Some Results on \( (1,2n-1) \)-Odd Factors

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1. Introduction

We consider finite undirected graph without loops and multiple edges. Let \( G \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). Given \( x \in V(G) \), the set of vertexes adjacent to \( x \), denoted by \( N_G(x) \), and \( d_G(x) = |N_G(x)| \) is called the degree of \( x \). If there exists a spanning subgraph \( F \) such that \( d_F(x) \in \{1,3,\cdots,2n-1\} \), then \( F \) is called to be \( (1,2n-1) \)-odd factor of \( G \). Some sufficient and necessary conditions are given for \( G - U \) to have \( (1,2n-1) \)-odd factor where \( U \) is any subset of \( V(G) \) such that \( |U| = k \).

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graph with \((1,2n-1)\)-odd factor we have
\[ o(G' - B') \leq (2n-1)|B'|. \]

Noting that \(G' - B' = G - B\),
Therefore
\[ o(G - B) = o(G' - B') \leq (2n-1)|B'| = (2n-1)(|B| - k) \]
For any \(B \subseteq V(G)\) and \(|B| \geq k\) we have
\[ o(G - B) \leq (2n-1)(|B| - k), \]
the following that the set \(U\) with any \(k\) vertexes, \(G' = G - U\) has \((1,2n-1)\)-odd factor, i.e., for any \(B' \subseteq V(G')\), there \(o(G' - B') \leq (2n-1)|B'|.\)
Noting that \(B = U \cup B'\), of course \(|B| \geq k\).
By
\[ o(G - B) \leq (2n-1)(|B| - k), \]
and \(G' - B' = G - B\), we have
\[ o(G' - B') = o(G - B) \leq (2n-1)(|B| - k) = (2n-1)|B'|. \]

**Lemma 2** [9] Connected claw free graphs of even order have 1-factor.

**Lemma 3** Connected claw free graphs of even order have \((1,2n-1)\)-odd factor.

**Proof** If \(n = 1\), by lemma 2, the conclusion is proved. Assume that \(n \geq 2\).

By contradiction, we assume that \(G\) has no \((1,2n-1)\)-odd factor, i.e., \(\exists S \subseteq V(G)\) such that
\[ o(G - S) > (2n-1)|S| \geq 3|S|(n \geq 2). \]
then there exists \(x \in S\) such that \(x\) connecting with three components of \(G - S\) at least. If not, for \(\forall x \in S\), \(x\) connects with two components of \(G - S\) at most, consequently \(o(G - S) \leq 2|S|\), contradiction.

**Theorem 1** Let \(G\) be graph with \(p\) order, \(x, y\) are a couple of nonadjacent vertexes and satisfy
\[ d_G(x) + d_G(y) \geq p + k + 1, \]
then the sufficient and necessary condition for \(G\) removing any \(k\) vertexes with \((1,2n-1)\)-odd factor is that \(G + xy\) getting rid of any \(k\) vertexes with \((1,2n-1)\)-odd factor.

**Proof** The necessary condition is obvious, next we prove the sufficient condition.

By contradiction, let \(G + xy\) remove any \(k\) vertexes with \((1,2n-1)\)-odd factor, but there exist \(k\) vertexes after getting rid of the \(k\) vertexes of \(G\) without \((1,2n-1)\)-odd factor. By lemma 1, there exists
\[ B \subseteq V(G), |B| \geq k \]
such that
\[ o(G - B) > (2n-1)(|B| - k), \]
and
\[ o(G + xy - B) \leq (2n-1)(|B| - k). \]

at the same time, by \(o(G - B) + |B| = p (mod 2)\) and \(p = k (mod 2)\),
Thereby \(o(G - B) \geq (2n-1)(|B| - k) + 2.\)
Furthermore, by \(o(G + xy - B) \geq o(G - B) - 2,\)
Consequently
\[ (2n-1)(|B| - k) \geq o(G + xy - B) \geq o(G - B) - 2 \geq (2n-1)(|B| - k) + 2 - 2 \]
Accordingly
\[ o(G - B) = (2n-1)(|B| - k) + 2 \]
and
\[ o(G + xy - B) = (2n-1)(|B| - k). \]
It shows that \(x, y\) are part of two odd components \(C_1, C_2\) of \(G - B\) respectively. Thus
\[ d_G(x) + d_G(y) \leq |V(C_1)| - 1 + |V(C_2)| - 1 + 2|B|. \]
On the other hand, by hypothesis
\[ d_G(x) + d_G(y) \geq p + k - 1 \geq |B| + |V(C_1)| + |V(C_2)| \]
\[ + (2n-1)(|B| - k) + k - 1. \]
But
\[ (2n - 2)|B| > (2n - 2)k - 1. \]

Contradiction.

**Theorem 2** Let \(t \leq k + 1\) connected graph \(G\) be \(p\) order, \(x, y\) are a couple of any nonadjacent vertexes of \(G\), and satisfy
\[ |N_G(x)| |N_G(y)| \geq p - t + k - 1, \]
then the sufficient and necessary condition for \(G\) removing any \(k\) vertexes with \((1,2n-1)\)-odd factor is \(G + xy\) getting rid of any \(k\) vertexes with \((1,2n-1)\)-odd factor.

**Proof** \(G\) is a spanning subgraph of \(G + xy\), so the necessary condition is obvious.
Next we prove the sufficient condition. We suppose \(G + xy\) getting rid of any \(k\) vertexes with \((1,2n-1)\)-odd factor, but \(G\) is not, i.e. there exist
\[ B \subseteq V(G), |B| \geq k \]
such that
\[ o(G - B) > (2n - 1)(|B| - k). \]
Be similar to the discussion of theorem 1.
\[ o(G - B) > (2n - 1)(|B| - k) + 2 \]

and
\[ o(G + xy - B) = (2n - 1)(|B| - k). \]

thereby \( x, y \) are part of two odd components \( C_1, C_2 \) of \( G - B \) respectively.

Noting that
\[ |N_G(x) \cup N_G(y)| \leq |V(C_1)| - 1 + |V(C_2)| - 1 + |B| \]

(1)

By hypothesis
\[ |N_G(x) \cup N_G(y)| \geq p - t + k - 1 \geq |V(C_1)| + |V(C_2)| - t \]
\[ + (2n - 1)(|B| - k) - t + k - 1 \]

(2)

Combining (1) with (2)
\[ -2 \geq (2n - 1)(|B| - k) - t + k - 1 \]

Consequently
\[ \frac{t - k - 1}{2n - 1} + k \geq |B| \geq k, \]

but \( t \leq k + 1 \).

Contradiction.

**Theorem 3** Let \( G \) be claw free graphs, \( x \) be partial \( k \) connection point. \( G' \) be graph obtained by locally fully on \( G \) in \( x \) point, then for \( U \subseteq V(G), |U| = k \), the sufficient and necessary condition for \( G - U \) with \((1,2n-1)\)-odd factor is \( G' - U \) with \((1,2n-1)\)-odd factor.

**Proof** \( G \) is a spanning subgraph of \( G' \), so the necessary condition is obvious.

Next we prove the sufficient condition. Let \( G' - U \) have \((1,2n-1)\)-odd factor, \( G - U \) have no \((1,2n-1)\)-odd factor. \( G' - U \) has \((1,2n-1)\)-odd factor,
\[ |V(G')| \equiv k \pmod{2} \]  
so \[ |V(G)| \equiv k \pmod{2} \]  

On the other hand, \( G \) is claw free, so \( G - U \) is claw free.

By lemma 2, lemma 3, \( G - U \) has two odd components at least.

If \( x \not\in U \), let \( x \in C_0 \) ( \( C_0 \) is branch of \( G - U \)). Now, \( G - U \) has the same odd components as \( G' - U \), therefore, \( G - U \) has \((1,2n-1)\)-odd factor. which is contradiction.

Next let \( x \in U \), since \( G' - U \) has not odd components, for any odd components of \( G - U \),
\[ N_G(x) \cap V(C) \neq \Phi \]
is complete.

Let \( x_1, x_2 \) be adjacent vertexes of \( x \) in two odd components of \( G - U \) respectively.

Then \( x_1, x_2 \) is nonadjacent in the induced subgraph of \( N_G(U - \{x\}) \), which is contradiction to the fact that \( x \) is a locally \( k \) connected vertex, since
\[ |U - \{x\}| \leq k - 1 \]

The proof is complete.

**REFERENCES**


