Limit Theorems for a Storage Process with a Random Release Rule

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ABSTRACT
We consider a discrete time Storage Process $X_n$ with a simple random walk input $S_n$ and a random release rule given by a family $\{U_x, x \geq 0\}$ of random variables whose probability laws $\{\mu_x, x \geq 0\}$ form a convolution semigroup of measures, that is, $\mu_x \times \mu_y = \mu_{x+y}$. The process $X_n$ obeys the equation:

$$X_n(0) = 0, U_0 = 0, X_n = S_n - U_{S_n}, n \geq 1.$$ 

We will assume that for each $x$, $\mu_x$ is supported on the positive real line $\mathbb{R}_+$. Consequently, for $x \leq y$, the distribution of $U_y - U_x$ is the same as that of $U_x$, (see 2.2 (ii)).

Keywords: Storage Process; Random Walk; Strong Law of Large Numbers; Central Limit Theorem

1. Introduction and Assumptions
The formal structure of a general storage process displays two main parts: the input process and the release rule. The input process, mostly a compound Poisson process, plays the role of a generator that produces material, and the release rule, which determines when material flows out of the system when its content is $x$. So the state $X(t)$ of the system at time $t$ obeys the well known equation:

$$X(t) = X(0) + A(t) - \int_0^t r(X(s)) \, ds.$$ 

Limit theorems and approximation results have been obtained for the process $X(t)$ by several authors, see [1-5] and the references therein. In this paper we study a discrete time new storage process with a simple random walk input $S_n$ and a random release rule given by a family of random variables $\{U_x, x \geq 0\}$, where $U_x$ has to be interpreted as the amount of material removed when the state of the system is $x$. Hence the evolution of the system obeys the following equation:

$$X_n = 0, U_0 = 0, X_n = S_n - U_{S_n}, n \geq 1.$$ 

We will make the following assumptions:

1.1. The probability distributions $\{\mu_x, x \geq 0\}$ of the random variables $\{U_x, x \geq 0\}$ form a convolution semigroup of measures:

$$\forall x, y \geq 0, \mu_x \times \mu_y = \mu_{x+y}, \quad (1.1)$$

We will assume that for each $x$, $\mu_x$ is supported on the interval $[0, x]$, that is, $\forall x \in \mathbb{R}_+, \mu_x[0, x] = 1$. Consequently, for $x \leq y$, the distribution of $U_y - U_x$ is the same as that of $U_x$, (see 2.2 (ii)).

1.2. Also we will need some smoothness properties for the stochastic process $U_x, x \geq 0$. These will be achieved if we impose the following continuity condition:

$$\lim_{x \to \alpha} \mu_x = \delta_\alpha \quad (1.2)$$

where $\delta_\alpha$ is the unit mass at $\alpha$ and the limit is in the sense of the weak convergence of measures.

1.3. The two families of random variables $\{U_x, x \geq 0\}$ and $\{Z_n, n \geq 1\}$ are independent.

2. Construction of the Processes $\{Z_n, n \geq 1\}, \{U_x, x \geq 0\}$ and $\{X_n, n \geq 0\}$

2.1. Let $\lambda$ be a probability measure on the Borel sets $\mathcal{B}_{\mathbb{R}_+}$ of the positive real line $\mathbb{R}_+$ and form the infinite product space $\left(\Omega, \mathcal{F}, P\right) = \left(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+}, \mathcal{L}^{\infty}\right)$. Now, as usual define random variables $Z_n$ on $\Omega$ by:

$$Z_n(\omega) = \omega_k(n), \quad \text{if} \quad \omega_k = (\omega_k(k)) \in \Omega,$$

Then the $Z_n$ are independent identically distributed with common distribution $\lambda$. We will assume that $E(Z_1) = \alpha > 0$, and $\sigma_Z = \sqrt{\sigma^2} = \infty$.

2.2. Let $\{\mu_x, x \geq 0\}$ be a semigroup of convolution of probability measures on $\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+}$ with $\mu_0 = \delta_0$ and satisfying (1.2) then, it is well known, that there is a probability space $\left(\Omega, \mathcal{F}, P\right)$ and a family
\{U_x, x \geq 0\}$ of positive random variables defined on this space such that the following properties hold:

(i). Under $P_x$ the distribution of $U_x$ is $\mu_x$, $U_0 = 0$.

(ii). For $x \leq y$, the random variables $U_x - U_y$ and $U_{y-x}$ have under $P_x$ the same distribution $\mu_{y-x}$.

(iii). For every $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$, the increments $U_n, U_{n-1}, \cdots, U_{1-U_{x_1}}$ are independent.

(iv). For almost all $\omega \in \Omega$, the function $x \rightarrow U_x(\omega)$ is right continuous with left hand limit (cadlag).

From (iv) we deduce:

(v). The function $(x, \omega \rightarrow U_x(\omega))$ is measurable on the product space $\mathbb{R}_+ \times \Omega$.

2.3. The basic probability space for the storage process $X_n$ will be the product $(\Omega, \mathcal{F}, P) = (\Omega \times \Omega, \mathcal{F} \times \mathcal{F}, P \otimes P)$). Then we define $X_n$ by the following recipe:

$$X_0 = 0,$$

$$X_n(\omega) = S_n(\omega) - U_{S_n(\omega)}(\omega), \text{ if } \omega = (\omega_1, \omega_2) \in \Omega, n \geq 1,$$

where $S_n$ is the simple random walk with:

$$S_0 = 0, \text{ } S_n = Z_1 + Z_2 + \cdots + Z_n, \text{ } n \geq 1.$$

2.4. Since $S_n$ is a simple random walk, the random variables $S_k - S_{k-1}$ and $S_{n-k}$ have the same distribution for $k \leq n$.

3. The Main Results

The main objective is to establish limit theorems for the processes $U_s$ and $X_s$. Since the behavior of $S_n$ is well understood, we will focus attention on the structure of the process $U_s$. The outstanding fact is that $U_s$ itself is a simple random walk. First we need some preparation.

3.1. Proposition: For every measurable bounded function $f: \mathbb{R}_+ \rightarrow \mathbb{R}$, the function $x \rightarrow \mu_x(f) = \int \{f(t) \mu_x(\mathcal{D}T)\}$ is measurable. Thus for any Borel set $A$ of $\mathbb{R}_+$ the function $x \rightarrow \mu_x(A)$ is measurable.

Proof: Assume first $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ continuous and bounded, and then from (1.2) we have

$$\lim_{x \rightarrow x_0} \mu_x(f) = \delta_x(f) = f(0).$$

Now by (1.1) we have

$$\mu_x(\mathcal{D}T) = \int \{f(t) \mu_x(\mathcal{D}T)\} \mu_x(\mathcal{D}T)$$

$$= \int \{f(t+\epsilon) \mu_x(\mathcal{D}T)\} \mu_x(\mathcal{D}T)$$

$$\rightarrow \int \{f(t) \mu_x(\mathcal{D}T)\}, \text{ } y \downarrow 0.$$
Since $E_p = E_{R_1} \cdot E_{R_2}$, we can write:

$$E_p \left( f(U_{s_i} - U_{s_j}) \right) = \int_{\Omega_1} \int_{\Omega_2} f(U_{s_i(\omega_1)} - U_{s_j(\omega_2)}) P_1(\omega_1) \cdot P_2(\omega_2).$$

But for each fixed $\omega_1 \in \Omega_1$ we get from 2.2(ii):

$$\int_{\Omega_2} f(U_{s_i(\omega_1)}(\omega_2) - U_{s_j(\omega_1)}(\omega_2)) P_2(\omega_2) = \mu_{\omega_1}(\omega_2) = \mu_{s_i(\omega_1)-s_j(\omega_1)}(f).$$

Applying $E_p$ to both sides of this formula we get the first equality of (3.7). To get the second one, observe that the function $\alpha \rightarrow \mu_{\omega_1}(\omega_2)$ is measurable (Proposition 3.1) and use the fact that under $P_1$ the random variables $S_i - S_j$ and $S_{n-k}$ have the same probability distribution by 2.4.

**3.8. Theorem:** The process $U_{s_i}, n \geq 0$ is a simple random walk with:

$$U_{s_0} = U_0 = 0$$

and $P(U_{s_i} \in A) = \mu(A) \lambda(dz)$

**Proof:** We prove that for all integers $1 \leq i \leq j < k \leq n$, and all positive measurable functions $f, g, h: \mathbb{R} \rightarrow \mathbb{R}$ we have:

$$E_p \left( f(U_{s_k} - U_{s_j}) \cdot g(U_{s_j} - U_{s_i}) \cdot h(U_{s_i} - U_{s_k}) \right) = E_p(f(U_{s_k} - U_{s_j}) \cdot E_p(g(U_{s_j} - U_{s_i})) \cdot E_p(h(U_{s_i} - U_{s_k})))$$

(3.8)

Let $\alpha_1$ be fixed in $\Omega_1$. By 2.2 (ii), (iii), under $P_2$ the random variables

$$U_{s_k(\alpha_1)} - U_{s_k(\alpha_1)}, U_{s_j(\alpha_1)} - U_{s_j(\alpha_1)},$$

$$U_{s_j(\alpha_1)} - U_{s_i(\alpha_1)},$$

are independent. Therefore, applying first $E_{R_2}$ in the L.H.S of (3.8), we get the formula:

$$E_{R_2} \left( \left[ f(U_{s_k(\alpha_1)} - U_{s_j(\alpha_1)})) \cdot g(U_{s_j(\alpha_1)} - U_{s_i(\alpha_1)})) \cdot h(U_{s_i(\alpha_1)} - U_{s_k(\alpha_1)}) \right] \right)$$

(3.9)

But $U_{s_k(\alpha_1)} - U_{s_k(\alpha_1)}, U_{s_j(\alpha_1)} - U_{s_j(\alpha_1)}, U_{s_j(\alpha_1)} - U_{s_i(\alpha_1)}$ have distributions $\mu_{s_k(\alpha_1)-s_k(\alpha_1)}, \mu_{s_j(\alpha_1)-s_j(\alpha_1)}$, respectively. Thus:

$$E_{R_2} \left( f(U_{s_k(\alpha_1)} - U_{s_j(\alpha_1))}) = \mu_{s_k(\alpha_1)-s_j(\alpha_1)}(f) \right)$$

$$E_{R_2} \left( g(U_{s_j(\alpha_1)} - U_{s_i(\alpha_1)})) = \mu_{s_j(\alpha_1)-s_i(\alpha_1)}(g) \right)$$

$$E_{R_2} \left( h(U_{s_i(\alpha_1)} - U_{s_k(\alpha_1)}) = \mu_{s_i(\alpha_1)-s_k(\alpha_1)}(h) \right)$$

By Proposition 3.1, the R.H.S of these equalities are random variables of $\omega_1$, independent under $P_1$ since they are measurable functions of the independent random variables $S_k - S_j$, $S_j - S_i$, $S_i - S_k$. Therefore, applying $E_{R_1}$ to both sides of formula (3.9) we get the proof of (3.8):

$$E_{R_1}E_{R_2} \left( \left[ f(U_{s_k} - U_{s_j}) \cdot g(U_{s_j} - U_{s_i}) \cdot h(U_{s_i} - U_{s_k}) \right] \right) = E_{R_1} \left( E_{R_2} \left( f(U_{s_k} - U_{s_j}) \right) \right) \cdot E_{R_2} \left( g(U_{s_j} - U_{s_i}) \right) \cdot E_{R_2} \left( h(U_{s_i} - U_{s_k}) \right)$$

To achieve the proof, write $U_{s_i}$ as follows:

$$U_{s_i} = \sum_{j=1}^{n} (U_{s_j} - U_{s_{j+1}}),$$

where the $U_{s_j} - U_{s_{j+1}}$ are independent with the same distribution given by

$$P(U_{s_j} \in A) = \int_{\mathbb{R}} \mu_\lambda(A) \lambda(dz)$$

according to (3.5).

**3.9. Proposition:** For every positive measurable function $f: \mathbb{R} \rightarrow \mathbb{R}$, we have:

$$E_p \left( f(U_{s_k}) \right) = \int_{\mathbb{R}} f(t) \cdot \mu_\lambda(\lambda^\ast(t)) \lambda^\ast(ds) \bigg|_{\mathbb{R}}$$

(3.9)

$\lambda^\ast$ being the n-fold convolution of the probability $\lambda$. In particular the distribution law of the process $U_{s_k}$ is given by:

$$B \in \mathbb{B}_{\mathbb{R}}, \quad P(U_{s_k} \in B) = \int_{\mathbb{R}} \mu_\lambda(B) \lambda^\ast(ds)$$

and its expectation is:

$$E_p(U_{s_k}) = \int_{\mathbb{R}} t \mu_\lambda(\lambda^\ast(t)) \lambda^\ast(ds)$$
Proof: We have:

\[ E_P \left( f \left( U_{S_n} \right) \right) = E_{R^n} \left( f \left( U_{S_n(\omega_n)} (\omega_2) \right) \right) \]

\[ = E_{R^n} \left( \int f(t) \mu_{S_n(\omega_n)} (dt) \right) \]

and, by Proposition 3.1, the function

\[ \omega_2 \rightarrow \int f(t) \mu_{S_n(\omega_n)} (dt) \]

is a measurable function of

\[ S_n(\omega_2). \] Since \( S_n = Z_1 + Z_2 + \cdots + Z_n \) is a simple random walk with the \( Z_n \) having distribution \( \lambda \), the random variable \( S_n \) has the distribution \( S_n^{-\lambda} \). So, by a simple change of variable we get:

\[ E_{R^n} \left( \int f(t) \mu_{S_n(\omega_n)} (dt) \right) = \int \int f(t) \mu_{S_n(\omega_n)} (dt) \lambda^n (ds) \]

So formula (3.9) is proved. To get the distribution law of the process \( U_{S_n} \), take \( f \) equal to the characteristic function of some Borel set \( B \).

3.10. Remark: Let \( \nu \) be the distribution of \( U_{Z_1} \), that is \( \nu(A) = \int \mu_\nu (A) \lambda (dz) \) and let

\[ \beta = E_P \left( U_{Z_1} \right) = \int _{R^+} t \mu_\nu (dt) \cdot \lambda (dz), \]

then as a direct consequence of theorem 3.8,

\[ P \left( U_{S_n} \in B \right) = \nu^n (B) \]

\[ E_P \left( U_{S_n} \right) = n \cdot \beta \]

Now we turn to the structure of the process \( X_n \). We need the following technical lemma:

3.11. Lemma: For every Borel positive function

\[ F : R_+ \times R_+ \rightarrow R \],

the function \( \varphi : s \rightarrow \int F(s,t) \mu_s (dt) \) is measurable.

Proof: Start with \( F = I_{A \times B} \), the characteristic function of the measurable rectangle \( A \times B \), in which case we have \( \varphi(s) = I_A(s) \mu_s (B) \). Since by proposition 3.1, the function \( s \rightarrow \mu_s (B) \) is measurable we deduce that \( \varphi \) is measurable in this case. Next consider the family

\[ \Gamma = \left\{ B \in B_{R_+} : s \rightarrow \int B (s,t) \mu_s (dt), \text{measurable} \right\} \]

It is easy to check that \( \Gamma \) is a monotone class closed under finite disjoint union. Since it contains the measurable rectangles, we deduce that \( \Gamma = B_{R_+ \times R_+} \). Finally consider the following class of Borel positive functions

\[ \mathcal{P} = \left\{ F : R_+ \times R_+ \rightarrow R, \varphi(s) = \int F(s,t) \mu_s (dt) \text{is Borel} \right\} \]

It is clear that \( \mathcal{P} \) is closed under addition and, by the step above, it contains the simple Borel positive functions. By the monotone convergence theorem, \( \mathcal{P} \) is exactly the class of all Borel positive functions.

3.12. Theorem: The random variables

\[ Z_k - (U_{S_k} - U_{S_{k-1}}), k = 1,2, \cdots \]

are independent with the same distribution given by:

\[ P \left( Z_k - (U_{S_k} - U_{S_{k-1}}) \in B \right) \]

\[ = \int \mu_\nu (s - B) \cdot \lambda (ds) \]

(3.12)

Consequently the storage process

\[ X_n = S_n - U_{S_n} = \sum_{i=1}^n Z_i - (U_{S_i} - U_{S_{i-1}}) \]

is a simple random walk with the basic distribution (3.12).

Proof: For each integer \( r \geq 0 \), and each \( (\omega_1, \omega_r) \in \Omega_1 \times \Omega_r \), put:

\[ W_r (\omega_1, \omega_r) = Z_r(\omega_r) - U_{S_r(\omega_r)} (\omega_r) \]

So it is enough to prove that for all \( 0 \leq i < j \leq k \) and all Borel positive functions \( f, g, h : R_+ \rightarrow R \), we have:

\[ E_P \left( f(W_i) \cdot g(W_j) \cdot h(W_k) \right) \]

\[ = E_P \left( f(W_i) \right) \cdot E_P \left( g(W_j) \right) \cdot E_P \left( h(W_k) \right) \]

(3.13)

From the construction of the process \( U_{S_n} \), we know that for \( \omega_k \) fixed, the random variables \( W_r (\omega_1, \omega_k) \), \( r = i, j, k \), are independent under \( P_k \) (see 2.2 (iii)). So, applying \( E_{P_k} \) to \( f(W_i) \cdot g(W_j) \cdot h(W_k) \), we get:

\[ E_{P_k} \left( f(W_i) \cdot g(W_j) \cdot h(W_k) \right) \]

\[ = E_{P_k} \left( f(W_i) \right) \cdot E_{P_k} \left( g(W_j) \right) \cdot E_{P_k} \left( h(W_k) \right) \]

(3.14)
\[
P(X_n \in B) = \int_{\mathbb{R}_+} \mu_\alpha (s-B) \lambda^n (ds)
\]

\[
E_p(X_n) = n \cdot (\alpha - \beta)
\]

For the proof, use the formula \(X_n = S_n - U_{S_n}\) and routine integration.

**3.16. Example:** Let \(0 < c < 1\) and let us take as measure \(\mu_\alpha\) the unit mass at the point \(cs\), that is, the Dirac measure \(\mu_\alpha = \delta_{cs}\), \(s \in \mathbb{R}_+\). It easy to check that \(\mu_{\alpha c \beta} = \mu_\alpha \times \mu_\beta\) for all \(s,t\) in \(\mathbb{R}_+\). Then for every probability measure \(\lambda\) on \(\mathbb{R}_+\), we have:

\[
P(U_{S_n} \in B) = \int_{\mathbb{R}_+} \mu_\alpha (B) \lambda^n (ds) = \lambda^n (c^{-1}B).
\]

This gives the distribution of the release process in this case:

\[
P(U_{S_n} \in B) = \int_{\mathbb{R}_+} \mu_\alpha (B) \lambda^n (ds) = \lambda^n (c^{-1}B).
\]

Since we have \(\lambda^n (c^{-1}B) = P(cS_n \in B)\), we deduce that the release rule consists in removing from \(S_n\) the quantity \(cS_n\).

Likewise it is straightforward, from Proposition 3.14, that

\[
P(X_n \in B) = \int_{\mathbb{R}_+} \mu_\alpha (s-B) \lambda^n (ds)
\]

\[
= \left[ \delta_{cs} (s-B) \lambda^n (ds) \right]
\]

\[
= \int_{\mathbb{R}_+} \left(1-c^{-1}\right)^n (s) \lambda^n (ds)
\]

\[
= \lambda^n \left((1-c)^{-1}B\right)
\]

from which we deduce that the distribution of the storage process is

\[
P(X_n \in B) = P((1-c)S_n \in B).
\]

One can give more examples in this way by choosing \(\lambda\) or/and the semigroup \(\{\mu_x, x \geq 0\}\). Consider the following simple example:

**3.17. Example:** Take \(\lambda\) the 0 - 1 Bernoulli distribution with probability of success \(p\). In this case the semigroup \(\{\mu_x, x \geq 0\}\) is a sequence \(\mu_x\) of probabilities with \(\mu_x\) supported by \(\{1, 2, \ldots, n\}\) for \(n \geq 1\) and \(\lambda^n\) is the Binomial distribution. So we get from proposition 3.9

\[
P(U_{S_n} \in B) = \int_{\mathbb{R}_+} \mu_\alpha (B) \lambda^n (ds)
\]

\[
= \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) p^k (1-p)^{n-k} \mu_\beta (B)
\]

Likewise we get the distribution of \(X_n\) from proposition 3.15 as:

\[
P(X_n \in B) = \int_{\mathbb{R}_+} \mu_\alpha (s-B) \lambda^n (ds)
\]

\[
= \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) p^k (1-p)^{n-k} \mu_\beta (s-B).
\]
\[ \sigma^2_{X_k} = \int_{R_k} (s-t)^2 \mu_s (dr) \cdot \lambda (ds) - (\alpha - \beta)^2 \]

and

\[ \int_{R_k} (s-t)^2 \mu_s (dr) \cdot \lambda (ds) = \int_{R_k} \left( s^2 + t^2 \right) \mu_s (dr) \cdot \lambda (ds) - 2 \int_{R_k} s \cdot t \mu_s (dr) \cdot \lambda (ds) \]

Since the variance \( \sigma^2 \) of \( Z_n \) is finite we have
\[ \int_{R_k} s^2 \mu_s (dr) \cdot \lambda (ds) = \int_{R_k} s^2 \cdot \lambda (ds) < \infty \], so the conclusion follows. \( \blacksquare \)

Finally we get under the conditions of proposition 4.2:
4.3. Theorem: Assume the conditions of proposition 4.2. Then the normalized sequences of random variables:
\[ T_n = \frac{U_{S_n} - n \cdot \beta}{\sigma \sqrt{n}} \] and \[ R_n = \frac{X_n - n \cdot (\alpha - \beta)}{\sigma_n \sqrt{n}} \]
both converge in distribution to the Normal law \( N(0,1) \).

Proof: The condition of the theorem insures the finiteness of the variances \( \sigma^2 \) and \( \sigma^2_{X_n} \). Now the conclusion results from the fact that \( U_{S_n} \) and \( X_n \) are simple random walks and the Lindberg Central Limit Theorem. To see this, we use the method of characteristic functions. Let us denote by \( f_{\theta} \) the characteristic function of the random variable \( \theta \). Since by Theorem 3.8 the components \( U_{S_n} - U_{S_{n-1}} \) of \( U_{S_n} \) have the same distribution as \( Z_{\theta} \), we have
\[ f_{\theta} (t) = \exp (it T_n) \]
\[ \frac{U_{S_n} - \beta}{\sigma \sqrt{n}} \]
\[ \frac{X_n - (\alpha - \beta)}{\sigma_n \sqrt{n}} \]
\[ \exp \left( \frac{-t^2}{2} \right) \]
where the second equality comes from the Taylor expansion of \( f_{U_{S_n}} \). It is well known that this limit is the characteristic function of the random variable \( N(0,1) \). The same proof works for \( R_n \), using the components of the process \( X_n \) as given in Theorem 3.12. \( \blacksquare \)

In some storage systems, the changes due to supply and release do not take place regularly in time. So it would be more realistic to consider the time parameter \( n \) as random. We will do so in what follows and will consider the asymptotic distributions of the processes \( U_{S_n} \) and \( X_n \), when properly normalized and random-normalized. First let us put for each \( k, \)
\[ A_k = \frac{U_{S_n} - U_{S_{n-1}} - \beta}{\sigma U} \] and
\[ B_k = \frac{Z_k - (U_{S_n} - U_{S_{n-1}} - (\alpha - \beta))}{\sigma_{X_k}} \]

Then we have:
4.4. Theorem: Let \( \{ N_n : n \geq 1 \} \) be a sequence of independent random variables, independent of the \( A_k \) and \( B_k \).

If \( \frac{N_n}{n} \) converges in probability to 1, as \( n \to \infty \), then the randomized processes:
\[ \frac{\sum_{i=1}^{N_n} A_k}{\sqrt{n}} \] and \[ \frac{\sum_{i=1}^{N_n} B_k}{\sqrt{n}} \]
both converge in distribution to the Normal law \( N(0,1) \).

Proof: It is a simple adaptation of [7], VIII.4, Theorem 4, p. 265. \( \blacksquare \)

5. Conclusion
In this paper, we presented a simple stochastic storage process \( X_n \) with a random walk input \( S_n \) and a natural release rule \( U_{S_n} \). Realistic conditions are prescribed which make this process more tractable when compared to those models studied elsewhere (see Introduction). In particular the conditions led to a simple structure of random walk for the processes \( U_{S_n} \) and \( X_n \), which has given explicitly their distributions, and a rather good insight on their asymptotic behavior since a SLLN and a CLT has been easily established for each of them. Moreover, a slightly more general limit theorem has been obtained when time is adequately randomized and both processes \( U_{S_n} \) and \( X_n \) properly normalized.

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REFERENCES

