An Application of Eulerian Graph to PI on $M_n(C)$

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ABSTRACT

We obtain a new class of polynomial identities on the ring of $n \times n$ matrices over any commutative ring with 1 by using the Swan’s graph theoretic method [1] in the proof of Amitsur-Levitzki theorem. Let $\Gamma$ be an Eulerian graph with $k$ vertices and $d$ edges. Further let $n \geq 1$ be an integer and assume that $d \geq 2kn$. We prove that

$$\sum_{x \in \Pi(\Gamma)} \text{sgn}(\pi) x_{a(1)}^x x_{a(2)} \cdots x_{a(d)} = 0$$

is an PI on $M_n(C)$. Standard and Chang [2] Giambruno-Sehgal [3] polynomial identities are the special examples of our conclusions.

Keywords: Eulerian Graph; Eulerian Path; Admissible; Polynomial Identity

1. Introduction

Let $\Gamma$ be a finite directed graph with multiple edges allowed, and let $V(\Gamma) = \{1, \cdots, k\}$ denote the vertex set of $\Gamma$ and $E(\Gamma) = \{e_1, \cdots, e_d\}$ the edges set of $\Gamma$. Let $\sigma$ and $\tau$ be the functions from $E(\Gamma)$ to $V(\Gamma)$ defined by $(\sigma(e), \tau(e)) = (i, j)$ where $e_i$ is an edge from vertex $i$ to vertex $j$. For a vertex $i \in V(\Gamma)$ we put

$$\phi_i(i) = \|e_i \sigma(e_i) = i\|, \phi_i(i) = \|e_i \tau(e_i) = i\|$$

and

$$\gamma(i) = \max \{\phi_i(i), \phi_j(i)\}$$

We say that $e_{a(1)} e_{a(2)} \cdots e_{a(n)}$ is an Eulerian path of $\Gamma$ if $\pi$ is an element of $\text{Sym}(d)$ (the symmetric group acting on the set $\{1, \cdots, d\})$ and $\tau(e_{a(i)}) = \sigma(e_{a(i+1)})$ for $i = 1, \cdots, d-1$.

It is well known that a connected graph $\Gamma$ has an Eulerian path starting at vertex $p$ and ending at vertex $q$ if and only if one of the following two conditions applies:

1) $p = q$ and $\phi_i(i) = \phi_j(i)$ for each $i = 1, \cdots, k$;

2) $p \neq q$ and $\phi_i(p) = \phi_i(q) + 1$, $\phi_j(p) = \phi_j(q) + 1$ and $\phi_i(i) = \phi_j(i)$ for each $i = 1, \cdots, k \setminus \{p, q\}$.

A directed connected graph $\Gamma_{p,q}$ with fixed vertices $p$ and $q$ is called Eulerian if either condition 1) or 2) is satisfied. We note that if $\Gamma_{p,q}$ is an Eulerian graph of type (b), then the vertices $p, q$ are uniquely determined, but in the other case we may choose any vertex $p = q$.

For an Eulerian graph $\Gamma_{p,q}$ denote by

$$\Pi(\Gamma_{p,q}) = \{\pi \in \text{Sym}(d) \mid e_{a(1)} \cdots e_{a(d)} \text{ is an Eulerian path of } \Gamma_{p,q} \text{ starting at vertex } p \text{ and ending at vertex } q\}$$

Thus $f_\tau(X)$ is a multilinear polynomial in the set $X = \{x_1, \cdots, x_\ell\}$ of non-commuting indeterminates.

Let $n \geq 1$ be an integer, $C$ a commutative ring with 1 and $T : X \rightarrow \{E_{u,v} \mid 1 \leq u, v \leq n\}$ a set map where the $E_{u,v}$’s are the standard matrix units over $C$. It is clear that $T$ can be viewed as a substitution. We shall define a directed graph $\Gamma_T$ induced from $\Gamma$ by $T$. First consider the directed graph on the vertex set $V \times \{1, \cdots, n\}$ with edge set $(\sigma, \tau) \rightarrow (\sigma, \tau + 1)$ and distinguished points $p, q$. Now we define $\Gamma_T$ by restricting the vertex set to $\bigcup_{\gamma(i) = \max \{\phi_i(i), \phi_j(i)\}} \text{sgn}(\pi) x_{a(1)}^x x_{a(2)} \cdots x_{a(d)}$ of non-commuting indeterminates.

2. Main Results

Let $\Gamma$ be an Eulerian graph with $d$ edges $e_1, e_2, \cdots, e_d$ and distinguished points $p$ and $q$, the polynomial $f_\tau(X)$ associated with $\Gamma$ is defined as follows:

$$f_\tau(X) = \sum_{\pi \in \Pi(\Gamma)} \text{sgn}(\pi) x_{a(1)}^x x_{a(2)} \cdots x_{a(d)}$$

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path of $\overline{\Gamma_r}$. For the remainder of this section, we introduce Swan’s theorem and our main results.

Swan [1]. Let $\Gamma$ be an Eulerian graph with $d$ edges and $k$ vertices satisfying $d \geq 2k$. Then $\Pi(\Gamma)$ has the same number of odd and even permutations (with respect to the fixed order)

**Theorem 1.** Let $\Gamma$ be an Eulerian graph with vertex set $V = \{1, 2, \ldots, k\}$ and $d$ edges. Further let $n \geq 1$ be an integer such that

$$d \geq 2\left(\sum_{i=1}^{k} \min\{n, \gamma(i)\}\right).$$

Then $f_r(\Gamma) = 0$ is a polynomial identity on the ring $M_n(C)$ of $n \times n$ matrices over a commutative ring $C$ with 1

**Corollary 2.** Let $\Gamma$ be an Eulerian graph with $k$ vertices and $d$ edges. Further let $n \geq 1$ be an integer and assume that $d \geq 2kn$. Then $f_r(\Gamma) = 0$ is a polynomial identity on $M_n(C)$.

3. Proof of Theorem 1

Since $f_r(X)$ is multilinear, it suffices to show that $f_r(X^T) = 0$ for any substitution $T$ of $n \times n$ matrix units over $C$. Fix such an $T$ and put $x_i^r = E_{u(r)(i)}$, $1 \leq r \leq d$. Then

$$f_r(X^T) = \sum_{\pi \in \Pi(\Gamma)} \text{sgn}(\pi) E_{u(\pi(1))} \cdots E_{u(\pi(d))} \quad (*)$$

Now consider $\overline{\Gamma_r}$. Clearly, and summand in (*) vanishes unless, for the given $\Gamma \in \Pi(\Gamma)$, $v(\pi(r)) = u(\pi(r+1))$

for all $1 \leq r \leq N - 1$, i.e., if $\pi$ is admissible. If so, on multiplying the matrix units, we obtain

$$\text{sgn}(\pi) E_{u(\pi(1))} \cdots E_{u(\pi(d))}.$$ 

It follows that

$$f_r(X^T) = \sum_{u,v} \left(\sum \text{sgn}(\pi) E_{uv}\right),$$

where the inner sum is taken over all admissible permutations with $u(\pi(1)) = u$ and $v(\pi(d)) = v$. If no such admissible $\pi$ exists, the inner sum is 0 by definition. We want to prove that this inner sum is 0 anyway. It is readily seen that for any choice of $u$ and $b$, a sum and $\text{sgn}(\pi)$ in the inner sum arises precisely if $\pi$ lifts to an Eulerian path of $\overline{\Gamma_r}$ from $(r, u)$ to $(q, v)$. Thus, on applying Swan’s theorem to $\overline{\Gamma_r}$ with $E(\overline{\Gamma_r}) = d$ and $\left|\overline{\Gamma_r}\right| = \sum_{i=1}^{k} \min\{n, \gamma(i)\}$, we find that the number of even and odd admissible permutations $\pi$ with $u(\pi(1)) = u$ and $v(\pi(d)) = v$ coincide whence the inner sum is 0 for any choice of $u$ and $v$. This completes the proof.

4. Applications

1) Let $\Gamma$ be the Eulerian graph on one vertex with $d$ loops. Then $\Pi(\Gamma) = \text{Sym}(d)$ and

$$f_r(\Gamma) = \sum_{\pi \in \text{Sym}(d)} \text{sgn}(\pi) x_{u(\pi(1))} \cdots x_{u(\pi(d))}$$


More generally, let $\Gamma$ be the Eulerian graph on $k$ vertices with distinguished points $p = q = 1$ and the number $\alpha(i, j)$ of edges from vertex $i$ to $j$:

$$\alpha(i, j) = \begin{cases} m & \text{if } j = i + 1 \text{ and } 1 \leq i \leq k - 1 \\ m & \text{if } i = k \text{ and } j = 1 \\ 0 & \text{otherwise} \end{cases}$$

Now clearly $d = km$ and

$$\Pi(\Gamma) = \text{sym}(m) \times \cdots \times \text{sym}(m),$$

$k$ times. On putting $\pi = \pi_1 \times \cdots \times \pi_k$ and labelling the indeterminates, corresponding to the edges from $i$ to $i+1$ by $x_{u(\pi(i))}^{(1)}, \ldots, x_{u(\pi(i))}^{(\pi(i))}$, from the corollary 2 it follows that

$$f_r(X) = \sum_{\pi \in \Pi(\Gamma)} \text{sgn}(\pi) \left(\prod_{r=1}^{m} x_{u(\pi(r))}^{(1)} \cdots x_{u(\pi(r))}^{(\pi(r))}\right) = 0$$

is a polynomial identity on $M_n(C)$ [3] if $km = d \geq 2kn$, i.e., if $m \geq 2n$.

2) For $\pi \in \Pi(\Gamma)$ we define a sequence $g(1), g(2), \ldots, g(d+1)$ of staircase steps, and the staircase height $g(\pi) = \max\{g(1), g(2), \ldots, g(d+1)\}$. We will construct a substitution $T$, such that $\pi$ lifts to the unique converging directed path of $\overline{\Gamma_s}$ (i.e., $\Pi(\overline{\Gamma_s}) = \{\pi\}$). First define a function $\pi^*: \{1, 2, \ldots, d+1\} \to A$ by

$$\pi^*(1) = \sigma(e_{u(1)}); \ \pi^*(r) = \sigma(e_{u(r)}) = \tau(e_{u(r-1)}); \ 2 \leq r \leq d; \ \pi^*(d+1) = \tau(e_{u(1)}).$$

Next we define by recursion the sequence of pair $(g(r), w_i)$, $1 \leq r \leq d+1$, where $g(r)$ is a natural number and $w_i$ is a subset of $\{1, 2, \ldots, d+1\}$. We put $g(1) = 1$ and $w_1 = \emptyset$. Having $(g(1), w_1), \ldots, (g(R), w_{R})$ in hand ($1 \leq r \leq d$). There are three cases to consider: a) $\pi^*(r+1) \neq \pi^*(r)$, $1 \leq r \leq d - 1$; b) $\pi^*(r+1) = \pi^*(r)$, $\pi^*(r+1) \neq \pi^*(r)$, $1 \leq r \leq d - t$, $t \leq w_i$; c) $\pi^*(r+1) = \pi^*(r)$, $\pi^*(r+1) \neq \pi^*(r)$, $1 \leq r \leq d - t$, $t \leq w_i$.

We now put
\[ g(r+1) = \begin{cases} 1 & \text{in case (1)} \\ g(t)+1 & \text{in case (2)} \\ g(i) & \text{in case (3)} \end{cases} \]

and

\[ w_{r+1} = \begin{cases} w_r & \text{in case (1) and (2)} \\ w_r \cup \{t, t+1, \ldots, r, r+1\} & \text{in case (3)} \end{cases} \]

Let \( n \geq g(r) \) for all \( 1 \leq r \leq d+1 \), it is clear that \( x^r = E_{g(r)(g(r) + 1)} \) gives a substitution of \( n \times n \) matrix units over \( C \). Now

\[
\begin{align*}
(p^{e(1)}, g(1)) & \xrightarrow{\pi_{e(1)}} (p^{e(2)}, g(2)) \rightarrow \cdots \\
(p^{e(d)}, g(d)) & \xrightarrow{\pi_{e(d)}} (p^{e(d+1)}, g(d+1))
\end{align*}
\]

is the unique covering directed path of \( \Gamma_s \) from \( (p^{e(1)}, g(1)) \) to \( (p^{e(d+1)}, g(d+1)) \) [4-7]. Since the \((g(1), g(d+1))\) entry of the \( n \times n \) matrix \( f_e(X^T) \) is \( sgn(\pi) \), we have

**Theorem 3.** Let \( \Gamma \) be an Eulerian graph and \( \pi \in \Pi(\Gamma) \). If \( n \geq g(\pi) \), then \( f_{\pi}(X) = 0 \) is not a polynomial identity on the ring \( M_n(C) \) of \( n \times n \) matrices over a commutative ring \( C \) with 1.

Remark. It is an obvious consequence of the above theorem that if \( n \geq \min \{ g(\pi) \mid \pi \in \Pi(\Gamma) \} \) is not the least integer \( n \geq 1 \) for which \( f_{\pi}(X) = 0 \) is not a polynomial identity on \( M_n(C) \).

We note that, in general \( \min \{ g(\pi) \mid \pi \in \Pi(\Gamma) \} \) is not the least integer \( n \geq 1 \) for which \( f_{\pi}(X) = 0 \) is not a polynomial identity on \( M_n(C) \).

Let \( \Gamma \) be the Eulerian graph on one vertex \( d \) loops. It is easily seen that

\[
g(1) = g(2) = 1, g(3) = g(4) = 2, \ldots, g(2s+1) = g(2s+2) = s+1, \ldots
\]

Thus \( g(\pi) = [d/2] + 1 \) for all \( \pi \in \text{sym}(d) \) and the minimality assertion of the Amitsur-Levitzki theorem follows; the main part is an immediate consequence of the corollary.

Let \( \Gamma \) be the Eulerian graph on \( k \) vertices with distinguished points \( p = q = 1 \) and the number \( \alpha(i,j) \) of edges from vertex \( i \) to \( j \):

\[
\alpha(i,j) = \begin{cases} m & \text{if } j = i+1 \text{ and } 1 \leq i \leq k-1 \\ m & \text{if } i = k \text{ and } j = 1 \\ 0 & \text{otherwise} \end{cases}
\]

Analogously, for any \( \pi \in \Pi(\Gamma) \) we have

\[
\begin{align*}
g(1) &= g(2) = \cdots = g(k+1) = 1, \\
g(k+2) &= g(k+3) = \cdots = g(2k+2) = 2, \\
\cdots &= g(s(k+1)+1) = g(s(k+1)+2) = \\
&\cdots = g((s+1)(k+1)) = s+1.
\end{align*}
\]

In consequence \( g(\pi) = m - \left( (m-1)/(k+1) \right) \) for all \( \pi \in \Pi(\Gamma) \).

For \( k = 2 \) we get the double Capelli polynomial; it is known, however, that in this case \( m - \left( (m-1)/3 \right) \) is not the smallest \( n \) for which \( f_{\pi}(X) = 0 \) is not a polynomial identity on \( M_n(C) \).

When \( k = 3 \) we use \( x, y, \) and \( z \) instead of the symbols \( x^{(1)}, x^{(2)}, \) and \( x^{(3)} \) respectively to denote the indeterminates of the triple Capelli polynomial and continue to write \( m \) for the number of edges from vertex \( i \) to \( i+1 \). Thus the triple Capelli polynomial is

\[
C_M(X, Y, Z) = \sum_{\pi \in \Pi(\Gamma)} sgn(\pi)x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}x_{\pi(4)}x_{\pi(5)}
\]

**REFERENCES**


