The $H$-Decomposition Problem for Graphs

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ABSTRACT

The concept of $H$-decompositions of graphs was first introduced by Erdős, Goodman and Pósa in 1966, who were motivated by the problem of representing graphs by set intersections. Given graphs $G$ and $H$, an $H$-decomposition of $G$ is a partition of the edge set of $G$ such that each part is either a single edge or forms a graph isomorphic to $H$. Let $\phi(n,H)$ be the smallest number $\phi$, such that, any graph of order $n$ admits an $H$-decomposition with at most $\phi$ parts. The exact computation of $\phi(n,H)$ for an arbitrary $H$ is still an open problem. Recently, a few papers have been published about this problem. In this survey we will bring together all the results about $H$-decompositions. We will also introduce two new related problems, namely Weighted $H$-Decompositions of graphs and Monochromatic $H$-Decompositions of graphs.

Keywords: Graph Decompositions; Weighted Graph Decompositions; Monochromatic Graph Decompositions; Turán Graph; Ramsey Numbers

1. Introduction

1.1. Terminology and Notations

For notation and terminology not discussed here the reader is referred to [1]. A graph is a (finite) set $V = V(G)$, called the vertices of $G$ together with a set $E = E(G)$ of (unordered) pairs of vertices of $G$, called the edges. We do not allow loops and multiple edges.

The number of vertices of a graph is its size and is denoted by $|V(G)|$. A vertex $v$ is incident with an edge $e$ if $v \in e$ and the two vertices incident with an edge are called its endpoints. Two vertices $x, y$ of $G$ are said to be adjacent or neighbors if $\{x, y\}$ is an edge of $G$. The degree of a vertex $v$ is the number of edges incident with $v$ and will be denoted by $\deg v$ or simply by $\deg v$ if it is clear which graph is being considered. The complete graph (clique) of order $n$ will be denoted by $K_n$, the complete bipartite graph with parts of size $m$ and $n$ will be denoted by $K_{m,n}$ and the cycle of length $n$ will be denoted by $C_n$.

The Turán graph of order $n$, denoted by $T_{r-1}(n)$, is the unique complete $(r-1)$-partite graph on $n$ vertices where every partite class has either $\left\lfloor \frac{n}{r-1} \right\rfloor$ or $\left\lceil \frac{n}{r-1} \right\rceil$ vertices. The well-known Turán's Theorem [2] states that $T_{r-1}(n)$ is the unique graph on $n$ vertices that has the maximum number of edges and contains no complete subgraph of order $r$. We let $t_{r-1}(n)$ denote the number of edges in $T_{r-1}(n)$.

Finally, a proper colouring or simply a colouring of the vertices of $G$ is an assignment of colours to the vertices in such a way that adjacent vertices have distinct colours; $\chi(G)$ is then the minimum number of colours in a (vertex) colouring of $G$. For example, $\chi(K_r) = r$, $\chi(C_n) = 2$ and $\chi(C_{2r-1}) = 3$.

1.2. Motivation and Definitions

Given two graphs $G$ and $H$, an $H$-decomposition of $G$ is a partition of the edge set of $G$ such that each part is either a single edge or forms an $H$-subgraph, i.e., a graph isomorphic to $H$. We allow partitions only, that is, every edge of $G$ appears in precisely one part. Let $\phi(G,H)$ be the smallest possible number of parts in an $H$-decomposition of $G$. It is easy to see that $\phi(G,H) = e(G) - p_H(G)(e(H) - 1)$, where $p_H(G)$ is the maximum number of pairwise edge-disjoint $H$-subgraphs that can be packed into $G$. Building upon a body of previous research, Dor and Tarsi [3] showed that if $H$ has a component with at least 3 edges, then the problem of checking whether an input graph $G$ is perfectly decomposable into $H$-subgraphs is NP-complete. Hence, it is NP-hard to compute the function $\phi(G,H)$ for such $H$. Therefore, the aim is to study the function
\[ \phi(n, H) = \max \{ \phi(G, H) | v(G) = n \}, \]

which is the smallest number such that any graph \( G \) of order \( n \) admits an \( H \)-decomposition with at most \( \phi(n, H) \) parts.

This function was first studied, in 1966, by Erdős, Goodman and Pósa [4], who were motivated by the problem of representing graphs by set intersections. They proved that \( \phi(n, K_r) = t_r(n) \). A decade later, this result was extended by Bollobás [5], who proved that \( \phi(n, K_r) = t_{r-1}(n) \), for all \( n \geq r \geq 3 \).

General graphs \( H \) were only considered recently by Pikhurko and Sousa [6]. In Section 2 we will present known results about the exact value of the function \( \phi(n, H) \) for some special graphs \( H \) and its asymptotic value for arbitrary \( H \). In Sections 3 and 4 two new \( H \)-decomposition problems will be introduced, namely the weighted version and the monochromatic version respectively.

2. \( H \)-Decompositions of Graphs

In 1966, Erdős, Goodman and Pósa [4], who were motivated by the problem of representing graphs by set intersections, proved that \( \phi(n, K_r) = t_r(n) \) and a decade later Bollobás [5] proved that \( \phi(n, K_r) = t_{r-1}(n) \), for all \( n \geq r \geq 3 \). Recently, Pikhurko and Sousa [6] studied the function \( \phi(n, H) \) for arbitrary graphs \( H \). They proved the following result.

**Theorem 2.1.** [6] Let \( H \) be any fixed graph with chromatic number \( r \geq 3 \). Then,

\[ \phi(n, H) = t_{r-1}(n) + o(n^2). \]

Let \( \text{ex}(n, H) \) denote the maximum number of edges in a graph of order \( n \), that does not contain \( H \) as a subgraph. Recall that \( \text{ex}(n, K_r) = t_{r-1}(n) \). The same authors also made the following conjecture.

**Conjecture 2.2.** For any graph \( H \) with chromatic number at least 3, there is \( n_0 = n_0(H) \) such that \( \phi(n, H) = \text{ex}(n, H) \) for all \( n \geq n_0 \).

The exact value of the function \( \phi(n, H) \) is far from being known, however, this conjecture has been verified for some special graphs. The following results have been proved by Sousa.

**Theorem 2.3.** [7] For all \( n \geq 6 \) we have

\[ \phi(n, C_5) = t_2(n) = \left\lceil n^2/4 \right\rceil. \]

**Theorem 2.4.** [8] For all \( n \geq 10 \) we have

\[ \phi(n, C_7) = t_2(n) = \left\lceil n^2/4 \right\rceil. \]

For \( r \geq 3 \), a clique-extension of order \( r+1 \) is a connected graph that consists of a \( K_r \) plus another vertex, say \( x \), adjacent to at most \( r-1 \) vertices of \( K_r \). For \( i = 1, \ldots, r-1 \) the \( H_{i/r} \) be the clique-extension of order \( r+1 \) that has \( \text{deg} x = i \).

**Theorem 2.5.** [9] For all \( n \geq 4 \) and \( i = 1, 2 \) we have

\[ \phi(n, H_{i/r}) = t_{i}(n) = \left\lceil n^2/4 \right\rceil. \]

**Theorem 2.6.** [9] Let \( r \geq 4 \) and let \( H \) be any clique-extension of order \( r+1 \). For all \( n \geq r+1 \) we have

\[ \phi(n, H) = t_{r-1}(n). \]

A graph \( H \) is said to be edge-critical if there exists an edge \( e \in E(H) \) whose deletion decreases the chromatic number, that is, \( \chi(H) > \chi(H - e) \). Cliques and odd-cycles are examples of edge-critical graphs. Özkahya and Person [10] were able to prove that Pikhurko and Sousa’s conjecture is true for all edge-critical graphs. Their result is the following.

**Theorem 2.7.** [10] Let \( H \) be any edge-critical graph with chromatic number \( r \geq 3 \). Then, there exists \( n_0 \) such that \( \phi(n, H) = \text{ex}(n, H) \), for all \( n \geq n_0 \). Moreover, the only graph attaining \( \phi(n, H) \) is the Turán graph \( T_{r-1}(n) \).

The case when \( H \) is a bipartite graph has been less studied. Pikhurko and Sousa [6] determined \( \phi(n, H) \) for any fixed bipartite graph with an \( O(1) \) additive error. For a non-empty graph \( H \), let \( \text{gcd}(H) \) denote the greatest common divisor of the degrees of \( H \). For example, \( \text{gcd}(K_{5,4}) = 2 \), while for any tree \( T \) with at least 2 vertices we have \( \text{gcd}(T) = 1 \). They proved the following result.

**Theorem 2.8.** [6] Let \( H \) be a bipartite graph with \( m \) edges and let \( d = \text{gcd}(H) \). Then there is \( n = n_H(H) \) such that for all \( n \geq n_H \) the following statements hold.

If \( d = 1 \), then if \( \left\lfloor \frac{n}{2} \right\rfloor = m-1(\text{mod} m) \),

\[ \phi(n, H) = \phi(n, K_n) = \left\lceil \frac{n(n-1)}{2m} \right\rceil + m - 1, \]

otherwise,

\[ \phi(n, H) = \phi(n, K_n^*) = \left\lceil \frac{n(n-1)}{2m} \right\rceil + m - 2 \]

where \( K_n^* \) denotes any graph obtained from \( K_n \) after deleting at most \( m-1 \) edges in order to have \( e(K_n^*) = m-1(\text{mod} m) \). Furthermore, if \( G \) is extremal then \( G \) is either \( K_n \) or \( K_n^* \).

If \( d \geq 2 \), then

\[ \phi(n, H) = \frac{nd}{2m} \left( \left\lfloor \frac{n}{d} \right\rfloor - 1 \right) + \frac{1}{2} n(d-1) + O(1). \]

Moreover, there is a procedure with running time polynomial in \( \log n \) which determines \( \phi(n, H) \) and
3. Weighted H-Decompositions of Graphs

In 2011, Sousa [11] introduced a weighted version of the $H$-decomposition problem for graphs. More precisely, let $G$ and $H$ be two graphs and $b$ a positive number. A weighted $(H,b)$-decomposition of $G$ is a partition of the edge set of $G$ such that each part is either a single edge or forms an $H$-subgraph, i.e., a graph isomorphic to $H$. We assign a weight of $b$ to each $H$-subgraph in the decomposition and a weight of 1 to single edges. The total weight of the decomposition is the sum of the weights of all elements in the decomposition. Let $\phi(G,H,b)$ be the smallest possible weight in an $(H,b)$-decomposition of $G$.

As before, the goal is to study the function

$$\phi(n,H,b) = \max \{\phi(G,H,b) \mid \nu(G) = n\},$$

which is the smallest number such that any graph $G$ with $n$ vertices admits an $(H,b)$-decomposition with weight at most $\phi(G,H,b)$.

Note that when we take $b=1$ the original $H$-decomposition problem is recovered, hence, it suffices to consider the case when $b \neq 1$. Furthermore, when $b \geq e(H)$ we easily have $\phi(n,H,b) = \left\lfloor \frac{n}{2} \right\rfloor$. Therefore, one only has to consider the case when $0 \leq b \leq e(H)$ and $b \neq 1$. Sousa [11] obtained the asymptotic value of the function $\phi(n,H,b)$ for any fixed bipartite graph $H$ when $0 \leq b \leq e(H)$ and $b \neq 1$.

Recall that for a non-empty graph $H$, $\text{gcd}(H)$ denotes the greatest common divisor of the degrees of $H$. Sousa proved the following result.

**Theorem 3.1.** [11] Let $H$ be a bipartite graph with $m$ edges, let $d = \text{gcd}(H)$ and $0 < b < m$ with $b \neq 1$ a constant. Then there is $n_0 = n_0(H)$ such that for all $n \geq n_0$ the following statements hold.

If $d = 1$, then

$$\phi(n,H,b) = bn\left(\frac{n-1}{2m}\right) + O(1).$$

If $d \geq 2$, let $n-1 = qd+r$ where $0 \leq r \leq d-1$ is an integer.

If $r \neq 0$ and $d-1 \leq \frac{bd}{m} + r$, then

$$\phi(n,H,b) = \frac{b}{m}\left(\frac{n}{2}\right) + \frac{1}{2}n\left(r - \frac{br}{m}\right) + O(1).$$

If $r = 0$ and $d-1 \geq \frac{bd}{m} + r$, then

$$\phi(n,H,b) = \frac{b}{m}\left(\frac{n}{2}\right) + \frac{1}{2}n\left(d - 1 - \frac{br}{m}\right) + O(1).$$

If $r = 0$ and $\frac{b}{m} < 1 - \frac{5d^2}{5d^2 - 2}$, then

$$\phi(n,H,b) = \frac{b}{m}\left(\frac{n}{2}\right) + \frac{1}{2}n\left(d - 1 - \frac{bd}{m}\right) + O(1).$$

If $r = 0$ and $1 - \frac{5d^2}{5d^2 - 2} \leq \frac{b}{m} \leq 1 - \frac{1}{d}$, then

$$\frac{b}{m}\left(\frac{n}{2}\right) + \frac{1}{2}n\left(d - 1 - \frac{bd}{m}\right) - \frac{1}{2} \leq \phi(n,H,b)$$

and

$$\phi(n,H,b) \leq \frac{b}{m}\left(\frac{n}{2}\right) + \frac{m-b}{5md^2}n.$$

If $r = 0$ and $\frac{b}{m} \geq 1 - \frac{1}{d}$, then

$$\frac{b}{m}\left(\frac{n}{2}\right) \leq \phi(n,H,b) \leq \frac{b}{m}\left(\frac{n}{2}\right) + \frac{m-b}{5md^2}n.$$

The case when $H$ is not a bipartite graph is still an open problem.

4. Monochromatic H-Decompositions of Graphs

In this section the $H$-decomposition problem is extended to coloured versions of the graph $G$ and monochromatic copies of $H$. We define the problem more precisely.

A $k$-edge-colouring of a graph $G$ is a function $c: E(G) \rightarrow \{1, \ldots, k\}$. We think of $c$ as a colouring of the edges of $G$, where each edge is given one of $k$ possible colours. Given a fixed graph $H$, a graph $G$ of order $n$ and a $k$-edge-colouring of the edges of $G$, a monochromatic $H$-decomposition of $G$ is a partition of the edge set of $G$ such that each part is either a single edge or a monochromatic copy of $H$. Let $\phi_k(G,H)$ be the smallest number such that, for any $k$-edge-colouring of $G$, there exists a monochromatic $H$-decomposition of $G$ with at most $\phi_k(G,H)$ elements. The objective is to study the function

$$\phi_k(n,H) = \max \{\phi_k(G,H) \mid \nu(G) = n\},$$

which is the smallest number such that, any $k$-edge-coloured graph of order $n$ admits a monochromatic $H$-decomposition with at most $\phi_k(G,H)$ elements.

This function was introduced recently by Liu and Sousa [12] and they studied the function $\phi_k(n,K_r)$ for...
all \( k \geq 2 \) and \( r \geq 3 \). Their results involve the Ramsey numbers and the Turán numbers. Recall that for \( r \geq 3 \) and \( k \geq 2 \), the Ramsey number for \( K_r \), denoted by \( R_k (r) \), is the smallest value of \( s \), for which every \( k \)-edge-colouring of \( K_s \) contains a monochromatic \( K_r \). The Ramsey numbers are notoriously difficult to calculate, even though, it is known that their values are finite for all \( r \geq 3 \) and \( k \geq 2 \). In fact, for the Ramsey numbers \( R_k (r) \), only three of them are currently known. In 1955, Greenwood and Gleason [13] were the first to determine \( R_5 (3) = 6 \), \( R_5 (3) = 17 \) and \( R_4 (4) = 18 \). Liu and Sousa [12] proved the following results about monochromatic \( K_r \)-decompositions.

**Theorem 4.1.** [12] Let \( k = 2, 3 \). There is an \( n_0 \) such that, for all \( n \geq n_0 \), we have

\[
\phi_k (n, K_3) = t_{R_k (3)-1} (n).
\]

That is, \( \phi_k (n, K_3) = t_k (n) \) and \( \phi_k (n, K_3) = t_{tk} (n) \). Moreover, the only \( k \)-edge-coloured graph \( G \) attaining \( \phi_k (n, K_3) \) is the Turán graph \( t_{R_k (3)-1} (n) \).

**Theorem 4.2.** [12] For all \( k \geq 4 \) we have

\[
\phi_k (n, K_3) = t_{R_k (3)-1} (n) + o(n^2).
\]

The same authors also made the following conjecture.

**Conjecture 4.3.** Let \( k \geq 4 \). Then

\[
\phi_k (n, K_3) = t_{R_k (3)-1} (n) \quad \text{for} \quad n \geq R_k (3).
\]

Larger cliques were also studied by Liu and Sousa and they obtained the exact value of the function \( \phi_k (n, K_r) \) for all \( k \geq 2 \) and \( r \geq 4 \). Recall that the Ramsey number \( R_2 (4) = 18 \) is also well-known.

**Theorem 4.4.** [12] Let \( r \geq 4 \), \( k \geq 2 \). There is an \( n_0 = n_0 (r, k) \) such that, for all \( n \geq n_0 \), we have

\[
\phi_k (n, K_r) = t_{R_k (r)-1} (n).
\]

In particular, \( \phi_k (n, K_4) = t_{t_7} (n) \). Moreover, the only graph attaining \( \phi_k (n, K_r) \) is the Turán graph \( T_{R_k (r)-1} (n) \).

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