldots, v_r\}$.

We construct $Q \in P_s(V, d)$ so that $L_Q(n, 2) = r$ for
Let $m = r + r = 2r$, then $m \leq d$. Let

$$E_k = \{(v_i, v_j), (v_j, v_{j+1}), (v_{m-w-1}, v_k), (v_k, v_i) \mid i = 1, \ldots, d, j = 1, 2, \ldots, m-1, k = m+1, \ldots, n, s = d+1, \ldots, n, t = m+1, \ldots, d\}$$

$G(Q_k) = G(V, E_k)$. It is obvious that $Q_k \in P_n(V, d)$ and $L_{Q_k}(n, 2) = r$.

The proof is now completed.

**Remark 3.6** By Theorem 3.5, we obtain that the set of the 2-common consequent of primitive digraphs of order $n$ with exact $d$ vertices having loop is

$$\{1, 2, \ldots, n - \left[d \over 2\right]\}.$$

But, in Theorem 3.5, $Q \in P_n(V, d)$ is not unique.

**Example.**

Let $n = 7, d = 5, r = 3, V = \{1, 2, 3, 4, 5, 6, 7\}$,

$$E_3 = \{(i, i), (j, j+1), (6, 1), (7, 1), (5, 7) \mid i = 1, 2, \ldots, 5, j = 1, \ldots, 5\}$$

$$E_6 = \{(i, i), (j, j+1), (6, 1), (7, 1), (5, 7), (3, 2) \mid i = 1, 2, \ldots, 5, j = 1, \ldots, 5\}$$

$G(Q_3) = G(V, E_3), i = 5, 6$.

Obviously,

$$Q_3 \in P_5(V, 5), i = 5, 6. \quad L_{Q_3}(7, 2) = 3,$$

but $M_3 = M(Q_3)$ and $M_6 = M(Q_6)$ are not isomorphic digraph.

**REFERENCES**


