On Certain Connected Resolving Parameters of Hypercube Networks*

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ABSTRACT
Given a graph \( G = (V, E) \), a set \( W \subseteq V \) is a resolving set if for each pair of distinct vertices \( u, v \in V(G) \) there is a vertex \( w \in W \) such that \( d(u, w) \neq d(v, w) \). A resolving set containing a minimum number of vertices is called a minimum resolving set or a basis for \( G \). The cardinality of a minimum resolving set is called the resolving number or dimension of \( G \) and is denoted by \( \dim(G) \). A resolving set \( W \) is said to be a star resolving set if it induces a star, and a path resolving set if it induces a path. The minimum cardinality of these sets, denoted respectively by \( \text{sr}(G) \) and \( \text{pr}(G) \), are called the star resolving number and path resolving number. In this paper we investigate these resolving parameters for the hypercube networks.

Keywords: Resolving Set; Basis; Path Resolving Set; Star Resolving Set; Hypercube Network

1. Introduction
A query at a vertex \( v \) discovers or verifies all edges and non-edges whose endpoints have different distance from \( v \). In the network verification problem [1], the graph is known in advance and the goal is to compute a minimum number of queries that verify all edges and non-edges. This problem has previously been studied as the problem of placing landmarks in graphs or determining the metric dimension of a graph [2]. Thus, a graph-theoretic interpretation of this problem is to provide representations for the vertices of a graph in such a way that distinct vertices have distinct representations. This is the subject of the papers [3-5].

For an ordered set \( W = \{w_1, w_2, \ldots, w_k\} \) of vertices and a vertex \( v \) in a connected graph \( G \), the code or representation of \( v \) with respect to \( W \) is the \( k \)-vector
\[
C_W(v) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))
\]
where \( d(x, y) \) is the distance between the vertices \( x \) and \( y \). The set \( W \) is a resolving set for \( G \) if distinct vertices of \( G \) have distinct codes with respect to \( W \). Equivalently, for each pair of distinct vertices \( u, v \in V(G) \) there is a vertex \( w \in W \) such that \( d(u, w) \neq d(v, w) \). The minimum cardinality of a resolving set for \( G \) is called the resolving number or dimension and is denoted by \( \dim(G) \).

2. An Overview of the Paper
The concept of resolvability in graphs has previously appeared in literature. Slater [4,5] introduced this concept, under the name locating sets, motivated by its application to the placement of a minimum number of sonar detecting devices in a network so that the position of every vertex in the network can be uniquely determined in terms of its distance from the set of devices. He referred to a minimum resolving set as a reference set and called the cardinality of a minimum resolving set as the location number. Independently, Harary and Melter [3] discovered this concept, but used the term metric dimension, rather than location number. Later, Khuller et al. [2] also discovered these concepts independently and used the term metric dimension. These concepts were rediscovered by Chartrand et al. [6] and also by Johnson [7] while attempting to develop a capability of large datasets of chemical graphs.

It was noted in [8] that determining the metric dimension of a graph is \( NP \)-complete. It has been proved that the metric dimension problem is \( NP \)-hard [2] for general graphs. Manuel et al. [9] have shown that the problem remains \( NP \)-complete for bipartite graphs. There are many applications of resolving sets to problems of...
network discovery and verification [1], pattern recognition, image processing and robot navigation [2], geometrical routing protocols [10], connected joins in graphs [11] and coin weighing problems [12]. This problem has been studied for trees, multi-dimensional grids [2], Petersen graphs [13], torus networks [14], Benes networks [9], honeycomb networks [15], enhanced hypercubes [16] and Illiac networks [17].

Many resolving parameters are formed by combining resolving property with another common graph-theoretic property such as being connected, independent, or acyclic. The generic nature of conditional resolvability in graphs provides various ways of defining new resolving parameters by considering different conditions. In general, a connected graph \( G \) can have many resolving sets. It is interesting to study those resolving set whose vertices are located close to one another. A resolving set \( W \) of \( G \) is connected if the subgraph induced by \( W \) is a nontrivial connected subgraph of \( G \). The minimum cardinality of a connected resolving set is called connected resolving number and it is denoted by \( cr(G) \) [18]. In this paper we introduce a new resolving parameter called star resolving number. A resolving set \( W \) is said to be a star resolving set if the subgraph induced by \( W \) is a star and a path resolving set [19] if \( W \) induces a path. In this paper we show the existence of star and path resolving sets in hypercube networks.

3. Topological Properties of Hypercube Networks

The hypercube is a very popular, versatile and vertex-transitive interconnection network. When the dimension of hypercube increases, the cardinality of its vertex set increases exponentially. The effectiveness of parallel computers is often determined by its communication network. The interconnection network is an important component of a parallel processing system. A good interconnection network should have less topological network cost and meanwhile keep the network diameter as shorter as possible [20].

**Definition 3.1.** Let \( Q \) denote the graph of \( r \)-dimensional hypercube, \( r \geq 1 \). The vertex set \( V(Q^r) = \{(x_0,x_1,\ldots,x_{r-1}):x_i = 0 \text{ or } 1\} \). Two vertices \((x_0,x_1,\ldots,x_{r-1})\) and \((y_0,y_1,\ldots,y_{r-1})\) are adjacent if and only if they differ exactly in one position. See Figure 1.

The hypercube \( Q^r \) has \( 2^r \) vertices and \( r2^r \) edges. It is \( r \)-regular and its diameter is \( r \). Further it is bipartite, Hamiltonian if \( r \neq 1 \) and Eulerian if \( r \) is even [21]. It has been proved in [22] that \( dim(Q^r) \leq r \). The bound is tight for \( r \leq 4 \), and it is not tight for \( r = 5 \). A laborious calculation verifies that \( Q^4 \) is resolved by the 4-vertex set \{0000, 0011, 00101, 01001\}. Caceres et al. [22] have determined \( dim(Q^r) \) for small values of \( r \) by computer search; the values are shown in the following table:

\[
\begin{array}{cccccccccc}
  r & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
  dim(Q^r) & 2 & 3 & 4 & 4 & 5 & 6 & 7 & 7 & 7 \\
\end{array}
\]

4. Star Resolving Number

We begin this section by defining a star and a path resolving set.

**Definition 4.1.** An \( r \)-dimensional star, denoted by \( S_r \), is a graph with one vertex of degree \( r - 1 \) and \( r - 1 \) vertices of degree 1. The vertex of degree \( r - 1 \) is called the hub of \( S_r \).

**Definition 4.2.** A set \( W \subseteq V \) is said to be a star resolving set if \( W \) resolves \( G \) and it induces a star. The minimum cardinality of \( W \) is called the star resolving number and is denoted by \( sr(G) \).

**Remark 1.** It is clear that \( 1 \leq sr(G) \leq \Delta(G)+1 \) for any graph \( G \). In a star resolving set the maximum distance between any two locations (vertices) is 2.

We now proceed to identify a star resolving set in a hypercube network \( Q^r \). It is clear that there are four copies of \( Q^{r-2} \) in \( Q^r \). We denote them as \( Q_0^{r-2} \), \( Q_1^{r-2} \), \( Q_2^{r-2} \), and \( Q_3^{r-2} \). Figure 2 exhibits the four copies of \( Q^2 \) in \( Q^5 \).

Let \( x \in V(Q_0^{r-2}) \). A vertex \( x' \in V(Q_1^{r-2}) \) or \( V(Q_2^{r-2}) \) is called the image of \( x \) if \( d(x,x') = 1 \). Note that vertices in \( Q_0^{r-2} \), at distance 1 from \( x \) are not considered as images of \( x \). If \( x' \) is the image of \( x \) in \( Q_3^{r-2} \) then \( x \) is called the pre-image of \( x' \).

The next result which we state as Lemma 1 is crucial to our work. We omit the proof as this result has been proved in [16] for enhanced hypercubes.

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Figure 1. (a) Binary representation; (b) Decimal representation.

Figure 2. The hypercube with dimension 5.
Lemma 4.1. Let \( x \in V(Q_0^{r-2}) \) and let \( x' \in V(Q_1^{r-2}) \) be the image of \( x \). Let \( w \) be any vertex in \( Q_0^{r-2} \). Then \( d(x', w) = 1 + d(x, w) \).

Lemma 4.2. Let \( x \in V(Q_0^{r-2}) \). Let \( x'_1 \in V(Q_1^{r-2}) \) and \( x'_2 \in V(Q_1^{r-2}) \) be the images of \( x \). Then \( x'_1 \) and \( x'_2 \) are equidistant from every vertex of \( Q_0^{r-2} \).

Proof. Since the shortest paths from \( x'_1 \) and \( x'_2 \) to any vertex of \( Q_0^{r-2} \) pass through \( x \), the conclusion follows.

Lemma 4.3. Let \( G = Q^r \), \( r \geq 1 \). Then \( sr(G) \geq r \).

Proof. The subcube \( Q_0^{r-2} \) of \( Q^r \) is \( r-2 \)-regular and hence it contains \( S_{r-1} \). Now there exist vertices \( x \in Q_1^{r-2} \) and \( y \in Q_2^{r-2} \) such that \( x, y \) are equidistant from every vertex of \( Q_0^{r-2} \) and in particular from every vertex of \( S_{r-1} \). This implies \( sr(G) \geq r \).

Lemma 4.4. Let \( G = Q^r \). Then \( sr(G) \leq r \).

Proof. We prove the theorem by induction on \( r \).

Base Case: Let \( G = Q^3 \) and \( W = \{w_0, w_1, w_2\} \), where \( w_0 = 0, w_1 = 1 \) and \( w_2 = 2 \). It follows from the definition of hypercube edges that \( w_0 \) is adjacent to both \( w_1 \) and \( w_2 \). It is easy to check that \( W \) is a resolving set for \( Q^3 \). Figure 2 shows the distinct codes of vertices in \( Q^3 \), with respect to \( W = \{w_0, w_1, w_2\} \). Since \( W \) induces \( S_1 \), it is a resolving set for \( G^3 \).

Now assume that the result is true for the hypercube \( Q^k \). Let \( W_k = \{w_0\} \cup \{w_i : 1 \leq i \leq r-2\} \), where \( w_j = 2^{-j} \), be a star resolving set for \( Q^{r-1} \). Here \( w_0 \) is the hub and it is adjacent to all \( w_i, 1 \leq i \leq r-2 \). Moreover \( W_k \subset V(Q_0^{r-2}) \). Divide \( Q^r \) into four copies \( Q_0^{r-2}, Q_1^{r-2}, Q_2^{r-2}, Q_3^{r-2} \). There exist vertices \( x \in Q_1^{r-2} \) and \( y \in Q_2^{r-2} \) having the same codes with respect to every vertex of \( Q_0^{r-2} \) and in particular with respect to every vertex of \( W_k \). Hence \( W_k \) cannot resolve \( x \) and \( y \). We exhibit a resolving set for \( Q^r \). Define \( W = W_k \cup \{w_{r-1}\} \) where \( w_{r-1} = 2^{-r-2} \) is a vertex.

Figure 3. (a) Resolving set \( W_1 \) in \( Q_3 \); (b) Codes of vertices of \( Q_3 \) with respect to \( W_1 \).

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Figure 2. Four copies of \( Q_3 \) in \( Q_1 \).
Let \( x' \in V(Q^{-2}_6) \) and \( y' \in V(Q^{-2}_3) \) be the images of \( x \) and \( y \) respectively. Since \( Q^{-2}_6 \cup Q^{-2}_3 \) is resolved by \( W \), there exist a \( w \in W \) such that \( d(x', w) \neq d(y', w) \). This implies that \( d(x, w) \neq d(y, w) \).

Lemma 3 and 4 imply the following result.

**Theorem 4.1.** Let \( G = Q', r \geq 1 \). Then \( sr(G) = r \).

5. Path Resolving Number

In this section we determine a path resolving number for hypercube networks.

**Definition 5.1.** [19] A resolving set \( W \) of \( G \) is a path resolving set for \( G \) if the graph induced by \( W \) is a path. The minimum cardinality of \( W \) is called path resolving number and is denoted by \( pr(G) \).

**Lemma 5.1.** Let \( G = Q', r \geq 1 \). Then \( pr(G) \geq r \).

**Proof.** Let \( P \) be the path in \( Q^{-2}_r \). Now \( P \) cannot resolve \( Q' \) as there are vertices \( x \in Q^{-2}_r \) and \( y \in Q^{-2}_r \) such that they are equidistant from every vertex of \( Q^{-2}_r \) in particular from every vertex of \( P \) since there exists a path \( P \) in \( Q^{-2}_r \), \( pr(G) \geq r \).

**Lemma 5.2.** Let \( G = Q', r \geq 1 \). Then \( pr(G) \leq r \).

**Proof.** Proceeding as in Lemma 4 we conclude that \( W = \{2^{-1}, 1, 1 \leq i \leq r \} \) is a path resolving set for \( Q' \).

Lemma 5 and Lemma 6 imply the following result.

**Theorem 5.1.** Let \( G = Q', r \geq 1 \). Then \( pr(G) = r \).

6. Conclusion

In this paper we have introduced a new resolving parameter called a star resolving number. We have determined the star resolving number and path resolving number for hypercube networks. The problem is open for architectures like Benes and Butterfly networks.

**REFERENCES**


