On the Derivative of a Polynomial

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ABSTRACT

Certain refinements and generalizations of some well known inequalities concerning the polynomials and their derivatives are obtained.

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1. Introduction to the Statement of Results

Let $P_n(z)$ denote the space of all complex polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree $n$. If $P \in P_n$, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|$$

(1)

and

$$\max_{|z|=R>1} |P(z)| \leq R^n \max_{|z|=1} |P(z)|$$

(2)

Inequality (1) is an immediate consequence of S. Bernstein’s theorem (see [1]) on the derivative of a trigonometric polynomial. Inequality (2) is a simple deduction from the maximum modulus principle (see [2, p. 346] or [3, p. 137]).

Both the inequalities (1) and (2) are sharp and the equality in (1) and (2) holds if and only if $P$ has all its zeros at the origin. It was shown by Frappier, Rahman and Ruscheweyh [4, Theorem 8] that if $P \in P_n$, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(e^{i\alpha})|$$

(3)

Clearly (3) represents a refinement of (1), since the maximum of $|P(z)|$ on $|z|=1$ may be larger than the maximum of $|P(z)|$ taken over $(2n)$th roots of unity, as is shown by the simple example $P(z) = z^n + ia$, $a > 0$.

A. Aziz [5] showed that the bound in (3) can be considerably improved. In fact proved that if $P \in P_n$, then for every given real $\alpha$,

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} (M_\alpha + M_{\alpha+\pi})$$

(4)

where

$$M_\alpha = \max_{|z|=1} |P(e^{i\alpha} z)|$$

(5)

and $M_{\alpha+\pi}$ is obtained by replacing $\alpha$ by $\alpha + \pi$. The result is best possible and equality in (4) holds for $P(z) = z^n + re^{i\alpha}, -1 \leq r \leq 1$.

Clearly inequality (4) is an interesting refinement of inequality (3) and hence of Bernstein inequality (1) as well.

If we restrict ourselves to the class of polynomials $P \in P_n$ having no zero in $|z| < 1$, then the inequality (1) can be sharpened. In fact, P. Erdős conjectured and later P. D. Lax [6] (see also [7]) verified that if $P(z) \neq 0$ for $|z| < 1$, then (1) can be replaced by

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|$$

(6)

In this connection A. Aziz [5], improved the inequality (4) by showing that if $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every real $\alpha$,

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} (M_\alpha + M_{\alpha+\pi})^{1/2}$$

(7)

where $M_\alpha$ is defined by (5). The result is best possible and equality in (7) holds for $P(z) = z^n + e^{i\alpha}$.

A. Aziz [5] also proved that if $P \in P_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every real $\alpha$ and $R > 1$,

$$\max_{|z|=R} |P(Rz) - P(z)| \leq \frac{R^n - 1}{2} (M_\alpha^2 + M_{\alpha+\pi}^2)^{1/2}$$

(8)

In this paper, we first present the following result which is a refinement of inequality (7).

Theorem 1. If $P \in P_n$, $P(z)$ does not vanish in $|z| < 1$ and $m = \min_{|z|=1} |P(z)|$, then for every real $\alpha$,

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2}$$

(9)

where $M_\alpha$ is defined by (5). The result is best possible and equality in (9) holds for $P(z) = z^n + e^{i\alpha}$.
As an application of Theorem 1, we mention the corresponding improvement of (8).

**Theorem 2.** If \( P \in P_n \), and \( P(z) \neq 0 \) for \(|z|<1\) and \( m = \min_{|z|=1} |P(z)| \) then for every real \( \alpha \) and \( R > 1 \),

\[
|P(Rz) - P(z)| \leq \frac{R^n - 1}{2} \left( M^2_a + M^2_{a+k^n} - 2m^2 \right)^{1/2},
\]

where \( M_a \) is defined by (5). The result is best possible and equality in (10) holds for \( P(z) = z^n + e^{\alpha z} \).

Here we also consider the class of polynomials \( P \in P_n \) having no zero in \(|z|<k\), \( k > 0 \) and present some generalizations of the inequalities (9) and (10). First we consider the case \( k \geq 1 \) and prove the following result which is a generalization of inequality (9).

**Theorem 3.** If \( P \in P_n \) does not vanish in \(|z|<k\), \( k \geq 1 \) and \( m = \min_{|z|=1} |P(z)| \), then for every real \( \alpha \),

\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{2(1+k^2)} \left( M^2_a + M^2_{a+k^n} - 2m^2 \right)^{1/2},
\]

where \( M_a \) is defined by (5).

Next result is a corresponding generalization of the inequality (10).

**Theorem 4.** If \( P \in P_n \) does not vanish in \(|z|<k\), \( k \geq 1 \) and \( m = \min_{|z|=1} |P(z)| \), then for every real \( \alpha \) and \( R > 1 \),

\[
|P(Rz) - P(z)| \leq \frac{R^n - 1}{2} \left( M^2_a + M^2_{a+k^n} - 2m^2 \right)^{1/2},
\]

where \( M_a \) is defined by (5).

**Remark 1.** For \( k = 1 \), Theorem 3 and Theorem 4 reduces to the Theorem 1 and Theorem 2 respectively.

For the case \( k \leq 1 \), we have been able to prove:

**Theorem 5.** If \( P \in P_n \), \( P(z) \) has no zero in \(|z|<k\), \( k \leq 1 \) and \( m = \min_{|z|=1} |P(z)| \), then for every real \( \alpha \),

\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{2(1+k^2)} \left( M^2_a + M^2_{a+k^n} - 2m^2 \right)^{1/2},
\]

provided \( |P'(z)| \) and \( |Q'(z)| \) attain maximum at the same point on \(|z|=1\) where \( Q(z) = z^n P(1/z) \). The result is best possible and equality in (13) holds for \( P(z) = z^n + k^n \).

**Theorem 6.** If \( P \in P_n \), \( P(z) \) has no zero in \(|z|<k\), \( k \leq 1 \) and \( m = \min_{|z|=1} |P(z)| \), then for every real \( \alpha \) and \( R > 1 \),

\[
|P(Rz) - P(z)| \leq \frac{R^n - 1}{2} \left( M^2_a + M^2_{a+k^n} - 2m^2 \right)^{1/2},
\]

provided \( |P'(z)| \) and \( |Q'(z)| \) attain maximum at the same point on \(|z|=1\) where \( Q(z) = z^n P(1/z) \). The result is best possible and equality in (14) holds for \( P(z) = z^n + k^n \).

2. **Lemmas**

For the proofs of these theorems, we need the following lemmas. The first Lemma is due to A. Aziz [5].

**Lemma 1.** If \( P \in P_n \), then for \(|z|<1\) and every real \( \alpha \),

\[
|P'(z)| + |nP(z) - zP'(z)| \leq \frac{n^2}{2} \left( M^2_a + M^2_{a+k^n} \right)
\]

where \( M_a \) is defined by (5).

**Lemma 2.** If \( P \in P_n \) and \( P(z) \neq 0 \) for \(|z|<k\), \( k \geq 1 \), then for \(|z|=1\),

\[
k|P'(z)| \leq |nP(z) - zP'(z)| - nm
\]

where \( m = \min_{|z|=1} |P(z)| \).

Lemma 2 is a special cases of a result due to A. Aziz and N. A. Rather [8, Lemma 5].

**Lemma 3.** If \( P \in P_n \) does not vanish in \(|z|<k\), \( k \leq 1 \), then

\[
k^p |P'(z)| \leq \max_{|z|=1} |Q'(z)| \text{ for } |z|=1
\]

where \( Q(z) = z^n P(1/z) \).

This Lemma is due to N. K. Govil [9].

**Lemma 4.** If \( P(z) \) is a polynomial of degree \( n \) which does not vanish in \(|z|<k\), \( k \leq 1 \), then for \(|z|=1\)

\[
k^p |P'(z)| + n \min_{|z|=1} |P(z)| \leq \max_{|z|=1} |Q'(z)|
\]

where \( Q(z) = z^n P(1/z) \).

**Proof of Lemma 4.** Let \( m = \min_{|z|=1} |P(z)| \). If \( P(z) \) has a zero on \(|z|=k\), then \( m = 0 \) and the result follows from Lemma 3. Henceforth we assume that \( P(z) \) has no zero on \(|z|=k\), therefore \( m > 0 \) and \( m \leq |P(z)| \) for \(|z|=k\).

If \( \alpha \) is any real or complex number with \(|\alpha|<1\), then for \(|z|=k\),

\[
|am^{n-k^n}| \leq |P(z)|
\]

By Rouche’s Theorem, it follows that the polynomial \( F(z) = P(z) - am^{n-k^n} \) does not vanish in \(|z|<k\), for every real or complex number \( \alpha \) with \(|\alpha|<1\).
Applying (2) to the polynomial $P'(z)$ which is of degree $n-1$ and using Theorem 1, we obtain for $t \geq 1$ and $0 \leq \theta < 2\pi$, 

$$
|P'(te^{i\theta})| \leq t^{n-1} \max_{|z|=1} |P'(z)|.
$$

Hence for each $\theta, 0 \leq \theta < 2\pi$ and $R > 1$, we have 

$$
|P(Re^{i\theta}) - P(e^{i\theta})| = \left| \int_0^R t^{n-1} P'(te^{i\theta}) \, dt \right| \leq \int_0^R |P'(te^{i\theta})| \, dt \\
\leq \frac{1}{2} (M_a^2 + M_{a+x}^2 - 2m^2)^{1/2} \int_R^\infty nt^{n-1} \, dt.
$$

This implies  for $|z| = 1$ and $R > 1$, 

$$
|P(Rz) - P(z)| \leq R^n - \frac{1}{2} (M_a^2 + M_{a+x}^2 - 2m^2)^{1/2},
$$

which proves Theorem 2.

The proof of the Theorem 3 and 4 follows on the same lines as that of Theorems 1 and 2, so we omit the details.

Proof of Theorem 5. Since all the zeros of $P(z)$ lie in $|z| \geq k$, where $k \leq 1$, $m = \min_{|z|=k} |P(z)|$, by Lemma 4, we have 

$$
|P'(z)| + nm \leq \max_{|z|=k} |Q'(z)|,
$$

where $Q(z) = z^n P(1/z)$. Also by hypothesis $|P'(z)|$ and $|Q'(z)|$ become maximum at the same point on $|z| = 1$, if 

$$
\max_{|z|=k} |P'(z)| = |P'(e^{i\alpha})|, 
0 \leq \alpha < 2\pi,
$$

then 

$$
\max_{|z|=k} |Q'(z)| = |Q'(e^{i\alpha})|, 
0 \leq \alpha < 2\pi
$$

and it can be easily verified that 

$$
|Q'(z)| = |nP(z) - zP'(z)| \text{ for } |z| = 1.
$$

Therefore, by Lemma 1 

$$
\left| P'(e^{i\alpha}) \right|^2 + \left| Q'(e^{i\alpha}) \right|^2 \\
\leq \left| P'(e^{i\alpha}) + nP(e^{i\alpha}) - e^{i\alpha} P'(e^{i\alpha}) \right|^2 \\
\leq \frac{n^2}{2} (M_a^2 + M_{a+x}^2).
$$

This gives with the help of (18), (19) and (20) that 

$$
\left| P'(e^{i\alpha}) \right|^2 + \left| Q'(e^{i\alpha}) \right|^2 \\
\leq \left| k^n \right|^2 + \left| P'(e^{i\alpha}) + nm \right|^2 \\
\leq \left| P'(e^{i\alpha}) \right|^2 + \left| Q'(e^{i\alpha}) \right|^2 \leq \frac{n^2}{2} (M_a^2 + M_{a+x}^2),
$$
which implies,
\[ \left| P'(e^{i\theta}) \right|^2 + k^{2m} \left| P'(e^{i\theta}) \right|^2 + n^2m^2 \leq \frac{n^2}{2} \left( M_{\alpha}^2 + M_{\alpha+1}^2 \right). \]

Equivalently,
\[ \left| P'(e^{i\theta}) \right|^2 \leq \frac{n^2}{2 \left( 1 + k^{2m} \right)} \left( M_{\alpha}^2 + M_{\alpha+1}^2 - 2m^2 \right) \]

and hence
\[ \max_{|\theta| = \frac{n}{4}} \left| P'(x) \right| \leq \frac{n}{\sqrt{2 \left( 1 + k^{2m} \right)}} \left( M_{\alpha}^2 + M_{\alpha+1}^2 - 2m^2 \right)^{1/2}. \]

This completes the proof of Theorem 5.

Theorem 6 follows on the same lines as that of Theorem 2, so we omit the details.

REFERENCES


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