Dynamics of a Discrete Predator-Prey System with Beddington-DeAngelis Function Response

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Received February 7, 2012; revised March 9, 2012; accepted March 16, 2012

ABSTRACT
This paper discusses the dynamic behaviors of a discrete predator-prey system with Beddington-DeAngelis function response. We first show that under some suitable assumption, the system is permanent. Furthermore, by constructing a suitable Lyapunov function, a sufficient condition which guarantee the global attractivity of positive solutions of the system is established.

Keywords: Discrete; Beddington-DeAngelis Functional Response; Permanence; Global Attractivity

1. Introduction
Since the end of the 19th century, many biological models have been established to illustrate the evolutionary of species, among them, predator-prey models attracted more and more attention of biologists and mathematicians. There are many different kinds of predator-prey models in the literature. In 1975, Beddington [1] and DeAngelis [2] proposed the predator-prey system with the Beddington-DeAngelis functional response as follows

$$\left\{ \begin{array}{l}
    x' = x \left( a - bx - \frac{cy}{m_1 + m_2 x + m_3 y} \right), \\
    y' = y \left( -d + \frac{fx}{m_1 + m_2 x + m_3 y} \right).
\end{array} \right. \quad (1.1)$$

Recently, Li and Takeuchi [3] proposed the following model with both Beddington-DeAngelis functional response and density dependent predator

$$\left\{ \begin{array}{l}
    x' = x \left( a - bx - \frac{cy}{m_1 + m_2 x + m_3 y} \right), \\
    y' = y \left( -d - ey + \frac{fx}{m_1 + m_2 x + m_3 y} \right),
\end{array} \right. \quad (1.2)$$

and discussed the dynamic behaviors of the model.

On the other hand, when the size of the population is rarely small or the population has non-overlapping generation, the discrete time models are more appropriate than the continuous ones. Discrete time models can also provide efficient computational models of continuous models for numerical simulations.

In [4], Qin and Liu studied the dynamic behavior of the following discrete time competitive system

$$\left\{ \begin{array}{l}
    x(n+1) = x(n) \exp \left( a(n) - b(n)x(n) - \frac{c(n)y(n)}{y(n)+1} \right), \\
    y(n+1) = y(n) \exp \left( -d(n) - e(n)y(n) + \frac{f(n)x(n)}{1+x(n)} \right),
\end{array} \right. \quad (1.3)$$

In [5], Wu and Li considered the following discrete time predator-prey system with hassell-varley type functional response

$$\left\{ \begin{array}{l}
    x(n+1) = x(n) \exp \left( a(n) - b(n)x(n) - \frac{c(n)y(n)}{m(n)y'(n)+x(n)} \right), \\
    y(n+1) = y(n) \exp \left( -d(n) + \frac{f(n)x(n)}{m(n)y'(n)+x(n)} \right),
\end{array} \right. \quad (1.4)$$

some sufficient conditions for the permanence and global attractivity of system (1.4) are obtained. For more work on this direction, one could refer to [6-14].

Based on the above discussion, in this paper, we consider the discrete analogous of (1.2), one can easily derive the discrete analogue of system (1.2), which takes the form of

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In this paper, we always assume that \( \{a(n)\}, \{b(n)\}, \{c(n)\}, \{d(n)\}, \{e(n)\}, \{f(n)\}, \{m_i(n)\}, \{m_z(n)\}, \{m_i(n)\} \) are all positive bounded sequences and
\[
0 < a' \leq a(n) \leq a^*, \quad 0 < b' \leq b(n) \leq b^*,
\]
\[
0 < c' \leq c(n) \leq c^*, \quad 0 < d' \leq d(n) \leq d^*,
\]
\[
0 < e' \leq e(n) \leq e^*, \quad 0 < f' \leq f(n) \leq f^*,
\]
\[
0 < m_i(n) \leq m_i(n) \leq m_i^*, \quad i = 1, 2, 3.
\]
Here, for any bounded sequence \( \{f(n)\} \),
\[
f'^* = \sup_{n \in N} f(n), \quad f^* = \inf_{n \in N} f(n).
\]
From the view point of biology, we will focus our discussion on the positive solutions of system (1.4). So it is assumed that the initial conditions of (1.4) are of the form
\[
x(0) > 0, \quad y(0) > 0.
\]
It is easily to see that the solutions of (1.4) with the initial condition (1.5) are defined and remain positive for all \( k \in N \).

2. Permanence

**DEFINITION 2.1.** System (1.5) is said to be permanent, if there are positive constants \( r_1, r_2, R_1, R_2 \) such that each positive solution \( (x(n), y(n)) \) of system (1.5) satisfies
\[
r_1 \leq \lim \inf_{n \to \infty} x(n) \leq \lim \sup_{n \to \infty} x(n) \leq R_1,
\]
\[
r_2 \leq \lim \inf_{n \to \infty} y(n) \leq \lim \sup_{n \to \infty} y(n) \leq R_2.
\]

By Lemma 2.1, we obtain
\[
\lim \sup_{n \to \infty} x(n) \leq G_1 = \left( \exp(a^* - 1) \right) / b'.
\]
Similarly, from the second equation of (1.5), it follows that
\[
\lim \inf_{n \to \infty} y(n) \geq G_2 = \frac{1}{e} \exp \left( \frac{f^*}{m_z^*} - d' - e^* \right).
\]

**LEMMA 2.1.** [6] Assume that \( \{x(n)\} \) satisfies \( x(n) > 0 \) and
\[
x(n+1) \leq x(n) \exp \left\{ a(n) - b(n) x(n) \right\}
\]
for all \( n \geq n_0 \), where \( \{a(n)\}, \{b(n)\} \) are positive sequences. Then
\[
\lim \sup_{n \to \infty} x(n) \leq \frac{\exp(a^* - 1)}{b'}.
\]

**LEMMA 2.2.** [6] Assume that \( \{x(n)\} \) satisfies
\[
x(n+1) \geq x(n) \exp \left\{ a(n) - b(n) x(n) \right\}, n \geq n_0
\]
\[
\lim \sup_{n \to \infty} x(n) \leq D_i = \frac{\exp(a^* - 1)}{b'}.
\]

**LEMMA 2.3.** Assume that \( \frac{f'}{m_z} - d'^* > 0 \) holds, then for any positive solution \( (x(n), y(n)) \) of system (1.4), one has
\[
\lim \inf_{n \to \infty} x(n) \geq \frac{a^*}{b'} \exp \left( \frac{f^*}{m_z^*} - d'^* - 1 \right).
\]

**Proof.** Let \( (x(n), y(n)) \) be any positive solution of system (1.5), from the first equation of (1.5), it follows that
\[
x(n+1) = x(n) \exp \left\{ a(n) - b(n) x(n) \right\} \frac{c(n)}{m_i(n) + m_z(n) x(n) + m_i(n) y(n)}
\]
\[
\leq x(n) \exp \left\{ (a(n) - b(n) x(n)) \right\}.
\]

Similarly, from the second equation of (1.5), it follows that
\[
y(n+1) = y(n) \exp \left\{ -d(n) - e(n) y(n) \frac{f(n)}{m_i(n) + m_z(n) x(n) + m_i(n) y(n)} \right\}
\]
\[
\leq y(n) \exp \left\{ \frac{f(n)}{m_i(n)} - d(n) - e(n) y(n) \right\} \leq y(n) \exp \left\{ \frac{f^*}{m_z^*} - d' - e^* y(n) \right\}.
\]
Under the assumption \( \frac{f'}{m^2} - d^* > 0 \), by Lemma 2.1, we obtain
\[
\limsup_{n \to \infty} y(n) \leq G_2 = \frac{1}{e} \exp \left( \frac{f}{m^2} - d^* - 1 \right).
\]
This completes the proof of Lemma 2.3.

**Lemma 2.4.** Assume that \( h_1 > 0, h_2 > 0 \). Then for any positive solution \( (x(n), y(n)) \) of system (1.5), one has
\[
\liminf_{n \to \infty} x(n) \geq g_1, \limsup_{n \to \infty} y(n) \geq g_2,
\]
where
\[
h_1 = a' - e^* / m_1', \quad h_2 = -d^* + f' g_1 / \left( m_1'' + m_2'' G_1 + m_3'' G_2 \right), \quad g_1 = \frac{h_1 \exp(h_1 - b^* G_1)}{b^*}, \quad g_2 = \frac{h_2 \exp(h_2 - e^* G_2)}{e^*}.
\]

**Proof.** Let \( (x(n), y(n)) \) be any positive solution of system (1.5), from the first equation of (1.5), it follows that
\[
x(n+1) = x(n) \exp \left[ a(n) - b(n) x(n) - \frac{c(n) y(n)}{m_1(n) + m_2(n) x(n) + m_3(n) y(n)} \right] \geq x(n) \exp \left[ a(n) - \frac{c(n)}{m_3(n)} - b(n) x(n) \right] \geq x(n) \exp \left[ a' - \frac{e^*}{m_3} - b^* x(n) \right].
\]
Under the assumption \( h_1 > 0 \), by Lemma 2.2 and Lemma 2.3, we obtain
\[
\liminf_{n \to \infty} x(n) \geq g_1 = \frac{h_1 \exp(h_1 - b^* G_1)}{b^*}.
\]
Similarly, from the second equation of (1.5) and Lemma 2.3, it follows that
\[
y(n+1) = y(n) \exp \left[ -d(n) - e(n) y(n) + \frac{f(n) x(n)}{m_1(n) + m_2(n) x(n) + m_3(n) y(n)} \right] \geq y(n) \exp \left[ -d^* - e^* y(n) + \frac{f' g_1}{m_1'' + m_2'' G_1 + m_3'' G_2} \right] = y(n) \exp \left[ h_2 - e^* y(n) \right].
\]

By Lemma 2.2 and Lemma 2.3, we have
\[
\liminf_{n \to \infty} y(n) \geq g_2 = \frac{h_2 \exp(h_2 - e^* G_2)}{e^*}.
\]
From Lemma 2.3 and Lemma 2.4, we obtain the following theorem.

**Theorem 2.1.** Assume that
\[
-d^* + \frac{f'}{m^2} > 0, \quad a' - e^* / m_1' > 0 \quad (2.1)
\]
\[
\frac{f' g_1}{m_1'' + m_2'' G_1 + m_3'' G_2} - d^* > 0 \quad (2.2)
\]
hold, then system (1.5) is permanent.

This section devotes to study the global attractivity of the positive solution of system (1.5).

**Definition 3.1.** A positive solution \( (x^*(n), y^*(n)) \) of system (1.5) is said to be globally attractive if each other positive solution \( (x(n), y(n)) \) of (1.5) satisfies
\[
\lim_{n \to \infty} |x(n) - x^*(n)| = 0, \quad \lim_{n \to \infty} |y(n) - y^*(n)| = 0.
\]

**Theorem 3.1.** In addition to (2.1) and (2.2), assume further that there exist positive constants \( \alpha, \beta \) and \( \delta \) such that
\[
\alpha \min \left\{ b', \frac{2}{G_1}, \frac{2}{G_2} \right\} - \frac{\alpha c^* G_3^{2/3} \left( m_1^* \right)^{1/3}}{9 \left( m_1^* m_2^* G_1 \right)^{2/3} G_3^{1/3}} - \frac{\beta f^* \left( m_1'' \right)^{1/3}}{9 \left( m_1'' m_2'' G_1 G_2 \right)^{2/3} G_3^{1/3}} \beta f^* G_2^{2/3} \left( m_1^{*2/3} \right) > \delta \quad (3.1)
\]
and

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Then the positive solution of system (1.5) is globally attractive.

**Proof.** From (3.1) and (3.2), there exists an enough small positive constant \( \varepsilon < \min \{ g_1/2, g_2/2 \} \) such that

\[
\alpha \min \left\{ b', \frac{2}{G_1 + e} - b^* \right\} - \frac{\alpha e^*(m_i^*)^{\alpha/3}}{9(m_i^*)^{2/3} g_i^{1/3}} - \frac{\alpha c^*(m_i^*)^{\alpha/3}}{9(m_i^*)^{2/3} g_i^{1/3}} > \delta.
\]  

(3.2)

and

\[
\beta \min \left\{ c^*, \frac{2}{G_1 + e} - c^* \right\} - \frac{\beta f^*(m_i^*)^{\beta/3}}{9(m_i^*)^{2/3} g_i^{1/3}} > \delta.
\]  

(3.3)

For any positive solutions \( (x_1(k), y_1(k)) \) and \( (x_2(k), y_2(k)) \) of system (1.4), it follows from Lemma 2.3 and Lemma 2.4 that

\[
g_1 - \varepsilon \leq x_i(k) \leq G_1 + \varepsilon \\
g_2 - \varepsilon \leq y_i(k) \leq G_2 + \varepsilon \quad (i = 1, 2)
\]  

(3.6)

Let

\[ V_i(k) = \ln x_i(k) - \ln x_i(k) \]

(3.5)

and

\[ A = m_1(k) + m_2(k)x_1(k) + m_3(k)y_2(k), \]

\[ B = m_1(k) + m_2(k)x_1(k) + m_3(k)y_1(k). \]

In view of (3.5), for above \( \varepsilon \), there exists an integer \( k_i > 0 \) such that, for all \( k > k_i \),

\[
\Delta V_i(k) = V_i(k + 1) - V_i(k) = \left[ \ln x_i(k + 1) - \ln x_i(k + 1) \right] - \left[ \ln x_i(k) - \ln x_i(k) \right] + c(k) m_1(k) \frac{y_1(k) - y_2(k)}{AB} + c(k) m_2(k) \frac{x_1(k) - y_1(k)}{AB}
\]

By the mean value theorem, we have

\[ x_i(k) - x_2(k) = \exp[\ln x_i(k) - \ln x_2(k)], \]

(3.7)

where \( \xi_1(k) \) lies between \( x_1(k) \) and \( x_1(k) \). It follows from (3.7) that

\[
\Delta V_i(k) \leq - \frac{1}{\xi_1(k)} \left[ 1 - b(k) \right] \left| x_i(k) - x_2(k) \right| + \left( \frac{c(k) m_1(k) y_1(k) - y_2(k)}{9m_1^2(k) m_2^2(k) x_1^3(k) x_2^3(k) y_1^2(k) y_2^2(k)} \right) + \frac{c(k) m_1(k) y_2(k) - y_1(k)}{9m_1^2(k) m_2^2(k) x_1^3(k) x_2^3(k) y_1^2(k) y_2^2(k)}.
\]
and so, for \( k > k_1 \)
\[
\Delta V_1(k) \leq -\min \left\{ b', \frac{2}{G_1 + \varepsilon} - b'' \right\} |y_1(k) - y_2(k)|
\]
\[\quad \quad + \frac{e^\alpha (m_2^e)^{3/2} (G_2 + \varepsilon)^{23/3} (G_2 - \varepsilon)^{23/3}}{9 \left[ m_1^m m_2^m (g_1 - \varepsilon) \right]^{23/3}} |y_1(k) - y_2(k)|
\]
\[\quad \quad + \frac{e^\alpha (m_1^e)^{3/2} |y_2(k) - y_1(k)|}{9 \left[ m_2^m (g_1 - \varepsilon) \right]^{23/3}}
\]
\[\quad \quad + \frac{e^\alpha (m_2^e)^{3/2} (G_2 + \varepsilon)^{23/3}}{9 \left[ m_2^m (g_1 - \varepsilon) \right]^{23/3}} |y_2(k) - y_1(k)|.
\]
(3.8)

Let
\[ V_1(k) = \ln y_1(k) - \ln y_2(k). \]

From the second equation of system (1.5), we have
\[
\Delta V_2(k) = V_2(k + 1) - V_2(k)
\]
\[\quad \quad = \ln y_1(k + 1) - \ln y_2(k + 1) - \ln y_1(k) + \ln y_2(k)
\]
\[\quad \quad \leq \ln y_1(k) - \ln y_2(k) - e(k) \left[ y_3(k) - y_3(k) \right]
\]
\[\quad \quad - \ln y_1(k) - \ln y_2(k)
\]
\[\quad \quad + f(k) \left[ m_3^e (k) x_3(k) \right] y_1(k) - y_1(k) \right]
\]
\[\quad \quad + f(k) \left[ m_3^e (k) x_3(k) \right] x_3(k) - x_3(k) \right] \frac{AB}{AB}
\]
\[\quad \quad + f(k) \left[ m_3^e (k) y_3(k) \right] x_3(k) - x_3(k) \right] \frac{AB}{AB}
\]
\[\quad \quad + f(k) \left[ m_3^e (k) y_3(k) \right] x_3(k) - x_3(k) \right].
\]

By the mean value theorem, we have
\[ y_1(k) - y_2(k) = \exp(\ln y_1(k)) - \exp(\ln y_2(k)) = \xi(k) [\ln y_1(k) - \ln y_2(k)], \]
(3.9)

where \( \xi(k) \) lies between \( y_1(k) \) and \( y_2(k) \). It follows from (3.9) that
\[
\Delta V_2(k) \leq \left[ \frac{1}{\xi(k)} \right] \left[ \frac{1}{\xi(k)} - e(k) \right] |y_1(k) - y_2(k)| + \frac{f^\alpha (m_2^e)^{3/2} (G_2 + \varepsilon)^{23/3} (G_2 - \varepsilon)^{23/3}}{9 \left[ m_3^m m_2^m (g_1 - \varepsilon) \right]^{23/3}} |y_1(k) - y_2(k)|
\]
\[\quad \quad + \frac{f^\alpha (m_2^e)^{3/2} (G_2 + \varepsilon)^{23/3}}{9 \left[ m_2^m (g_1 - \varepsilon) \right]^{23/3}} |y_2(k) - y_2(k)| + \frac{f^\alpha (m_2^e)^{3/2} (G_2 + \varepsilon)^{23/3}}{9 \left[ m_2^m (g_1 - \varepsilon) \right]^{23/3}} |y_2(k) - y_2(k)|.
\]
(3.10)

and so, for \( k > k_1 \)
\[
\Delta V_2(k) \leq -\min \left\{ b', \frac{2}{G_1 + \varepsilon} - b'' \right\} |y_1(k) - y_2(k)| + \frac{f^\alpha (m_2^e)^{3/2} (G_2 + \varepsilon)^{23/3} (G_2 - \varepsilon)^{23/3}}{9 \left[ m_3^m m_2^m (g_1 - \varepsilon) \right]^{23/3}} |y_1(k) - y_2(k)|
\]
\[\quad \quad + \frac{f^\alpha (m_2^e)^{3/2} (G_2 + \varepsilon)^{23/3}}{9 \left[ m_2^m (g_1 - \varepsilon) \right]^{23/3}} |y_2(k) - y_2(k)| + \frac{f^\alpha (m_2^e)^{3/2} (G_2 + \varepsilon)^{23/3}}{9 \left[ m_2^m (g_1 - \varepsilon) \right]^{23/3}} |y_2(k) - y_2(k)|.
\]
(3.10)

Now we define a Lyapunov function as follows:
\[ V(k) = \alpha V_1(k) + \beta V_2(k). \]

Calculating the difference of \( V(k) \) along the solution of system (1.5), for \( k > k_1 \), it follows from (3.8) and (3.10) that
\[
\Delta V(k) = \alpha \Delta V_1(k) + \beta \Delta V_2(k) \leq -\alpha \min \left\{ b', \frac{2}{G_1 + \varepsilon} - b'' \right\} |y_1(k) - y_2(k)| + \frac{\alpha f^\alpha (m_2^e)^{3/2} (G_2 + \varepsilon)^{23/3} (G_2 - \varepsilon)^{23/3}}{9 \left[ m_3^m m_2^m (g_1 - \varepsilon) \right]^{23/3}} |y_1(k) - y_2(k)|
\]
\[\quad \quad - \frac{\alpha f^\alpha (m_2^e)^{3/2} (G_2 + \varepsilon)^{23/3} (m_2^e)^{3/2}}{9 \left[ m_3^m m_2^m (g_1 - \varepsilon) \right]^{23/3}} |y_1(k) - y_2(k)|
\]
\[\quad \quad - \beta \min \left\{ b', \frac{2}{G_1 + \varepsilon} - b'' \right\} |y_1(k) - y_2(k)| + \frac{\beta f^\alpha (G_2 + \varepsilon)^{23/3} (m_2^e)^{3/2}}{9 \left[ m_3^m m_2^m (g_1 - \varepsilon) \right]^{23/3}} (G_2 - \varepsilon)^{23/3} |y_1(k) - y_2(k)|
\]
\[\quad \quad + \frac{\beta f^\alpha (G_2 + \varepsilon)^{23/3}}{9 \left[ m_3^m m_2^m (g_1 - \varepsilon) \right]^{23/3}} (g_2 - \varepsilon)^{23/3} |y_1(k) - y_2(k)|.
\]

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It follows from (3.3) and (3.4) that
\[ \Delta V(k) \leq -\delta \left[ k_1(k) - x_2(k) \right] \left[ y_2(k) - y_1(k) \right]. \]

Summating both sides of the above inequalities from \( k_i \) to \( k \), we have
\[ \sum_{k_i}^k \Delta V(i) \leq -\delta \sum_{k_i}^k \left[ k_1(i) - x_2(i) \right] \left[ y_2(i) - y_1(i) \right]. \]

Which implies
\[ \sum_{k_i}^k \left[ k_1(i) - x_2(i) \right] \left[ y_2(i) - y_1(i) \right] \leq \frac{V(k)}{\delta}. \]

Then
\[ \sum_{k_i}^\infty \left[ k_1(i) - x_2(i) \right] \left[ y_2(i) - y_1(i) \right] < +\infty. \]

Therefore,
\[ \lim_{k \to \infty} \left[ k_1(i) - x_2(i) \right] \left[ y_2(i) - y_1(i) \right] = 0. \]

That is
\[ \lim_{k \to \infty} k_1(k) - x_2(k) = 0, \lim_{k \to \infty} y_1(k) - y_2(k) = 0. \]

This completes the proof of Theorem 3.1.

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