Two Implicit Runge-Kutta Methods for Stochastic Differential Equation

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ABSTRACT
In this paper, the Itô-Taylor expansion of stochastic differential equation is briefly introduced. The colored rooted tree theory is applied to derive strong order 1.0 implicit stochastic Runge-Kutta method (SRK). Two fully implicit schemes are presented and their stability qualities are discussed. And the numerical report illustrates the better numerical behavior.

Keywords: Stochastic Differential Equation; Implicit Stochastic Runge-Kutta Method; Order Condition

1. Introduction
In this paper, we want to obtain numerical methods for strong solution of Stochastic Differential Equations of Itô type.

\[ dy = f(y(t))dt + g(y(t))dW(t), y \in \mathbb{R} \quad (1.1) \]

Note that \( f \) is a slowly varying continuous component function, which is called drift coefficient, \( g \) is the rapidly varying continuous function called the diffusion coefficient. \( W(t) \) is a Wiener process.

Recently, many scholars have successfully derived some methods for SDEs for both Itô and Stratonovich forms. Burrage and Burrage [1-3] established the colored rooted tree theory and Stochastic B-series expansion. Tian and Burrage [2,4,5] derived some strong order 1.0 2-stage Stochastic Runge-Kutta methods, including semi-implicit and implicit methods. Wang P. [6] derived some strong order 1.0 3-stage semi-implicit methods. Wang ZY [7] mainly considered the strong order SRKs for the SDEs of Itô form. In his PhD thesis he offered us the Colored Rooted tree theory for Itô type, and constructed some 2-stage and 3-stage explicit methods. Along this line, I will construct some implicit SRKs for SDEs of Itô type. In Section 2, the colored rooted tree theory for deriving SRK for SDEs of Itô type is briefly introduced and the 2 2-stage fully implicit SRKs are obtained. In Section 3 we will discuss their stability property. And in Section 4, we will report the numerical experiments.

2. 2-Stage Implicit SRK and Order Conditions
Many scholars, including Burrage [2], offered the definition of the order of numerical methods in their thesis.

Definition 2.1. Let \( \tilde{y}_N \) be the numerical approximation to \( y(t_N) \) after \( N \) steps with constant stepsize \( (t_N - t_0)/N \); then \( \tilde{y}_N \) is said to be converge strongly to \( y \) with order \( p \) if

\[ E([\tilde{y}_N - y(t_N)]] \leq C h^p, \quad h \in (0, \delta) \quad (2.1) \]

Note that \( C \) is a constant that independent of \( h \) and \( \delta > 0 \).

Butcher presented the Rooted Tree theory, after which this theory was extended into stochastic area. Burrage [2] presented Colored Rooted Tree theory in her PhD thesis, and Wang [7] did the research especially for Itô SDEs. Similar to the deterministic condition, the definition of the elementary differential can be associated with \( \forall t \in T \)

\[ F(\phi)(y) = y \]
\[ F(\tau)(y) = f(y) \]
\[ F(\delta)(y) = g(y) \]
\[ F(t)(y) = f^{(m)}[F(t_1)(y), \ldots, F(t_n)(y)], t = \{t_1, \ldots, t_n\} \]
\[ F(t)(y) = g^{(m)}[F(t_1)(y), \ldots, F(t_n)(y)] \quad \forall t \in T \]

Here \( \phi \) stands for the trees having order 0.

Wang [7] deduced the Itô-Taylor series for SDEs. Firstly let’s introduce two operators

\[ L_0 = \frac{\partial}{\partial t} + f \cdot \frac{\partial}{\partial x} + \frac{1}{2} g^2 \cdot \frac{\partial^2}{\partial x^2} \]
\[ L_1 = g \cdot \frac{\partial}{\partial x} \]
Now we introduce a very important proposition from Kloeden and Platen [8].

**Proposition 2.1.** if $A \subset M$, $h : \mathbb{R} \to \mathbb{R}$ is sufficiently derivative, and let $X(t)$ be the solution of the equation

$$\frac{dX(t)}{dt} = f(X(t)) + g(X(t))dW(t), \quad t > 0$$

$$X(0) = X_0$$

then

$$h(X(t)) = \sum_{a \in A}^{} h_a(X_0) + \sum_{a \in A}^{} I_a[h_a(X(t))]_{1} \tag{2.2}$$

Letting $h(X(t)) = X(t)$, then

$$X(t)$$

$$= X_0 + L^0 X_0 I_0 + L^1 L^0 I_1 + L^0 L^2 X_0 I_{11} + L^0 X_0 I_{10}$$

$$+ L^1 L^2 X_0 I_{01} + L^1 L^0 X_0 I_{01} + \cdots$$

$$= X_0 + \beta_0 + \beta_1 + g\beta_1 + g\beta_1 I_{10} + \left( \frac{g\beta_1 + 1}{2} \right) I_{01} + \frac{g\beta_1^2 + g\beta_1 \beta_1}{2} I_{11} + \cdots$$

And from the definition of the elementary differential we can know

$$X(t) = F(\phi)(X_0) + F(\phi)(X_0) X_0 + F(\phi)(X_0) I_0 + F(\phi)(X_0) I_1$$

$$+ F(\phi)(X_0) I_{11} + \frac{1}{2} F(\phi)(X_0) I_{11} + \cdots$$

$$= X_0 + \beta_0 + \beta_1 + g\beta_1 + g\beta_1 I_{10} + \left( \frac{g\beta_1 + 1}{2} \right) I_{01} + \frac{g\beta_1^2 + g\beta_1 \beta_1}{2} I_{11} + \cdots$$

$$= \sum_{j=1}^5 \alpha(t) F(t) I(t) + \cdots$$

Like the conclusion of Burrage [2], the Taylor-series of the actual solution of the SDEs is

$$X(t) = \sum_{i=t}^5 \alpha(t) F(t) I(t) \tag{2.3}$$

The structure of Stratonovich-Taylor series is similar to the Itô-Taylor expansion, however, the stochastic calculations of these two types are different. **Table 1** presents the trees and the corresponding elementary differentials. Especially, in order to illustrate the difference between Itô type and stratonovich type, we list all the stochastic calculations of trees having order \leq 2.

Now we show general form of Runge-Kutta methods for SDEs of Itô form. Let the stepsize of the methods is a constant $h = \frac{T}{N}$, $t_n = nh(n = 0, \ldots, N)$, $y_n$ is the numerical solution of $X(t)$, then

$$y_i = y_n + \sum_{j=1}^s Z_{ij}(0) \cdot f(Y_j) + \sum_{j=1}^s Z_{ij}(1) \cdot g(Y_j) \tag{2.4}$$

Note that

$$Z_{ij}(0) = h \cdot \alpha_i, \quad i, j = 1, \ldots, s$$

$$Z_{ij}(1) = \sum_{j=1}^s \beta_i \cdot \theta_j, \quad i, j = 1, \ldots, s$$

where the $\theta(i = 1, \ldots, p)$ is random variables.

Using the Butcher Table, SRK can be written as

$$\begin{bmatrix}
A \\
\alpha \end{bmatrix} = 
\begin{bmatrix}
B(0) \\
B(1) \\
\vdots \\
B(s)
\end{bmatrix}$$

Wang [7] deduced the Taylor series for the SRK of Itô form. And offered the definition of Elementary Weight, which has the same form of Burrage’s conclusion [2].

**Definition 2.2.**
\( L_n = \sum_{i \in T} \left( I(i) - \frac{\Phi(i)}{l(i)!} \right) \alpha(i) F(t)(y(t_n)) \)
\[
= \sum_{i \in T} e(i) \alpha(i) F(t)(y(t_n))
\]

Proposition 2.2, given by Burrage and Burrage [3], gives the necessary conditions of the methods.

**Proposition 2.2.** \( L_n \) is the local truncation error of the numerical methods at \( t = t_n \), \( \varepsilon_n \) is the global error at \( t = t_N \), if \( f \) and \( g \) is sufficiently derivative, and \( \forall n = 1, \ldots, N \)

\[
\left( E \left[ \| L_n \|^2 \right] \right)^{1/2} = O \left( h^{p+\frac{1}{2}} \right)
\]
\[
E \left[ L_n \right] = O \left( h^{p+1} \right)
\]

then
\[
E \left[ \varepsilon_n \right] = O \left( h^p \right)
\]

From the Proposition 2.2, the Runge-Kutta methods of the strong order 1.0 have to satisfy

1) \( \forall t \) that \( \rho(t) \leq 1 \)
\[
\left( E \left[ \left( e(t) \right)^2 \right] \right)^{1/2} = 0
\]
\[
\Rightarrow E \left[ \left( e(t) \right)^2 \right] = 0 \quad (2.6)
\]

2) \( \forall t \) that \( \rho(t) \leq 1.5 \)
\[
E \left[ e(t) \right] = 0
\]

obviously, \( \forall t \), \( E \left[ \left( e(t) \right)^2 \right] = 0 \Rightarrow E \left[ e(t) \right] = 0 \), thus in

2) We just need to consider the condition when \( \rho(t) = 1.5 \).

Now we introduce the random variables
\( \theta_1 = I_1, \theta_2 = \sqrt{h} \). And we note \( e = A \cdot e, b = B^{(1)} \cdot e, d = B^{(2)} \cdot e, \lambda = b - I_1 + d \cdot \sqrt{h} \)

Now let’s start to construct the methods of strong order 1.0.

1) For tree \( \sigma \)
\[
E \left[ \left( I_1 - z^{(1)} e \right)^2 \right] = E \left[ \left( I_1 - \left( 1 - \gamma^{(1)} \right) e - \sqrt{h} \cdot \gamma^{(2)} \right) e \right]^2 = 0
\]

namely
\[
(1 - \gamma^{(1)} \right) e \cdot h^2 + (\gamma^{(2)} \right) e \cdot h = 0
\]
\[
\Rightarrow \gamma^{(1)} e = 1, \gamma^{(2)} \right) e = 0
\]

2) For tree \( \tau \)
\[
E \left[ \left( I_0 - z^{(0)} e \right)^2 \right] = (1 - \alpha^2) e \cdot h^2 = 0
\]
\[
\Rightarrow \alpha^2 e = 1
\]

3) For tree \( \{ \sigma \} \)

<table>
<thead>
<tr>
<th>( \rho(t) )</th>
<th>( t )</th>
<th>( I(t) )</th>
<th>( \rho(t) )</th>
<th>( t )</th>
<th>( I(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \phi )</td>
<td>1</td>
<td>2</td>
<td>( {\sigma, \tau} )</td>
<td>( I_{ii} )</td>
</tr>
<tr>
<td>0.5</td>
<td>( \sigma )</td>
<td>( I_i )</td>
<td>2</td>
<td>( {\tau, \sigma} )</td>
<td>( I_{ii} )</td>
</tr>
<tr>
<td>1</td>
<td>( \tau )</td>
<td>( I_{ii} )</td>
<td>2</td>
<td>( {\tau} )</td>
<td>( I_{ii} )</td>
</tr>
<tr>
<td>1</td>
<td>( {\sigma} )</td>
<td>( I_{ii} )</td>
<td>2</td>
<td>( {\sigma, \sigma} )</td>
<td>( I_{ii} + \frac{1}{2} I_{ii} + \frac{1}{2} I_{ii} )</td>
</tr>
<tr>
<td>1.5</td>
<td>( {\sigma} )</td>
<td>( I_{ii} )</td>
<td>2</td>
<td>( {{\sigma}} )</td>
<td>( I_{i0} )</td>
</tr>
<tr>
<td>1.5</td>
<td>( {\tau} )</td>
<td>( I_{ii} )</td>
<td>2</td>
<td>( {{\tau}} )</td>
<td>( I_{ii} )</td>
</tr>
<tr>
<td>1.5</td>
<td>( {{\sigma}} )</td>
<td>( I_{ii} )</td>
<td>2</td>
<td>( {{\sigma, \sigma}} )</td>
<td>( I_{ii} + \frac{1}{2} I_{ii} )</td>
</tr>
<tr>
<td>2</td>
<td>( {\sigma, \sigma} )</td>
<td>( I_{ii} + \frac{1}{2} I_{ii} )</td>
<td>2</td>
<td>( {{\sigma, \sigma}} )</td>
<td>( I_{ii} + \frac{1}{2} I_{ii} )</td>
</tr>
</tbody>
</table>

**Table 1. Trees and the corresponding elementary differentials.**

<table>
<thead>
<tr>
<th>( \rho(t) )</th>
<th>( t )</th>
<th>( \Phi(t) )</th>
<th>( \rho(t) )</th>
<th>( t )</th>
<th>( \Phi(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \phi )</td>
<td>( e )</td>
<td>1.5</td>
<td>( {\sigma} )</td>
<td>( 2z^{(0)} Z^{(0)} e )</td>
</tr>
<tr>
<td>0.5</td>
<td>( \sigma )</td>
<td>( z^{(0)} e )</td>
<td>1.5</td>
<td>( {\tau} )</td>
<td>( 2z^{(0)} Z^{(0)} e )</td>
</tr>
<tr>
<td>1</td>
<td>( \tau )</td>
<td>( z^{(0)} e )</td>
<td>1.5</td>
<td>( {{\sigma}} )</td>
<td>( 6z^{(0)} Z^{(0)} e )</td>
</tr>
<tr>
<td>1</td>
<td>( {\sigma} )</td>
<td>( 2z^{(0)} Z^{(0)} e )</td>
<td>1.5</td>
<td>( {\sigma, \sigma} )</td>
<td>( 3z^{(0)} (Z^{(0)} e) )</td>
</tr>
</tbody>
</table>

**Table 2. Trees and the corresponding elementary weights.**
\[
E \left[ \left( I_{11} - z^{(1)f} Z^{(1)} e \right)^2 \right] = 0
\]

namely
\[
E \left[ \left( I_1 - \frac{1}{2} - \gamma^{(1)r} b \right) - I_1 \left( \gamma^{(1)r} d + \gamma^{(2)r} b \right) - h \left( \frac{1}{2} + \gamma^{(2)r} d \right)^2 \right] = 0
\]

\Rightarrow \alpha^T \cdot d = 0

where
\[
X = \left( 1 - \gamma^{(1)r} b, \gamma^{(1)r} d + \gamma^{(2)r} b, \frac{1}{2} + \gamma^{(2)r} d \right)
\]

and
\[
D = \begin{pmatrix}
3 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix}
\]

\Rightarrow \alpha^T \cdot d = 0

Here we gained the conditions for the methods with strong order 1.0, theoretically we can construct any-stage methods, both explicit and implicit. And now we consider the 2-stage implicit methods.

\[
\begin{pmatrix}
\gamma^{(1)r} e = 1 \\
\gamma^{(2)r} e = 0 \\
\alpha^T e = 1 \\
\gamma^{(1)r} b = \frac{1}{2} \\
\gamma^{(1)r} d + \gamma^{(2)r} b = 0 \\
\gamma^{(2)r} d = -\frac{1}{2} \\
\alpha^T \cdot d = 0 \\
\gamma^{(2)r} c = 0 \\
\gamma^{(1)r} B^{(1)d} + \gamma^{(2)r} B^{(2)} b + \gamma^{(2)r} b^{(2)} b + \gamma^{(2)r} B^{(2)} d = 0 \\
2\gamma^{(2)r} bd + \gamma^{(2)r} b^2 + \gamma^{(2)r} d^2 = 0
\end{pmatrix}
\]

4) For tree \{\sigma\}
\[
E \left[ I_{10} - z^{(1)f} Z^{(1)} e \right] = E \left[ I_{10} - \alpha^T \cdot h \left( b \cdot I_1 + d \cdot \sqrt{h} \right) \right] = 0
\]

\Rightarrow \alpha^T \cdot d = 0

5) For tree \{\epsilon\}
\[
E \left[ I_{01} - z^{(1)f} Z^{(0)} e \right] = E \left[ I_{01} - h \left( \gamma^{(1)r} c \cdot I_1 + \gamma^{(2)r} c \cdot \sqrt{h} \right) \right] = 0
\]

\Rightarrow \gamma^{(2)r} c = 0

6) For tree \{\sigma, \epsilon\}
\[
E \left[ I_{111} - \left( \gamma^{(1)r} I_1 + \gamma^{(2)r} \sqrt{h} \right) \left( B^{(1)} I_1 + B^{(2)} \sqrt{h} \right) \left( b I_1 + d \sqrt{h} \right) \right] = 0
\]

\Rightarrow \gamma^{(1)r} B^{(1)} d + \gamma^{(1)r} B^{(2)} b + \gamma^{(2)r} B^{(1)} b + \gamma^{(2)r} B^{(2)} d = 0

7) For tree \{\sigma, \epsilon\}
\[
E \left[ \frac{1}{2} I_{01} + I_{111} - \frac{1}{2} z^{(1)f} \cdot Z^{(1)} e \right] = 0
\]

\Rightarrow 2\gamma^{(1)f} bd + \gamma^{(2)f} b^2 + \gamma^{(2)f} d^2 = 0

Thus, the 2-stage implicit SRKs should satisfy the system

3. Stability
Saito and Mitzi [9] introduced the definition of mean-square (MS) stability, and the scholars such as Burrage [2] and Tian [4, 5] researched it and gave some improve-
Consider the linear test equation of Itô type of SDEs.
\[ dy = \lambda y dt + \mu y dw(t) \quad (3.1) \]
and we use one-step scheme
\[ y_{n+1} = R(h, \lambda, \mu, I)y_n \]
where \( h \) is the step size, \( I \) is the random variable in the numerical scheme.

Satio and Mitzui [9] introduced the definition

**Definition 3.1.** If for \( \lambda, \mu, h \),
\[ R(h, \lambda, \mu) = E\left( R^2(h, \lambda, \mu, I) \right) < 1 \]
then the numerical scheme is said to be MS stable, and the \( R(h, \lambda, \mu) \) is said to be the MS-stability function.

1) For \( \text{Imp}_1 \), we can obtain the MS-stability function
\[ y_{n+1} = R(h, \lambda, \mu, I_n)y_n \]
where
\[ R(h, \lambda, \mu, I_n) = 1 + R_1 \cdot p + \frac{1}{2} (R_1 + R_2) \cdot q \cdot I_n + \frac{1}{2} (R_1 - R_2) \cdot q \]
Note that
\[ R_1 = -2(5q \cdot I_n + 3p - 2q - 6) \]
\[ R_2 = -2(-q \cdot I_n - 6 - 8q + 3p) \]
\[ R_1 = \frac{1}{4q^2 I_n^2 - 4q^2 + 12 - 12p - 10q \cdot I_n + 3p^2 + 5pq \cdot I_n + 4p - 2pq} \]
and
\[ p = \lambda h, q = \mu \sqrt{h}, I_n \text{ is the standard Gaussian variable } \sim N(0,1) \]

**Figure 1** describes the stable region of \( \text{Imp}_1 \).

2) For the method \( \text{Imp}_2 \), we obtain that
\[ y_{n+1} = R(h, \lambda, \mu, I_n)y_n \]
where
\[ R(h, \lambda, \mu, I_n) = 1 + pR_1 + q \left( R_1 \left( \frac{1}{2} I_n + \frac{1}{2} \right) + R_2 \left( \frac{1}{2} I_n - \frac{1}{2} \right) \right) \]
Note that
\[ R_1 = \frac{1}{1 - \frac{1}{2} p} \]
\[ R_2 = \frac{1 + qR_1}{1 - \frac{1}{2} p - qI_n} \]
and

**Figure 2** represents the stable region of \( \text{Imp}_2 \).

### 4. Numerical Results

Now we report the numerical results of the schemes derived in this paper. At first we will use the points of numerical simulation in a single trajectory to compare the absolute error Ms of five different schemes—explicit Euler-Maruyama scheme, explicit milstein scheme, explicit two-stage scheme \( \text{Imp}_2 \) which is designed by Wang [7], \( \text{Imp}_1 \) and \( \text{Imp}_2 \)—for a same non-linear system (4.1).

After which we will simulate 100 trajectories of each scheme and then compare their absolute error Ms. Errors for the (4.1) is given by
\[ M = \frac{1}{k} \sum_{i=1}^{k} \| x_i - y(t_i) \| \]
Note that \( x_i \) is the exact value at step point \( t_i \) and \( y(t_i) \) is the numerical simulation at that point, \( k \) is the number of the points chosen in the trajectories. And the non-linear system (4.1) is given by
\[
\begin{align*}
dX(t) &= \left( \frac{1}{2} X(t) + \sqrt{X^2(t) + 1} \right) \cdot dt \\
&+ \sqrt{X^2(t) + 1} \cdot dw(t). \quad t \in [0, 5] \\
X(0) &= 0
\end{align*}
\]
And the analytical solution of the system 10 is
\[ X(t) = \sinh(t + w(t)) \quad (4.2) \]

Firstly, we compare the error Ms in a single trajectory. From the Table 3, we can know that in a random trajectory(actually we choose the first one), the \( \text{Imp}_1 \) is obviously better than all the other schemes, and also,
Figure 2. Stable region of Imp$_2$.

Table 3. The absolute error Ms in a single trajectory.

<table>
<thead>
<tr>
<th>Stepsize</th>
<th>$2^{-4}$</th>
<th>$2^{-5}$</th>
<th>$2^{-6}$</th>
<th>$2^{-7}$</th>
<th>$2^{-8}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler</td>
<td>1.52</td>
<td>32.33</td>
<td>1.36</td>
<td>11.31</td>
<td>11.02</td>
</tr>
<tr>
<td>Milstein</td>
<td>2.98</td>
<td>8.59</td>
<td>0.49</td>
<td>1.01</td>
<td>0.33</td>
</tr>
<tr>
<td>Imp$_1$</td>
<td>4.96</td>
<td>10.74</td>
<td>0.40</td>
<td>1.34</td>
<td>0.42</td>
</tr>
<tr>
<td>Imp$_2$</td>
<td>5.58</td>
<td>8.27</td>
<td>0.88</td>
<td>1.93</td>
<td>0.74</td>
</tr>
</tbody>
</table>

Now let’s contrast the absolute error Ms of 100 trajectories.

Table 4. Mean of the absolute error Ms in 100 trajectories.

<table>
<thead>
<tr>
<th>stepsize</th>
<th>$2^{-4}$</th>
<th>$2^{-5}$</th>
<th>$2^{-6}$</th>
<th>$2^{-7}$</th>
<th>$2^{-8}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler</td>
<td>12.97</td>
<td>7.47</td>
<td>1.87</td>
<td>0.98</td>
<td>5.38</td>
</tr>
<tr>
<td>Milstein</td>
<td>14.75</td>
<td>7.24</td>
<td>3.44</td>
<td>1.90</td>
<td>8.68</td>
</tr>
<tr>
<td>Imp$_1$</td>
<td>14.68</td>
<td>7.19</td>
<td>3.37</td>
<td>1.85</td>
<td>8.73</td>
</tr>
<tr>
<td>Imp$_2$</td>
<td>1.86</td>
<td>1.52</td>
<td>0.34</td>
<td>0.15</td>
<td>2.72</td>
</tr>
<tr>
<td></td>
<td>38.01</td>
<td>13.63</td>
<td>4.30</td>
<td>2.16</td>
<td>5.38</td>
</tr>
</tbody>
</table>

Imp$_2$ has a same accuracy with $I_{21}$ scheme and milstein scheme.

From the Table 4, we can conclude that Imp$_1$ is obviously better than all the other schemes, especially when $h = 2^{-4}, 2^{-6}, 2^{-7}$. Still, Imp$_2$ always has a same accuracy with $I_{21}$ scheme and milstein scheme. It shows that Imp$_1$ is better than other schemes, and Imp$_2$ is also a proper scheme for solving stochastic differential equations.

REFERENCES


