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# Sinc-Collocation Method for Solving Linear and Nonlinear System of Second-Order Boundary Value Problems

**Mohamed El-Gamel**
Department of Mathematical Sciences, Faculty of Engineering, Mansoura University, Mansoura, Egypt
Email: gamel_eg@yahoo.com

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## ABSTRACT

Sinc methods are now recognized as an efficient numerical method for problems whose solutions may have singularities, or infinite domains, or boundary layers. This work deals with the sinc-collocation method for solving linear and nonlinear system of second order differential equation. The method is then tested on linear and nonlinear examples and a comparison with B-spline method is made. It is shown that the sinc-collocation method yields better results.

**Keywords:** Sinc Function; Collocation Method; System; Numerical Solution

## 1. Introduction

Numerous problems in physics, chemistry, biology and engineering science are modelled mathematically by systems of ordinary differential equations, e.g. series circuits, mechanical systems with several springs attached in series lead to a system of differential equations (for example see [1,2]). However, many classical numerical methods used with second-order initial value problems cannot be applied to second-order boundary value problems (BVPs).

Most realistic systems of ordinary differential equations do not have exact analytic solutions, so approximation and numerical techniques must be used. There are many publications dealing with the linear system of second-order boundary value problems. They introduced various numerical methods. For instance, a finite difference method has been proposed in recent works [3-8]. For a nonlinear system of second- order BVPs, there are few valid methods to obtain numerical solutions. Geng *et al*. have studied the numerical solution of a nonlinear system of second-order boundary value problems in the reproducing kernel space [9]. Lu considered the variational iteration method to solve a nonlinear system of second-order boundary value problems [10]. Recently, Bataineh *et al*. [11] represented modified homotopy method for solving systems of second-order boundary value problems. Sinc-collocation method was applied to solve nonlinear systems of second order boundary value problems in [12].

In this paper, we discuss the use of sinc-collocation method for solving a class of linear and non-linear system of differential equations

$$\begin{aligned}&\sum_{i=0}^{2}\mu_i(x)u_1^{(i)}(x)+\sum_{i=0}^{2}\kappa_i(x)u_2^{(i)}(x)=f_1(x,u_1,u_2)\\&\sum_{i=0}^{2}\tau_i(x)u_1^{(i)}(x)+\sum_{i=0}^{2}\sigma_i(x)u_2^{(i)}(x)=f_2(x,u_1,u_2),\\&\qquad x\in J=[a,b]\end{aligned}\tag{1}$$

subject the boundary conditions

$$\begin{aligned}u_1(a)&=u_1(b)=0,\\u_2(a)&=u_2(b)=0.\end{aligned}\tag{2}$$

where $u_1(x)$, $u_2(x)$, $f_1(x,u_1,u_2)$, $f_2(x,u_1,u_2)$, and $\mu_i(x)$, $\kappa_i(x)$, $\sigma_i(x)$, and $\tau_i(x)$, for $i=0,1,2$, are analytic functions. It will always be assumed that (1) possesses a unique solution $u\in C^n(J)$.

Numerical examples including regular, singular as well as singularly perturbed problems are considered. On the basis of these examples, the results reveal that the method is very effective and convenient.

The paper is organized into five sections. Section 2 contains notation, definitions and some results of sinc function theory. In Section 3, the sinc-collocation method is developed for linear second-order system of differential equation with homogeneous boundary conditions. The method is developed for nonlinear second-order system of differential equation in Section 4. Some numerical examples are presented in Section 5. Finally, Section 6 provides conclusions of the study.

## 2. Sinc Function

In recent years, a lot of attention has been devoted to the study of the sinc method to investigate various scientific models. The efficiency of the method has been formally proved by many researchers [13-22].

A general review of sinc function approximation is given in [23,24]. Hence, only properties of the sinc function that are used in the sequel.

If $f(x)$ is defined on the real line, then for $h>0$ the Whittaker cardinal expansion of $f$ is given by:

$$f_m(x)=\sum_{k=-N}^{N} f_k S(k,h)(x), \quad m=2N+1 \tag{3}$$

where $f_k=f(x_k)$, $x_k=hk$, and the mesh size is given by

$$h=\sqrt{\frac{\pi d}{\alpha N}}, \quad 0<\alpha\le 1, \quad d\le\frac{\pi}{2} \tag{4}$$

where $N$ is suitably chosen and $\alpha$ depends on the asymptotic behavior of $f(x)$. The n-th derivative of the function f at the sampling points $x_k=kh$ can be approximated using a finite number of terms as:

$$f^{(n)}(x_k)\cong h^{-n}\sum_{k=-N}^{N}\delta_{jk}^{(n)} f_k$$

where

$$\delta_{jk}^{(n)}=\frac{\mathrm{d}^n}{\mathrm{d}x^n}S(j,h)(x)\Big|_{x=x_k}$$

In particular,

$$\delta_{jk}^{(0)}=\left[S(j,h)(x)\right]\Big|_{x=x_k}=\begin{cases}1, & j=k,\\ 0, & j\ne k,\end{cases} \tag{5}$$

$$\delta_{jk}^{(1)}=\frac{\mathrm{d}}{\mathrm{d}x}\left[S(j,h)(x)\right]\Big|_{x=x_k}=\begin{cases}0, & j=k,\\ \dfrac{(-1)^{k-j}}{k-j}, & j\ne k,\end{cases} \tag{6}$$

and

$$\delta_{jk}^{(2)}=\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[S(j,h)(x)\right]\Big|_{x=x_k}=\begin{cases}\dfrac{-\pi^2}{3}, & j=k,\\ \dfrac{-2(-1)^{k-j}}{(k-j)^2}, & j\ne k.\end{cases} \tag{7}$$

We note that

$$\delta_{kj}^{(0)}=\delta_{jk}^{(0)}, \quad \delta_{kj}^{(2)}=\delta_{jk}^{(2)} \quad \text{and} \quad \delta_{kj}^{(1)}=-\delta_{jk}^{(1)}.$$

The interpolation formula for $f(x)$ over $[a,b]$ takes the form

$$f(x)\approx\sum_{k=-N}^{N} f_k S(k,h)\circ\phi(x), \tag{8}$$

where the basis functions on $(a,b)$ are then given by

$$S(k,h)\circ\phi(x)=\operatorname{sinc}\left(\frac{\phi(x)-kh}{h}\right)$$

and the transformation function

$$\phi(x)=\ln\left(\frac{x-a}{b-x}\right) \tag{9}$$

transforms $[a,b]$ to the infinite range $[-\infty,\infty]$. The interpolation points $\{x_k\}$ are then given by:

$$x_k=\frac{a+b\mathrm{e}^{kh}}{1+\mathrm{e}^{kh}}$$

The $n$-th derivative of the function $f$ at points $x_k$ can be approximated using a finite number of terms as

$$f^{(n)}(x)\approx\sum_{k=-N}^{N} f_k\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[S(k,h)\circ\phi(x)\right]. \tag{10}$$

Setting

$$\frac{\mathrm{d}^i}{\mathrm{d}\phi^i}\left[S(j,h)\circ\phi(x)\right]=S_j^{(i)}(x), \quad 0\le i\le 2,$$

and noting that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[S(j,h)\circ\phi(x)\right]=S_j^{(1)}(x)\phi'(x)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[S(j,h)\circ\phi(x)\right]=S_j^{(2)}(x)\left[\phi'(x)\right]^2+S_j^{(1)}(x)\phi''(x)$$

and

$$\delta_{jk}^{(n)}=h^n\frac{\mathrm{d}^n}{\mathrm{d}\phi^n}\left[S(j,h)\circ\phi(x)\right]_{x=x_k}$$

which will be used later.

## 3. System of Linear Second Order Equations

Consider a linear, system of linear second order equations of the form

$$\begin{aligned}&\sum_{i=0}^{2}\mu_i(x)u_1^{(i)}(x)+\sum_{i=0}^{2}\kappa_i(x)u_2^{(i)}(x)=f_1(x)\\ &\sum_{i=0}^{2}\tau_i(x)u_1^{(i)}(x)+\sum_{i=0}^{2}\sigma_i(x)u_2^{(i)}(x)=f_2(x),\\ &x\in J=[a,b]\end{aligned} \tag{11}$$

We assume that $u_1(x)$ and $u_2(x)$ the solutions of (11) and (2), is approximated by the finite expansion of Sinc basis functions

$$u_{1m}(x)=\sum_{j=-N}^{N}c_j S_j(x), \quad m=2N+1 \tag{12}$$

and

$$u_{2m}(x)=\sum_{j=-N}^{N}d_j S_j(x), \quad m=2N+1 \tag{13}$$

 

where $S_j(x)$ is the function $S(j,h)\circ\phi(x)$ for some fixed step size $h$. If we replace each term of (11) with its corresponding approximation given by the right-hand side of (10) and (8) we have

$$\begin{aligned}
&\sum_{j=-N}^{N}\left(\left[\sum_{i=0}^{2}\mu_i(x)\frac{\mathrm{d}^i}{\mathrm{d}x^i}S(j,h)\circ\phi(x)\right]c_j + \left[\sum_{i=0}^{2}\kappa_i(x)\frac{\mathrm{d}^i}{\mathrm{d}x^i}S(j,h)\circ\phi(x)\right]d_j\right) = f_1(x)\\
&\sum_{j=-N}^{N}\left(\left[\sum_{i=0}^{2}\tau_i(x)\frac{\mathrm{d}^i}{\mathrm{d}x^i}S(j,h)\circ\phi(x)\right]c_j + \left[\sum_{i=0}^{2}\sigma_i(x)\frac{\mathrm{d}^i}{\mathrm{d}x^i}S(j,h)\circ\phi(x)\right]d_j\right) = f_2(x)
\end{aligned} \tag{14}$$

Substituting $x = x_k = \phi^{-1}(kh)$ in (14) and applying the collocation to it, we eventually obtain the following theorem.

**Theorem:** If the assumed approximate solution of problem (11) and (2) is (12) and (13) , then the discrete sinc-collocation system for the determination of the unknown coefficients is given by

$$\begin{aligned}
&\sum_{j=-N}^{N}\left(\left[\sum_{i=0}^{2}g_i(x_k)\frac{\delta_{jk}^{(i)}}{h^i}\right]c_j + \left[\sum_{i=0}^{2}\eta_i(x_k)\frac{\delta_{jk}^{(i)}}{h^i}\right]d_j\right) = f_{1k},\\
&\sum_{j=-N}^{N}\left(\left[\sum_{i=0}^{2}\varepsilon_i(x_k)\frac{\delta_{jk}^{(i)}}{h^i}\right]c_j + \left[\sum_{i=0}^{2}\zeta_i(x_k)\frac{\delta_{jk}^{(i)}}{h^i}\right]d_j\right) = f_{2k},\\
&k = -N, -N+1, \cdots, N
\end{aligned} \tag{15}$$

where

$$g_0(x_k) = \mu_0(x_k), \qquad g_2(x_k) = \mu_2(x_k)\left[\phi'(x_k)\right]^2,$$
$$g_1(x_k) = \mu_1(x_k)\phi'(x_k) + \mu_2(x_k)\phi''(x_k),$$

$$\eta_0(x_k) = \kappa_0(x_k), \qquad \eta_2(x_k) = \kappa_2(x_k)\left[\phi'(x_k)\right]^2,$$
$$\eta_1(x_k) = \kappa_1(x_k)\phi'(x_k) + \mu_2(x_k)\phi''(x_k),$$

$$\varepsilon_0(x_k) = \tau_0(x_k), \qquad \varepsilon_2(x_k) = \tau_2(x_k)\left[\phi'(x_k)\right]^2,$$
$$\varepsilon_1(x_k) = \tau_1(x_k)\phi'(x_k) + \mu_2(x_k)\phi''(x_k),$$

and

$$\zeta_0(x_k) = \sigma_0(x_k),$$
$$\zeta_2(x_k) = \sigma_2(x_k)\left[\phi'(x_k)\right]^2,$$
$$\zeta_1(x_k) = \sigma_1(x_k)\phi'(x_k) + \mu_2(x_k)\phi''(x_k).$$

Now, since $\delta_{jk}^{(2)} = \delta_{kj}^{(2)}$, $\delta_{jk}^{(1)} = -\delta_{kj}^{(1)}$ and $\delta_{jk}^{(0)} = \delta_{kj}^{(0)}$, we write the above equation in the form

$$\begin{aligned}
&\sum_{j=-N}^{N}\left(\left[\sum_{i=0}^{2}g_i(x_k)\frac{(-1)^i\delta_{kj}^{(i)}}{h^i}\right]c_j + \left[\sum_{i=0}^{2}\eta_i(x_k)\frac{(-1)^i\delta_{kj}^{(i)}}{h^i}\right]d_j\right) = f_{1k}\\
&\sum_{j=-N}^{N}\left(\left[\sum_{i=0}^{2}\varepsilon_i(x_k)\frac{(-1)^i\delta_{kj}^{(i)}}{h^i}\right]c_j + \left[\sum_{i=0}^{2}\zeta_i(x_k)\frac{(-1)^i\delta_{kj}^{(i)}}{h^i}\right]d_j\right) = f_{2k}\\
&k = -N, -N+1, \cdots, N
\end{aligned} \tag{16}$$

We now rewrite these equations in matrix form. The system in (16) takes the matrix form

$$\mathbf{Ac} = \Theta, \tag{17}$$

where

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \vdots & \mathbf{A}_{12} \\ \cdots & \cdots & \cdots \\ \mathbf{A}_{21} & \vdots & \mathbf{A}_{22} \end{pmatrix},$$

$$\mathbf{A}_{11} = \sum_{i=0}^{2}\frac{1}{h^i}\mathbf{I}^{(i)}\mathbf{D}(g_i), \quad \mathbf{A}_{12} = \sum_{i=0}^{2}\frac{1}{h^i}\mathbf{I}^{(i)}\mathbf{D}(\eta_i)$$
$$\mathbf{A}_{21} = \sum_{i=0}^{2}\frac{1}{h^i}\mathbf{I}^{(i)}\mathbf{D}(\varepsilon_i), \quad \mathbf{A}_{22} = \sum_{i=0}^{2}\frac{1}{h^i}\mathbf{I}^{(i)}\mathbf{D}(\zeta_i)$$

$$\Theta = \mathbf{D}(f)\mathbf{1} = \begin{pmatrix} f_{1-N} \\ f_{1-N+1} \\ \cdots \\ f_{1N-1} \\ f_{1N} \\ f_{2-N} \\ f_{2-N+1} \\ \cdots \\ f_{2N-1} \\ f_{2N} \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} c_{-N} \\ c_{-N+1} \\ \cdots \\ c_{N-1} \\ c_N \\ d_{-N} \\ d_{-N+1} \\ \cdots \\ d_{N-1} \\ d_N \end{pmatrix}$$

Now we have a linear system of $4N+2$ equations of the $4N+2$ unknown coefficients. We can obtain the coefficients of the approximate solution by solving this linear system. The system (17) may be easily solved by a variety of methods. In this paper we used the Q-R method. The solution $c$ gives the coefficients in the approximate sinc-collocation solutions $u_{1m}(x)$ and $u_{2m}(x)$ of $u_1(x)$ and $u_2(x)$.

## 4. System of Non-Linear Second Order Equations

Consider a nonlinear, system second-order equations of the form

$$\begin{aligned}&\sum_{i=0}^{2}\mu_i(x)u_1^{(i)}(x)+\sum_{i=0}^{2}\kappa_i(x)u_2^{(i)}(x)\\&+N_1(u_1,u_2)=f_1(x)\\&\sum_{i=0}^{2}\tau_i(x)u_1^{(i)}(x)+\sum_{i=0}^{2}\sigma_i(x)u_2^{(i)}(x)\\&+N_2(u_1,u_2)=f_2(x).\end{aligned} \tag{18}$$

where $N_1$ and $N_2$ are nonlinear functions of $u_1$ and $u_2$ is an analytic function and $N(u)$ may be a polynomial or a rational function, or exponential. Due to the large number of different possibilities, our work will be focused mainly on the following forms $N(u)$

1) $N(u)=u^n,\quad n>1$
2) $N(u)=\exp(\pm u),\cos(u),\sin(u),\sin h(u),\cos h(u),\cdots,$
3) $N(u)=\dfrac{1}{(1\pm u)^n},\dfrac{1}{(1\pm u^2)^n},\dfrac{1}{(u^2\pm 1)^n},\quad n\neq 0,$

or any analytic function of $u$ which has a power series expansion. We limit our study to the case

$$N_j(u_1,u_2)=\sum_{i=1}^{2}P_{ji}u_i^n,\quad j=1,2,$$

where $n$ is an integer, or a fraction.

We consider next applying of the sinc-collocation method to solve problem (18) and (2).

**Lemma:** The following relation holds

$$\left[f(x)\right]^n\approx\sum_{k=-N}^{N}f_k^nS(k,h)\circ\phi(x). \tag{19}$$

where $N$ and $h$ are now dependent on both $f(x)$ and $\left[f(x)\right]^n$.

Replacing the terms of (18) with the appropriate representation defined in (8), (10) and (19) and applying the collocation to it, we eventually obtain the following theorem.

**Theorem:** If the assumed approximate solution of problem (18) and (2) is (12) and (13), then the discrete sinc-collocation system for the determination of the unknown coefficients is given by

$$\begin{aligned}&\sum_{j=-N}^{N}\left(\left[\sum_{i=0}^{2}g_i(x_k)\frac{(-1)^i\delta_{kj}^{(i)}}{h^i}\right]c_j+\left[\sum_{i=0}^{2}\eta_i(x_k)\frac{(-1)^i\delta_{kj}^{(i)}}{h^i}\right]d_j\right)\\&+P_{11}(x_k)c_k^n+P_{12}(x_k)d_k^n=f_{1k},\\&\sum_{j=-N}^{N}\left(\left[\sum_{i=0}^{2}\varepsilon_i(x_k)\frac{(-1)^i\delta_{kj}^{(i)}}{h^i}\right]c_j+\left[\sum_{i=0}^{2}\zeta_i(x_k)\frac{(-1)^i\delta_{kj}^{(i)}}{h^i}\right]d_j\right)\\&+P_{21}(x_k)c_k^n+P_{22}(x_k)d_k^n=f_{2k}\end{aligned} \tag{20}$$

Let $\mathbf{c}^n$ be the $4N+2$-vector with *j*-th component given by $\mathbf{c}_j^n$. In this notation the system in (20) takes the matrix form

$$\mathbf{Ac}+\mathbf{Ec}^n=\Theta, \tag{21}$$

where

$$\mathbf{E}=\begin{pmatrix}\mathbf{P}_{11}&\mathbf{P}_{12}\\\mathbf{P}_{21}&\mathbf{P}_{22}\end{pmatrix},$$

$$P_{11}=\mathbf{D}(P_{ii}).$$

Now we have a nonlinear system of $4N+2$ equations in the $4N+2$ unknown coefficients. We can obtain the coefficients in the approximate solution by solving this nonlinear system by *Newton*'*s method*.

Starting from an initial estimate $\mathbf{c}_0$, the corrections are made using

$$\begin{aligned}\mathbf{c}_{j+1}&=\mathbf{c}_jJ^{-1}(\mathbf{c}_j)\left\{\Theta-\mathbf{Ac}_j-\mathbf{Ec}_j^n\right\}\\J(\mathbf{c}_j)&=\mathbf{A}+n\mathbf{Ec}_j^{n-1}.\end{aligned}$$

Here, $\mathbf{c}_j$ is the current iterate, and $\mathbf{c}_{j+1}$ is the new iterate. A common numerical practice is to stop the Newton iteration whenever the distance between two iterates is less than a given tolerance, *i.e.* when

$$\left\|\mathbf{c}_{j+1}-\mathbf{c}_j\right\|\le\varepsilon,$$

where the Euclidean norm is used. The solution $\mathbf{c}$ gives the coefficients in the approximate sinc-collocation solution $u_m(x)$ of $u(x)$.

## 5. Numerical Examples

In this section, some numerical examples are studied to demonstrate the accuracy of the present method. The results obtained by the method are compared with the exact solution of each example and are found to be in good agreement with each other.Comparison between sinc-collocation and other method shall be presented.

All computations were carried out using **Matlab** on a personal computer with a machine precision of $10^{-32}$. In all cases, $d$ is taken to be $d=\frac{\pi}{2}$. The selection of a larger $N$ yields more accuracy, but at the expense of a lengthier computation. We report absolute error which is defined as

$$\left\|E_c\right\|=\left|u_{\text{exact}}-u_{\text{sinc-collocation}}\right|$$

**Example 1:** [3,11] consider the linear system of second order boundary value problems

$$\begin{aligned}&\frac{\mathrm{d}^2u_1}{\mathrm{d}x^2}+(2x-1)\frac{\mathrm{d}u_1}{\mathrm{d}x}+\cos\pi x\frac{\mathrm{d}u_2}{\mathrm{d}x}=f_1(x)\quad 0\le x\le 1,\\&\frac{\mathrm{d}^2u_2}{\mathrm{d}x^2}+xu_1=f_2(x)\end{aligned}$$

 

where

$$f_1(x) = -\pi^2 \sin \pi x + (2x-1)(\pi+1)\cos \pi x$$
$$f_2(x) = 2 + x \sin \pi x$$

subject to the boundary conditions

$$u_1(0) = u_1(1) = 0, \quad u_2(0) = u_2(1) = 0$$

whose exact solutions are

$$u_1(x) = \sin \pi x \quad \text{and} \quad u_2(x) = x^2 - x.$$

Maximum absolute errors for $u_1$ and $u_2$ are tabulated in **Table 1** for the sinc-collocation method.

Maximum absolute error are tabulated in **Table 2** for sinc-collocation together with the analogous results of N. Caglar and H. Caglar [3].

**Example 2:** [11] Now we turn to a nonlinear problem

$$\frac{d^2u_1}{dx^2} + x\frac{du_1}{dx} + \cos(\pi x)\frac{du_2}{dx} = f_1(x), \quad 0 \le x \le 1,$$
$$\frac{d^2u_2}{dx^2} + x\frac{du_1}{dx} + xu_1^2 = f_2(x)$$

where

$$f_1(x) = \sin x + \left(x^2 - x + 2\right)\cos x + (1-2x)\cos \pi x$$
$$f_2(x) = -2 + x\sin x + x(x-1)^2 \sin^2 x + \left(x^2 - x\right)\cos x.$$

subject to the boundary conditions

$$u_1(0) = u_1(1) = 0, \quad u_2(0) = u_2(1) = 0,$$

whose exact solutions are

$$u_1(x) = (x-1)\sin x, \quad u_2(x) = x - x^2.$$

The computational results are summarized in **Table 3**.

**Example 3:** Now we turn to a singular problem,

$$\frac{d^2u_1}{dx^2} + \left(\frac{1}{x}\right)\frac{du_1}{dx} + \left(\frac{1}{x^2}\right)u_1 + u_2 = f_1(x), \quad 0 \le x \le 1,$$
$$\frac{d^2u_2}{dx^2} + \left(\frac{1}{x}\right)\frac{du_2}{dx} + \left(\frac{1}{x^2}\right)u_2 + u_1 = f_2(x)$$

where

$$f_1(x) = -5 + \frac{2}{x} + x\sqrt{1-x}$$
$$f_2(x) = x - x^2 - \frac{3}{2}\frac{1}{\sqrt{1-x}} - \frac{1}{4}\frac{x}{\sqrt{(1-x)^3}} + \frac{2\sqrt{1-x}}{x}.$$

subject to the boundary conditions

$$u_1(0) = u_1(1) = 0, \quad u_2(0) = u_2(1) = 0,$$

whose exact solutions are

$$u_1(x) = x(1-x), \quad u_2(x) = x\sqrt{1-x}.$$

**Table 1. Maximum absolute error for example 1.**

| $N$ | Max. absolut error in $u_1$ | Max. absolut error in $u_2$ |
|---|---|---|
| 20 | 3.128E–005 | 1.175E–006 |
| 40 | 1.829E–007 | 5.095E–009 |
| 60 | 3.573E–009 | 7.696E–011 |
| 80 | 1.287E–010 | 2.267E–012 |
| 100 | 6.839E–012 | 1.026E–013 |

**Table 2. Comparison between maximum absolute error for example 1.**

| | Max. absolute error in $u_1$ | Max. absolute error in $u_2$ |
|---|---|---|
| Sinc-collocation | 6.839E–012 | 1.026E–013 |
| B-spline method [3] | 2.109E–04 | 1.071E–05 |

**Table 3. Maximum absolute error for example 2.**

| $N$ | Max. absolute error in $u_1$ | Max. absolute error in $u_2$ |
|---|---|---|
| 10 | 4.059E–001 | 3.270E–002 |
| 20 | 1.698E–006 | 8.060E–007 |
| 30 | 8.140E–008 | 3.817E–008 |
| 40 | 6.401E–009 | 2.816E–009 |

The computational results are summarized in **Table 4**.

**Example 4:** Another example is also a singular problem

$$\frac{d^2u_1}{dx^2} + \frac{1}{x}\frac{du_1}{dx} + \frac{1}{x^2}u_1 + \frac{1}{x^2}u_2 + x^2u_1^2 = f_1(x),$$
$$\frac{d^2u_2}{dx^2} + \frac{1}{x}\frac{du_2}{dx} + \frac{1}{x^2}u_2 + \frac{1}{x^2}u_1 + x^2u_2^2 = f_2(x)$$

where

$$f_1(x) = 5 + x^4(x-1)^2 - \frac{2}{x} + \frac{\sin(\pi x)}{x^2}$$
$$f_2(x) = \left(\frac{1}{x^2} - \pi^2\right)\sin(\pi x)$$
$$+ \frac{\pi}{x}\cos(\pi x) + \sin(x)\sin^2(\pi x) + 1 - \frac{1}{x}.$$

subject to the boundary conditions

$$u_1(0) = u_1(1) = 0, \quad u_2(0) = u_2(1) = 0$$

whose exact solutions are

$$u_1(x) = x^2 - x \quad \text{and} \quad u_2(x) = \sin \pi x.$$

Maximum absolute errors for $u_1$ and $u_2$ are tabulated in **Table 5** for the sinc-collocation method.

**Example 5:** Our final example is the singularly perturbed problem

 

**Table 4. Maximum absolute error for example 3.**

| $N$ | Max. absolute error in $u_1$ | Max. absolute error in $u_2$ |
|---|---|---|
| 10 | 4.589E–004 | 4.629E–003 |
| 20 | 1.535E–004 | 7.790E–004 |
| 30 | 2.918E–005 | 1.608E–004 |
| 40 | 5.404E–006 | 5.265E–005 |

**Table 5. Maximum absolute error for example 4.**

| $N$ | Max. absolute error in $u_1$ | Max. absolute error in $u_2$ |
|---|---|---|
| 10 | 1.279E–003 | 2.517E–003 |
| 20 | 7.488E–006 | 3.419E–005 |
| 30 | 6.742E–006 | 7.779E–006 |
| 40 | 1.616E–007 | 5.188E–008 |

$$\varepsilon \frac{d^2 u_1}{dx^2} + 4x^2 u_1 - 2x u_2 = f_1(x), \quad 0 \le x \le 1,$$
$$\varepsilon \frac{d^2 u_2}{dx^2} + x^2 u_2 + x^2 u_1 = f_2(x)$$

where

$$f_1(x) = -2\varepsilon + 4x^3(1-x) - 2x\sin \pi x$$
$$f_2(x) = -\varepsilon\pi^2 \sin \pi x + x^2 \sin \pi x + x^3(1-x).$$

subject to the boundary conditions

$$u_1(0) = u_1(1) = 0, \quad u_2(0) = u_2(1) = 0,$$

whose exact solutions are

$$u_1(x) = x - x^2 \quad \text{and} \quad u_2(x) = \sin \pi x.$$

**Table 6** the maximum absolute errors obtained by using the sinc-collocation method for $N$ = 40 and different $\varepsilon$.

## 6. Conclusions

This paper described an efficient method for solving the system of second-order boundary value problems. Our approach was based on the sinc-collocation method. Properties of the sinc-collocation method are utilized to reduce the computation of this problem to some linear or nonlinear algebraic equations. The method is computationally attractive and applications are demonstrated through illustrative examples. Numerical examples including regular, singular as well as singularly perturbed problems are presented. As expected, the accuracy increases as the number of terms $N$ in the sinc expansion increases. The obtained results showed that this approach can solve the problem effectively.

The sinc-collocation method is a simple method with high accuracy for solving a large variety of linear and nonlinear system of differential equations. So it may be easily applied by researchers and engineers familiar with the sinc function. Extension of the method for solving systems of partial differential equations offers an excellent opportunity for future research.

**Table 6. Maximum absolute error for example 5.**

| $\varepsilon$ | Max. absolute error in $u_1$ | Max. absolute error in $u_2$ |
|---|---|---|
| $10^{-2}$ | 1.587E–007 | 1.773E–007 |
| $10^{-3}$ | 8.427E–007 | 6.749E–007 |
| $10^{-6}$ | 1.488E–006 | 8.433E–007 |
| $10^{-8}$ | 4.777E–007 | 1.483E–007 |
| $10^{-10}$ | 1.728E–006 | 1.553E–006 |

## REFERENCES


[1] K.W. Tomantschger, "Series Solutions of Coupled Differential Equations with One Regular Singular Point," *Journal of Computational and Applied Mathematics*, Vol. 140, No. 1-2, 2002, pp. 773-783. doi:10.1016/S0377-0427(01)00598-2

[2] C. Wafo Soh and F. M. Mahomed, "Linearization Criteria for a System of Second-Order Ordinary Differential Equations," *International Journal of Non-Linear Mechanics*, Vol. 36, No. 4, 2001, pp. 671-677. doi:10.1016/S0020-7462(00)00032-9

[3] N. Caglar and H. Caglar, "B-Spline Method for Solving Linear System of Second-Order Boundary Value Problems," *Computers & Mathematics with Applications*, Vol. 57, No. 5, 2009, pp. 757-762. doi:10.1016/j.camwa.2008.09.033

[4] S. H. Chen, J. Hu, L. Chen and C. P. Wang, "Existence Results for *n*-Point Boundary Value Problem of Second Order Ordinary Differential Equations," *Journal of Computational and Applied Mathematics*, Vol. 180, No. 2, 2005, pp. 425-432. doi:10.1016/j.cam.2004.11.010

[5] X. Y. Cheng and C. K. Zhong, "Existence of Positive Solutions for a Second-Order Ordinary Differential System," *Journal of Mathematical Analysis and Applications*, Vol. 312, No. 1, 2005, pp. 14-23. doi:10.1016/j.jmaa.2005.03.016

[6] A. Lomtatidze and L. Malaguti, "On a Two-Point Boundary Value Problem for the Second-Order Ordinary Differential Equations with Singularities," *Nonlinear Analysis: Theory, Methods & Applications*, Vol. 52, No. 6, 2003, pp. 1553-1567. doi:10.1016/S0362-546X(01)00148-1

[7] H. Thompson and C. Tisdell, "Boundary Value Problems for Systems of Difference Equations Associated with Systems of Second-Order Ordinary Differential Equations," *Applied Mathematics Letters*, Vol. 15, No. 6, 2002, pp. 761-766. doi:10.1016/S0893-9659(02)00039-3

[8] H. Thompson and C. Tisdell, "The Nonexistence of Spurious Solutions to Discrete, Two-Point Boundary Value Problems," *Applied Mathematics Letters*, Vol. 16, No. 1, 2003, pp. 79-84. doi:10.1016/S0893-9659(02)00147-7


 

[9] F. Z. Geng and M. G. Cui, "Solving a Nonlinear System of Second-Order Boundary Value Problems," *Journal of Mathematical Analysis and Applications*, Vol. 327, No. 2, 2007, pp. 1167-1181. doi:10.1016/j.jmaa.2006.05.011

[10] J. F. Lu, "Variational Iteration Method for Solving a Nonlinear System of Second-Order Boundary Value Problems," *Computers & Mathematics with Applications*, Vol. 54, No. 7-8, 2007, pp. 1133-1138. doi:10.1016/j.camwa.2006.12.060

[11] A. Bataineh, M. S. M. Noorani and I. Hashim, "Modified Homotopy Analysis Method for Solving Systems of Second-Order BVPs," *Communications in Nonlinear Science and Numerical Simulation*, Vol. 14, No. 2, 2009, pp. 430-442. doi:10.1016/j.cnsns.2007.09.012

[12] M. Dehghan and A. Saadatmandi, "The Numerical Solution of a Nonlinear System of Second-Order Boundary Value Problems Using the Sinc-Collocation Method," *Mathematical and Computer Modelling*, Vol. 46, No. 11-12, 2007, pp. 1434-1441. doi:10.1016/j.mcm.2007.02.002

[13] B. Bialecki, "Sinc-Collocation Methods for Two-Point Boundary Value Problems," *IMA Journal of Numerical Analysis*, Vol. 11, No. 3, 1991, pp. 357-375. doi:10.1093/imanum/11.3.357

[14] M. El-Gamel and A. I. Zayed, "Sinc-Galerkin Method for Solving Nonlinear Boundary-Value Problems," *Computers & Mathematics with Applications*, Vol. 48, No. 9, 2004, pp. 1285-1298. doi:10.1016/j.camwa.2004.10.021

[15] M. El-Gamel, J. Cannon and A. Zayed, "Sinc-Galerkin Method for Solving Linear Sixth Order Boundary-Value Problems," *Mathematics of Computation*, Vol. 73, No. 247, 2004, pp. 1325-1343.

[16] M. El-Gamel, S. H. Behiry and H. Hashish, "Numerical Method for the Solution of Special Nonlinear Fourth-Order Boundary Value Problems," *Applied Mathematics and Computation*, Vol. 145, No. 2-3, 2003, pp. 717-734. doi:10.1016/S0096-3003(03)00269-8

[17] M. El-Gamel and J. Cannon, "On the Solution of Second Order Singularly-Perturbed Boundary Value Problem by the Sinc-Galerkin Method," *Zeitschrift für Angewandte Mathematik und Physik* (*ZAMP*), Vol. 56, No. 1, 2005, pp. 45-58. doi:10.1007/s00033-004-3002-6

[18] A. Mohsen and M. El-Gamel, "A Sinc-Collocation Method for the Linear Fredholm Integro-Differential Equations," *Zeitschrift für Angewandte Mathematik und Physik* (*ZAMP*), Vol. 58, No. 3, 2007, pp. 380-390. doi:10.1007/s00033-006-5124-5

[19] A. Mohsen and M. El-Gamel, "On the Galerkin and Collocation Methods for Two-Point Boundary Value Problems Using Sinc Bases," *Computers & Mathematics with Applications*, Vol. 56, No. 4, 2008, pp. 930-941. doi:10.1016/j.camwa.2008.01.023

[20] A. Mohsen and M. El-Gamel, "On the Numerical Solution of Linear and Nonlinear Volterra Integral and Integro-Differential Equations," *Applied Mathematics and Computation*, Vol. 217, No. 7, 2010, pp. 3330-3337. doi:10.1016/j.amc.2010.08.065

[21] R. Smith, G. Bogar, K. Bowers and J. Lund, "The Sinc-Galerkin Method for Fourth-Order Differential Equations," *SIAM Journal on Numerical Analysis*, Vol. 28, No. 3, 1991, pp. 760-788. doi:10.1137/0728041

[22] G. Y. Yin, "Sinc-Collocation Method with Orthogonalization for Singular Poisson-Like Problem," *Mathematics of Computation*, Vol. 62, No. 205, 1994, pp. 21-40. doi:10.1090/S0025-5718-1994-1203738-7

[23] J. Lund and K. L. Bowers, "Sinc Methods for Quadrature and Differential Equations," Society for Industry and Applied Mathematics (SIAM), Philadelphia, 1992. doi:10.1137/1.9781611971637

[24] F. Stenger, "Numerical Methods Based on Sinc and Analytic Functions," Springer, New York, 1993. doi:10.1007/978-1-4612-2706-9