Analytical Solutions of Some Two-Point Non-Linear Elliptic Boundary Value Problems

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ABSTRACT
Several problems arising in science and engineering are modeled by differential equations that involve conditions that are specified at more than one point. The non-linear two-point boundary value problem (TPBVP) (Bratu’s equation, Troesch’s problems) occurs in engineering and science, including the modeling of chemical reactions, diffusion processes, and heat transfer. An analytical expression pertaining to the concentration of substrate is obtained using Homotopy perturbation method for all values of parameters. These approximate analytical results were found to be in good agreement with the simulation results.

Keywords: Two-Point Elliptic Boundary Value Problems; Bratu’s Equation; Troesch’s Problem; Non-Linear Equations; Homotopy Perturbation Method; Porous Catalyst; Numerical Simulation

1. Introduction
All chemical reactions are usually accompanied with mass and energy transfer, either homogeneously or heterogeneously. Mathematical modeling for these processes is based on material and energy balance. One can generate a set of differential equations known as the reaction-diffusion problem. Owing to the strong nonlinearity of the reaction rate, mainly from the effect of temperature, reaction-diffusion equations are paid more attention in analyzing and designing chemical and catalytic reactors [1]. The same phenomena exist in electrochemical processes, with the add complexity of a varying potential field, and considerable research has been reviewed for electrochemical reactions occurring in the porous electrode [2].

Linear and nonlinear phenomena are of fundamental importance in various fields of science and engineering. Most models of real-life problems are still very difficult to solve. Therefore, approximate analytical solutions such as Homotopy perturbation method (HPM) [3-12] were introduced. This method is the most effective and convenient ones for both linear and nonlinear equations. Perturbation method is based on assuming a small parameter. The majority of nonlinear problems, especially those having strong nonlinearity, have no small parameters at all and the approximate solutions obtained by the perturbation methods, in most cases, are valid only for small values of the small parameter. Generally, the perturbation solutions are uniformly valid as long as a scientific system parameter is small. However, we cannot rely fully on the approximations, because there is no criterion on which the small parameter should exist. Thus, it is essential to check the validity of the approximations numerically and/or experimentally. To overcome these difficulties, HPM have been proposed recently. In this paper we will apply Homotopy perturbation method (HPM) to the nonlinear Bratu’s problem, Troesch’s problem, and catalytic reactions in flat particles.

Systems of nonlinear differential equations arise in mathematical models throughout science and engineering. When an explicit condition that a solution must satisfy is specified at one value of the independent variable, usually its lower bound, this is referred to as an initial value problem (IVP). When the conditions to be satisfied occur at more than one value of the independent variable, this is referred to as a boundary value problem (BVP). If there are two values of the independent variable at which conditions are specified, then this is a two-point boundary value problem (TPBVP). TPBVPs occur in a wide variety of problems, including the modeling of chemical reactions, heat transfer, and diffusion. They are also of interest in optimal control problems.

There are many techniques available for the numerical solution of TPBVPs for ordinary differential equations [13]. The standard techniques can be divided into two classes. Typical of this class are various shooting and multi-shooting approaches. The other class involves converting the TPBVP into a system of algebraic equa-
2. Mathematical Formulation of the Problem

Many problems in science and engineering require the computational of family of solutions of a non linear system of the form [16]:

\[ G(y, \lambda) = 0, \quad y = y(\lambda) \]  

(1)

where \( G: \mathbb{R}^{m+1} \rightarrow \mathbb{R} \) is continuously differentiable function, \( y \) represents the solution and \( \lambda \) is a real parameter (i.e., Reynold’s number, load etc.). It is required to find a solution for some \( \lambda \)-interval, i.e., a path solutions, \( (y(\lambda), \lambda) \). Equations of the form (1) are called nonlinear elliptic eigenvalue problems if the operator \( G \) with \( \lambda \) fixed is an elliptic differential operator. For more details about this type of operators see [17]. As a typical example of nonlinear elliptic eigenvalue problems, we consider the following problem

\[ G(y, \lambda) = \Delta y + \lambda f(y), \quad \text{in } \Omega \]  

(2)

\[ y = 0, \quad \text{on } \partial \Omega \]  

(3)

where \( \Delta \) is Laplacian operator in one dimension.

Equation (2) arises in many physical problems. For example, in chemical reactor theory, radiative heat transfer, combustion theory, and in modelling the expansion of the universe. The function \( y \) could be a function of several variables and the domain \( \Omega \) is usually taken to be the unit interval \([0,1]\) in \( \mathbb{R} \), or the unit square \([0,1] \times [0,1]\) in \( \mathbb{R}^2 \), or the unit cube \([0,1] \times [0,1] \times [0,1]\) in \( \mathbb{R}^3 \). Equation (1) can take several forms, for example, Bratu equation is given by

\[ \Delta y + \lambda e^y = 0, \quad \text{in } \Omega \]  

(4)

\[ y = 0, \quad \text{on } \partial \Omega \]  

(5)

and a reaction-diffusion problem takes the form

\[ \Delta y + \lambda \exp\left(\frac{y}{1+\alpha y}\right) = 0, \quad \text{in } \Omega \]  

(6)

\[ y = 0, \quad \text{on } \partial \Omega \]  

(7)

There are no bifurcation points in the two problems above; all singular points are fold points. The behaviour of the solution near the singular points has been studied numerically [17-19] and theoretically [20-23]. For both one and two-dimensional cases, the Bratu problem has exactly one fold point, whereas the three-dimensional case has infinitely many fold points.

2.1. Bratu’s Equation and Its Solution

Bratu’s equation [24] was first studied as a simple case of a second-order ordinary differential equation by Bratu [25]. The equation arises when deriving the temperature distribution for a reaction in an infinite vessel with plane-parallel walls, and also in a simplification of a combustion reaction with a cylindrical vessel [26]. The differential equation is

\[ y'' + \lambda \exp(y) = 0, \quad t \in [0,1] \]  

(8)

with boundary conditions

\[ y(0) = y(1) = 0 \]  

(9)

The analytical solution of Equations (8) and (9) using Homotopy perturbation method (See Appendix A) is

\[ y(t) = -\left(\frac{7+b^2}{4} + \frac{5+b^2}{3} - \frac{\sqrt{12} b}{2} \right) \cos(\sqrt{\lambda} t) \]

\[ + \frac{1-b^2}{12} \cos(2\sqrt{\lambda} t) + \left(\frac{b \sin(2\sqrt{\lambda} t)}{6}\right) \]

\[ + \sin(\sqrt{\lambda} t) \left(b + \frac{t}{2}\right) + \left(\frac{1}{6} \sin(\sqrt{\lambda} t) \right) \left(\frac{3+b^2}{4}\right) \]

\[ - \left(\frac{b \sin(2\sqrt{\lambda} t)}{6} + \cos(\sqrt{\lambda} t) \left(\frac{b \sqrt{\lambda}}{2} - \frac{b^2+2}{3}\right) \right) \]

\[ \cos(\sqrt{\lambda} t) \left(\frac{b \sqrt{\lambda}}{2} - \frac{b^2+2}{3}\right) \]

\[ \right) \]

(10)

where

\[ b = \frac{1-\cos(\sqrt{\lambda})}{\sin(\sqrt{\lambda})} \]  

(11)

2.2. Reaction Diffusion Equation and Its Solution

Consider the reaction diffusion equation [16]

\[ y'' + \lambda \exp\left(\frac{y}{1+\alpha y}\right) = 0, \quad t \in (0,1) \]  

(12)

with the boundary conditions

\[ y(0) = y(1) = 0 \]  

(13)
The analytical solution of Equations (12) and (13) using Homotopy perturbation method (See Appendix C) is

\[ y(t) = \left( \frac{\alpha (b^2 + 3)}{2} - 1 \right) + \left( 1 - \frac{\alpha (2b^2 + 4)}{3} + \alpha b \sqrt{\lambda} \right) \cos(\sqrt{\lambda}t) \\
+ \frac{\alpha (b^2 - 1) \cos(2\sqrt{\lambda}t)}{3} - \left( \frac{\alpha b \sin(2\sqrt{\lambda}t) t}{6} \right) \\
+ \sin(\sqrt{\lambda}t) \left\{ (b - \alpha \sqrt{\lambda}t) + \left( \frac{\alpha}{\sin(\sqrt{\lambda})} \right) \sqrt{\lambda} \sin(\sqrt{\lambda}) \right\} \\
+ \frac{b \sin(2\sqrt{\lambda}) (b^2 + 3)}{2} - \left( \frac{b^2 - 1 \cos(2\sqrt{\lambda})}{6} \right) \\
+ \left( \frac{2b^2 + 4 - b \sqrt{\lambda}}{3} \right) \cos(\sqrt{\lambda}t) \right\} \]

where \( b \) is defined by Equation (11).

### 2.3. Troesch’s Problem and Its Solution

Troesch’s problem comes from the investigation of the confinement of a plasma column under radiation pressure. The problem was first described and solved by Weibel [27]. It has become a widely used test problem, and has been solved many times, including in analytical closed form [28] by using a shooting method [29], by using a Laplace transform decomposition technique [30] and most recently by using a modified Homotopy perturbation technique [31]. The differential equation is

\[ y'' = y \sinh(\alpha y), \quad t \in [0,1] \]  

with the boundary conditions

\[ y(0) = 0 \quad \text{and} \quad y(1) = 1 \]  

The known analytical, closed form solution [28] of Equations (15) and (16) is given by

\[ y(t) = \frac{2}{\lambda} \sinh^{-1} \left[ \frac{y'(0)}{2} - \frac{\sinh(\lambda, 1 - \frac{1}{4} y'(0)^2)}{\sinh(\lambda, 1 - m^2)} \right] \]  

where \( y'(0) = 2(1 - m)^2 \) is the derivative at \( t = 0 \) and the constant \( m \) is the solution to the equation

\[ \frac{\sinh(\frac{\lambda}{2})}{(1 - m)} = \text{sc}(\lambda, m) \]  

We have obtained the analytical solution of Equations (15) and (16) using Homotopy perturbation method (See Appendix F) is

\[ y(t) = \left( \frac{\sinh(\lambda t)}{\sinh(\lambda)} \right) + \frac{\lambda}{48 \sinh^3(\lambda)} \left[ \frac{\sinh(\lambda t)}{\sinh(\lambda)} \right] \left( 3 \cosh(\lambda) - \frac{\sinh(3\lambda)}{4} \right) - 3 \tanh(\lambda) \]

(19)

### 2.4. Catalytic Reactions in a Flat Particle and Its Solution

This example arises in a study of heat and mass transfer for a catalytic reaction within a porous catalyst flat particle [32]. The differential equation is the direct result of a material and energy balance. Assuming a flat geometry for the particle and that conductive heat transfer is negligible compared to convective heat transfer yields the differential equation.

\[ y'' = \lambda y \exp \left[ \frac{\gamma \beta (1 - y)}{1 + \beta (1 - y)} \right], \quad t \in [0,1] \]  

with boundary conditions

\[ y'(0) = 0 \quad \text{and} \quad y(1) = 1 \]  

The analytical solution of the Equations (20) and (21) using Homotopy perturbation method [33-41] (See Appendix H) is

\[ y(t) = \left[ 1 + \frac{\lambda \beta \gamma (\cosh(2k) - 3)}{6k^2(1 + \beta)^2 \cosh^2(k)} \left( \frac{\cosh(kt)}{\cosh(k)} \right) \right] \\
+ \frac{\lambda \gamma \beta (3 - \cosh(2kt))}{6(1 + \beta)^2 \cosh^2(k)} \]  

(22)

where

\[ k = \sqrt{\lambda + \frac{\lambda \beta \gamma}{(1 + \beta)}} \]  

(23)

### 3. Numerical Simulation

The non-linear equations [Equations (3), (7), (10) and (15)] for the given boundary conditions are solved by numerically. The function pdex4 in Matlab software is used to solve two-point boundary value problems (BVPs) for ordinary differential equations given in Appendix B, Appendix D, Appendix E, Appendix G, Appendix I, Appendix J and Appendix K. The numerical results are also compared with the obtained analytical expressions [Equations (5), (6), (9), (14), (17) and (18)] for all values of parameters \( \lambda, \alpha, \beta \) and \( \gamma \).
4. Results and Discussion

Figure 1 represents the dimensionless concentration $y(t)$ versus the dimensionless distance $t$ for different values of the dimensionless parameter $\lambda$. From this figure, it is evident that the values of the dimensionless concentration $y(t)$ increases when dimensionless parameter $\lambda$ increases. Figures 2(a)-(d) show the concentration $y(t)$ versus dimensionless distance $t$ for various values of dimensionless parameters $\alpha$ and $\lambda$. From these figures, it is obvious that the values of the dimensionless concentration $y(t)$ increases when dimensionless parameters $\lambda$ increases for the fixed values of $\alpha$. From the Figures 3(a) and (b), it is clear that the concentration $y(t)$ decreases for the different values of the dimensionless parameter $\alpha$, for the various values of $\lambda$. The dimensionless concentration $y(t)$ versus the dimensionless distance $t$ for different values of dimensionless parameter $\lambda$ is plotted in Figure 4.

![Figure 1](image1.png)

Figure 1. The curve is plotted for the influence of $\lambda$ on the dimensionless on concentration $y(t)$ versus the dimensionless distance $t$ obtained from the Equations (10) and (11).

![Figure 2](image2.png)

Figure 2. Influence of $\lambda$ on the dimensionless concentration $y(t)$ obtained from the Equation (14). The curve is plotted, when (a) $\alpha = 0.5$; (b) $\alpha = 1$; (c) $\alpha = 2$; (d) $\alpha = 3$. 
Figure 3. Influence of $\alpha$ on the dimensionless concentration $y(t)$ obtained from the Equation (14). The curve is plotted, when (a) $\lambda = 0.3$; (b) $\lambda = 1$.

Figure 4. The curve is plotted for the influence of $\lambda$ on the dimensionless concentration $y(t)$ versus the dimensionless distance $t$ from the Equation (19).

From this figure, it shows that the concentration $y(t)$ decreases for the various values of $\lambda$. Figures 5(a)-(d) shows the dimensionless concentration $y(t)$ in the reactor versus the dimensionless distance down the reactor $t$. From these figures it is clear that the concentration $y(t)$ decreases for the fixed values of $\alpha$ and $\gamma$ for the different values of $\lambda$.

Figures 6 and 7 shows the dimensionless concentration $y(t)$ versus the dimensionless distance $t$. From these figures it is clear that the concentration $y(t)$ decreases for the fixed values of $\lambda$ and $\gamma$ for the different values of $\alpha$.

5. Conclusion

The steady state non-linear reaction-diffusion equation has been solved analytically and numerically. The dimensionless concentrations $y(t)$ in the reactor at the position $t$ are derived by using the HPM. The primary result of this work is simple approximate calculations of concentration for all values of dimensionless parameters $\alpha$, $\beta$, $\gamma$ and $\lambda$. The HPM is an extremely simple method and it is also a promising method to solve other non-linear equations. This method can be easily extended to find the solution of all other non-linear equations.

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Figure 5. The curve is plotted for the influence of $\lambda$ on the dimensionless concentration $y$ versus the dimensionless distance down the reactor $t$ obtained from Equations (22) and (23), when (a) $\beta = 0.2$, $\gamma = 1$; (b) $\beta = 0.1$, $\gamma = 5$; (c) $\beta = 0.05$, $\gamma = 20$; (d) $\beta = 0.3$, $\gamma = 0.5$. 
Figure 6. The curve is plotted for the influence of $\beta$ on the dimensionless concentration $y$ versus the dimensionless distance down the reactor $t$ obtained from Equations (22) and (23), when (a) $\lambda = 1, \gamma = 10$; (b) $\lambda = 1, \gamma = 0.5$; (c) $\lambda = 1, \gamma = 1$; (d) $\lambda = 0.5, \gamma = 10$.

Figure 7. The curve is plotted for the influence of $\gamma$ on the dimensionless concentration $y$ versus the dimensionless distance down the reactor $t$ obtained from Equations (22) and (23), when (a) $\lambda = 1, \beta = 0.5$; (b) $\lambda = 1, \beta = 0.1$; (c) $\lambda = 1, \beta = 0.05$; (d) $\lambda = 2, \beta = 0.05$. 

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Appendix A: Solution of Bratu’s Equation Using HPM

In this Appendix, we indicate how the Equation (10) is derived. When \( y \) is small, Equation (8) is reduced to

\[
\frac{d^2 y}{dt^2} + \lambda \left[ 1 + y + \frac{y^2}{2} \right] = 0 \tag{A1}
\]

We construct the Homotopy for the Equation (A1) as follows:

\[
(1-p) \left[ \frac{d^2 y}{dt^2} + \lambda y + \lambda \right] + p \left[ \frac{d^2 y}{dt^2} + \lambda y + \frac{\lambda y^2}{2} \right] = 0 \tag{A2}
\]

The analytical solution of Equation (8) with Equation (9) is

\[
y = y_0 p y_1 + p^2 y_2 + \cdots \tag{A3}
\]

Substituting the Equation (A3) into an Equation (A2) we get

\[
(1-p) \left[ \frac{d^2 \left( y_0 p y_1 + p^2 y_2 + \cdots \right)}{dt^2} \right]
\]

\[
+ \lambda \left( y_0 + p y_1 + p^2 y_2 + \cdots \right) + \lambda
\]

\[
+ p \left[ \frac{d^2 \left( y_0 p y_1 + p^2 y_2 + \cdots \right)}{dt^2} \right]
\]

\[
+ \frac{\lambda}{2} \left( y_0 + p y_1 + p^2 y_2 + \cdots \right) + \lambda
\]

\[
\left[ \frac{\lambda}{2} \left( y_0 + p y_1 + p^2 y_2 + \cdots \right) \right]^2 = 0 \tag{A4}
\]

Comparing the coefficients of like powers of \( p \) in Equation (A4) we get

\[
p^0: \frac{d^2 y_0}{dt^2} + \lambda y_0 + \lambda = 0 \tag{A5}
\]

\[
p^1: \frac{d^2 y_1}{dt^2} + \lambda y_1 + \frac{\lambda y_0}{2} = 0 \tag{A6}
\]

The initial approximations are as follows

\[
y_0(0) = 0, \quad y_0(1) = 0, \tag{A7}
\]

\[
ya_i(0) = y_i(1) = 0, \quad i = 1, 2, 3, \cdots \tag{A8}
\]

Solving the Equation (A5) and the Equation (A6) and using the boundary conditions Equation (A7) and the Equation (A8) we obtain the following results:

\[
y_0 = \left( \cos \left( \sqrt{\lambda} t \right) - 1 \right) + b \sin \left( \sqrt{\lambda} t \right) \tag{A9}
\]

\[
y_i = \left[ \left( \frac{b^2 + 2}{3} - \frac{\sqrt{\lambda} b t}{2} \right) \cos \left( \sqrt{\lambda} t \right) + \left( \frac{1-b^2}{12} \right) \cos \left( 2 \sqrt{\lambda} t \right)
\]

\[
+ \left( \frac{b \sin(2 \sqrt{\lambda} t)}{6} - \frac{b^2 + 3}{4} \right) + \sin \left( \sqrt{\lambda} t \right) \right] b + \frac{\sqrt{\lambda} t}{2}
\]

\[
+ \frac{1}{\sin(\sqrt{\lambda})} \left[ \left( \frac{b^2 + 3}{4} \right) - \left( \frac{b^2 - 1}{12} \right) \cos \left( 2 \sqrt{\lambda} t \right)
\]

\[
- \lambda \sin \left( \sqrt{\lambda} \right) - \frac{b \sin(2 \sqrt{\lambda})}{6}
\]

\[
+ \cos \left( \sqrt{\lambda} \right) \left[ \left( \frac{b \sqrt{\lambda}}{2} \right) - \left( \frac{b^2 + 2}{3} \right) \right] \right] \right]
\]

where \( b \) is defined in Equation (9). According to the HPM, we can conclude that

\[
y = \lim_{p \to 1} y(t) = y_0 + y_1 \tag{A11}
\]

After putting the Equation (A9) and Equation (A10) into an Equation (A11) we obtain the solution in the text.

Appendix B: Matlab Program Is to Find the Numerical Solution of the Non Linear Differential Equations (8) and (9)

```matlab
function pdex4
m = 0;
x = linspace(0,1);
t = linspace(0,10000);
sol = pdepe(m,@pdex4pde,@pdex4ic,@pdex4bc,x,t);
u1 = sol(:,:,1);
figure
plot(x,u1(end,:))
title('u1(x,t)')
xlabel('Distance x')
ylabel('u1(x,2)')
end

function [c,f,s] = pdex4pde(x,t,u,DuDx)
c = 1;
f = DuDx;
lambda = 2;
F = lambda*exp(u);
s = F;
end

function u0 = pdex4ic(x)
u0 = 1;
end

function [pl,ql,pr,qr] = pdex4bc(xl,ul,xr,ur,t)
pl = ul;
ql = 0;
end
```
Appendix C: Solution of Reaction Diffusion Equation Using HPM

In this Appendix, we indicate how Equation (14) is derived. When \( \frac{y}{1+\alpha y} \) is small, Equation (12) reduces to

\[
\frac{d^2 y}{dt^2} + \lambda \left[ 1 + y - \alpha y^2 \right] = 0 \quad (C1)
\]

We construct the Homotopy for Equation (C1) as follows:

\[
(1 - p) \left[ \frac{d^2 y}{dt^2} + \lambda y + \lambda \right] + p \left[ \frac{d^2 y}{dt^2} + \lambda y + \lambda - \lambda \alpha y^2 \right] = 0 \quad (C2)
\]

The analytical solution of Equation (12) with Equation (13) is

\[
y = y_0 + py_1 + p^2 y_2 + \cdots \quad (C3)
\]

Substituting Equation (C3) into an Equation (C2) we get

\[
(1 - p) \left[ \frac{d^2 (y_0 + py_1 + p^2 y_2 + \cdots)}{dt^2} + \lambda (y_0 + py_1 + p^2 y_2 + \cdots) + \lambda \right]
\]

\[
- p \left[ \frac{d^2 (y_0 + py_1 + p^2 y_2 + \cdots)}{dt^2} + \lambda (y_0 + py_1 + p^2 y_2 + \cdots) \right]
\]

\[
- \lambda \alpha (y_0 + py_1 + p^2 y_2 + \cdots)^2 + \lambda \right] = 0 \quad (C4)
\]

Comparing the coefficients of like powers of \( p \) in Equation (C4) we get

\[
p^0: \frac{d^2 y_0}{dt^2} + \lambda y_0 + \lambda = 0 \quad (C5)
\]

\[
p^1: \frac{d^2 y_1}{dt^2} + \lambda y_1 - \lambda \alpha y_0^2 = 0 \quad (C6)
\]

The initial approximations are as follows:

\[
y_0(0) = 0, \quad y_0(1) = 0, \quad (C7)
\]

\[
y_1(0) = y_1(1) = 0, \quad i = 1, 2, 3 \cdots \quad (C8)
\]

Solving the Equations (C5) and (C6) and using the boundary conditions Equation (C7) and the Equation (C8) we obtain the following results:

\[
y_0 = \left( \cos (\sqrt{\lambda}t) - 1 \right) + b \sin (\sqrt{\lambda}t) \quad (C9)
\]

\[
y_1 = \left[ \frac{\alpha (b^2 + 3)}{2} \right] + \left[ \frac{1 - b^2}{12} \right] \cos (2\sqrt{\lambda}t)
\]

\[
+ \left[ \frac{b \sin (2\sqrt{\lambda}) + \alpha \sin (\sqrt{\lambda}t)}{2} \right] - \left( \frac{b^2 - 1}{12} \right) \cos (2\sqrt{\lambda}) \quad (C10)
\]

\[
- \left( \frac{\lambda \sin (\sqrt{\lambda})}{2} - \frac{\lambda \sin (2\sqrt{\lambda})}{6} \right)
\]

\[
+ \left[ \frac{b \sqrt{\lambda}}{2} \right] - \left( \frac{b^2 + 2}{3} \right) \right]
\]

where \( b \) is defined in the text Equation (6). According to the HPM, we can conclude that

\[
y = \lim_{p \to 1} y(\tau) = y_0 + y_1 \quad (C11)
\]

After putting Equation (C9) and Equation (C10) into an Equation (C11) we obtain the solution in the text.

Appendix D: Matlab Program Is to Find the Numerical Solution of the Non-Linear Differential Equations (12) and (13)

function pdex4
m = 0;
x = linspace(0,1);
t = linspace(0,10000);
sol = pdepe(m,@pdex4pde,@pdex4ic,@pdex4bc,x,t);
u1 = sol(:,:,1);
figure
plot(x,u1(end,:));
title('u1(x,t)')
xlabel('Distance x')
ylabel('u1(x,2)')

%------------------
function [c,f,s] = pdex4pde(x,t,u,DuDx)
c = 1;
f = DuDx;
lamda=1.5;
alpha=0.5;
F = lamda*exp(u(1+(alpha*u)));
s = F;
%--------------------------------------------------
function u0 = pdex4ic(x); %create a initial conditions
u0 = 1;
%--------------------------------------------------
function [pl, ql, pr, qr] = pdex4bc(xl, ul, xr, ur, t) % create a boundary conditions
pl = ul;
ql = 0;
pr = ur - 0;
qr = 0;

Appendix E: Matlab Program Is to Find the Numerical Solution of the Non Linear Differential Equations (12) and (13)

function pdex4
m = 0;
x = linspace(0, 1);
t = linspace(0, 10000);
sol = pdepe(m, @pdex4pde, @pdex4ic, @pdex4bc, x, t);
u1 = sol(:,:,1);
figure
plot(x, u1, 'b--
','LineWidth',1);
axis([0 1 0 1.5])
title('u1(x,t)')
xlabel('Distance x')
ylabel('u1(x,t)')

%-----------------------------------------------
%-----------------------------------------------
function [c, f, s] = pdex4pde(x, t, u, DuDx)
c = 1;
f = DuDx;
lamda = 0.3;
alpha = 30;
F = lamda * exp((1 + (alpha*u)));
s = F;

%-----------------------------------------------
function u0 = pdex4ic(x); % create a initial conditions
u0 = 1;

%-----------------------------------------------
function [pl, ql, pr, qr] = pdex4bc(xl, ul, xr, ur, t) % create a boundary conditions
pl = ul;
ql = 0;
pr = ur - 0;
qr = 0;

Appendix F: Solution of Troesch’s Problem Using HPM

In this Appendix, we indicate how the Equation (19) is derived.

When \( \lambda y \) is small, Equation (15) is reduces to

\[
\frac{d^2y}{dr^2} - \lambda \left[ \lambda y + \frac{\lambda^2 y^3}{6} \right] = 0
\]  

(E1)

We construct the Homotopy for the Equation (E1) is as follows:

\[
(1 - p) \left[ \frac{d^2y}{dr^2} - \lambda^2 y \right] + p \left[ \frac{d^2y}{dr^2} - \lambda^2 y - \frac{\lambda^4 y^3}{6} \right] = 0
\]  

(E2)

The analytical solution of Equation (15) with Equation (16) is

\[
y = y_0 + py_1 + p^2 y_2 + \cdots
\]  

(E3)

Substituting the Equation (E3) into an Equation (E2) we get

\[
(1 - p) \left[ \frac{d^2(y_0 + py_1 + p^2 y_2 + \cdots)}{dr^2} - \lambda^2 (y_0 + py_1 + p^2 y_2 + \cdots) \right]
\]

\[
- \lambda^2 \left( y_0 + py_1 + p^2 y_2 + \cdots \right)
\]

\[
+ p \left[ \frac{d^2(y_0 + py_1 + p^2 y_2 + \cdots)}{dr^2} - \lambda^2 (y_0 + py_1 + p^2 y_2 + \cdots) \right]
\]

\[
- \lambda^2 \left( y_0 + py_1 + p^2 y_2 + \cdots \right)
\]

\[
\frac{\lambda^4 (y_0 + py_1 + p^2 y_2 + \cdots)^3}{6}
\]  

(E4)

Comparing the coefficients of like powers of \( p \) in Equation (E4) we get

\[
p^0 : \frac{d^2y_0}{dr^2} - \lambda^2 y_0 = 0
\]  

(E5)

\[
p^1 : \frac{d^2y_1}{dr^2} - \lambda^2 y_1 - \frac{\lambda^4 y_0^3}{6} = 0
\]  

(E6)

The initial approximations are as follows

\[
y_0(0) = 0, y_1(0) = 1,
\]

(E7)

\[
y_i(0) = 0, y_i(1) = 0, i = 1, 2, 3\cdots
\]  

(E8)

Solving the Equation (E5) and the Equation (E6) and using the boundary conditions Equation (E7) and the Equation (E8) we obtain the following results:

\[
y_0 = \frac{\sinh(\lambda t)}{\sinh(\lambda)}
\]  

(E9)

\[
y_1 = \frac{\lambda^2}{48 \sinh^3(\lambda)} \left[ \frac{\sinh(\lambda) - \sinh(3\lambda)}{\sinh(\lambda) - \sinh(3\lambda)} \right]
\]  

(E10)

According to the HPM, we can conclude that

\[
y = \lim_{p \to 1} y(p) = y_0 + y_1
\]  

(E11)
After putting the Equation (E9) and the Equation (E10) into an Equation (E11) we obtain the solution in the text.

Appendix G: Matlab Program Is to Find the Numerical Solution of the Non Linear Differential Equations (15) and (16)

```matlab
function pdex4
m = 0;
x = linspace(0,1);
t=linspace(0,10000);
sol= pdepe(m,@pdex4pde,@pdex4ic,@pdex4bc,x,t);
u1 = sol(:,:,1);
figure
plot(x,u1(end,:))
title('u1(x,t)')
ylabel('u1(x,2)')

-----------------------------------------------------
ex4pde(x,t,u,DuDx)
-----------------------------------------------------
ix H: Solution of Catalytic Reactions
Appendix, we indicate how the Equation (22) is
numerical Solution of the Non Linear
Differential Equations (15) and (16)
function pdex4
m = 0;
x = linspace(0,1);
t=linspace(0,10000);
sol= pdepe(m,@pdex4pde,@pdex4ic,@pdex4bc,x,t);
u1 = sol(:,:,1);
figure
plot(x,u1(end,:))
title('u1(x,t)')
ylabel('u1(x,2)')
%-------------------
function [c,f,s] = pdex4pde(x,t,u,DuDx)
c = 1;
f = DuDx;
lamda=2.8;
F =-lamda*(sinh(lamda*u))
s = F;
%-------------
function u0 = pdex4ic(x); %create a initial conditions
u0 = 1;
%------------------------------------------------------------------
function[pl,ql,pr,qr]=pdex4bc(xl,ul,xr,ur,t) %create a
boundary conditions
pl = ul;
ql = 0;
pr = ur-1;
qr = 0;
%-------------------------------------------------------------------
```

Appendix H: Solution of Catalytic Reactions in a Flat Particle Using HPM

In this Appendix, we indicate how the Equation (22) is derived. When \( \frac{\gamma \beta (1-y)}{1+\beta (1-y)} \) is small, Equation (20) reduces to

\[
\frac{d^2 y}{dr^2} - \left( \frac{\lambda \gamma \beta}{1+\beta} \right) y + \left( \frac{\lambda \gamma \beta y^2}{(1+\beta)^2} \right) = 0
\]  
(H1)

We construct the Homotopy for the Equation (H1) is as follows:

\[
(1 - p) \left[ \frac{d^2 y}{dr^2} - \left( \frac{\lambda \gamma \beta}{1+\beta} \right) y \right] + p \left[ \frac{d^2 y}{dr^2} - \left( \frac{\lambda \gamma \beta y^2}{(1+\beta)^2} \right) y \right] = 0
\]  
(H2)

The analytical solution of Equation (20) with Equation (21) is

\[
y = y_0 + p y_1 + p^2 y_2 + \cdots
\]  
(H3)

Substituting the Equation (E3) into an Equation (E2) we get

\[
(1 - p) \left[ \frac{d^2 \left( y_0 + p y_1 + p^2 y_2 + \cdots \right)}{dr^2} - \left( 1 + \frac{\gamma \beta}{1+\beta} \right) \right] y_0 = 0
\]  
(H4)

Comparing the coefficients of like powers of \( p \) in Equation (H4) we get

\[
p^0 \left[ \frac{d^2 y_0}{dr^2} - \lambda \left( 1 + \frac{\gamma \beta}{1+\beta} \right) y_0 \right] = 0
\]  
(H5)

\[
p^1 \left[ \frac{d^2 y_1}{dr^2} - \lambda \left( 1 + \frac{\gamma \beta}{1+\beta} \right) y_1 + \frac{\lambda \gamma \beta y_0^2}{(1+\beta)^2} \right] = 0
\]  
(H6)

The initial approximations are as follows

\[
y_0(0) = 0, y_0(1) = 1,
\]  
(H7)

\[
y_i'(0) = 0, y_i(1) = 0, \quad i = 1, 2, 3, \ldots
\]  
(H8)

Solving the Equation (H5) and the Equation (H6) and using the boundary conditions Equation (H7) and the Equation (H8) we obtain the following result:

\[
y_0 = \frac{\cosh (k)}{\cosh (k)}
\]  
(H9)

\[
y_i = \frac{\lambda - \gamma \beta \gamma (\cosh (2k) - 3)}{6k^2 (1+\beta)^2 \cosh^2 (k)} \left( \frac{\cosh (k)}{\cosh (k)} \right) + \frac{\lambda\gamma \beta}{6(1+\beta)^2k^2 \cosh^2 (k)}
\]  
(H10)

where \( k \) is defined in the text Equation (23).

According to the HPM, we can conclude that

\[
y = \lim_{p \to 1} y_i(t) = y_0 + y_1
\]  
(H11)

After putting the Equation (H9) and the Equation (H10) into an Equation (H11) we obtain the solution in the text.
Appendix I: Matlab Program Is to Find the Numerical Solution of the Non Linear Differential Equations (20) and (21)

```matlab
function pdex4
m = 0;
x = linspace(0,1);
t = linspace(0,10000);
sol = pdepe(m,@pdex4pde,@pdex4ic,@pdex4bc,x,t);
figure
plot(x,sol(:,:,1))
title('u1(x,t)')
xlabel('Distance x')
ylabel('u1(x,2)')
```

beta=0.15;
F=-lamda*u*exp(beta*gamma*(1-u)/(1+beta*(1-u)));
s = F;

function u0 = pdex4ic(x);

u0 = 1;

Appendix J: Matlab Program Is to Find the Numerical Solution of the Non Linear Differential Equations (20) and (21)

```matlab
function pdex4
m = 0;
x = linspace(0,1);
t = linspace(0,10000);
sol = pdepe(m,@pdex4pde,@pdex4ic,@pdex4bc,x,t);
figure
plot(x,sol(:,:,1))
title('u1(x,t)')
xlabel('Distance x')
ylabel('u1(x,2)')
```

beta=0.1;
F=-lamda*u*exp(beta*gamma*(1-u)/(1+beta*(1-u)));
s = F;

function u0 = pdex4ic(x);

u0 = 1;

Appendix K: Matlab Program Is to Find the Numerical Solution of the Non Linear Differential Equations (20) and (21)

```matlab
function pdex4
m = 0;
x = linspace(0,1);
t = linspace(0,10000);
sol = pdepe(m,@pdex4pde,@pdex4ic,@pdex4bc,x,t);
figure
plot(x,sol(:,:,1))
title('u1(x,t)')
xlabel('Distance x')
ylabel('u1(x,2)')
```

beta=0.15;
F=-lamda*u*exp(beta*gamma*(1-u)/(1+beta*(1-u)));
s = F;

Appendix: L Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>t</td>
<td>Dimensionless distance down the reactor</td>
</tr>
<tr>
<td>y</td>
<td>Dimensionless concentration in the reactor</td>
</tr>
<tr>
<td>Symbol</td>
<td>Dimensionless Parameter</td>
</tr>
<tr>
<td>--------</td>
<td>-------------------------</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>Dimensionless parameter</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>Dimensionless parameter</td>
</tr>
</tbody>
</table>