Blow-Up Phenomena for a Class of Parabolic Systems with Time Dependent Coefficients

Lawrence E. Payne\(^1\), Gérard A. Philippin\(^2\)

\(^1\)Department of Mathematics, Cornell University, Ithaca, USA
\(^2\)Département de Mathématiques et de Statistique, Université Laval, Québec City, Canada

Email: gphilip@mat.ulaval.ca

Received February 8, 2012; revised March 7, 2012; accepted March 14, 2012

ABSTRACT

Blow-up phenomena for solutions of some nonlinear parabolic systems with time dependent coefficients are investigated. Both lower and upper bounds for the blow-up time are derived when blow-up occurs.

Keywords: Parabolic Systems; Blow-Up; Sobolev Type Inequality

1. Introduction

It is well known that the solutions of parabolic problems may remain bounded for all time, or may blow-up in finite or infinite time. When blow-up occurs at time \( t^* \), the evaluation of \( t^* \) is of great practical interest.

In a recent paper \([1]\) Payne and Schaefer have investigated the blow-up phenomena of solutions in some parabolic systems of equations under homogeneous Dirichlet boundary conditions. The contribution of this note is to extend their investigations to a class of parabolic systems with time dependent coefficients. The case of a single parabolic equation was investigated recently in \([2]\).

There is an abounding literature dealing with blow-up phenomena of solutions to parabolic partial differential equations. We refer the interested readers to \([3-5]\). A variety of physical, chemical, biological applications are discussed in \([5,6]\). Further references to the field are \([1,7-19]\). In this note we investigate the blow-up phenomena of the solution \((u,v)\) of the following parabolic system

\[
\begin{align*}
  u_t &= \Delta u + k_1(t) f_1(u), \quad x = (x_1,\cdots,x_N) \in \Omega, t \in (0,t^*) \\
  v_t &= \Delta v + k_2(t) f_2(v), \quad x \in \Omega, t \in (0,t^*) \\
  u(x,0) &= u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0, \quad x \in \Omega,
\end{align*}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N, N \geq 2 \). The initial data \((u_0,v_0)\) as well as the data \( k_1(t), k_2(t)\), \( f_1(t), f_2(t)\) are assumed nonnegative, so that the solution \((u,v)\) of (1.1) will be nonnegative by the maximum principle. More specific assumptions on the data will be made later.

In Section 2 we derive conditions on the data of problem (1.1) sufficient to guarantee that blow-up will occur, and derive under these conditions some upper bound for \( t^* \). In Section 3 we derive some lower bounds for the blow-up time \( t^* \) when blow-up occurs. However this section is limited to the case of \( \Omega \) in \( \mathbb{R}^2 \) and in \( \mathbb{R}^3 \) respectively, because our technique makes use of some Sobolev type inequalities available in \( \mathbb{R}^2 \) and in \( \mathbb{R}^3 \) only. For convenience we include the proof of one of these inequalities in Section 4.

2. Conditions for Blow-Up in Finite Time \( t^* \)

Let \( \lambda_1 \) be the first eigenvalue and \( \phi_1 \) be the associated eigenfunction of the Dirichlet-Laplace operator defined as

\[
\Delta \phi_1 + \lambda_1 \phi_1 = 0, \phi_1 > 0, x \in \Omega, \quad \phi_1 = 0, x \in \partial \Omega.
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N, N \geq 2 \). The initial data \((u_0,v_0)\) as well as the data \( k_1(t), k_2(t)\), \( f_1(t), f_2(t)\) are assumed nonnegative, so that the solution \((u,v)\) of (1.1) will be nonnegative by the maximum principle. More specific assumptions on the data will be made later.

In Section 2 we derive conditions on the data of problem (1.1) sufficient to guarantee that blow-up will occur, and derive under these conditions some upper bound for \( t^* \). In Section 3 we derive some lower bounds for the blow-up time \( t^* \) when blow-up occurs. However this section is limited to the case of \( \Omega \) in \( \mathbb{R}^2 \) and in \( \mathbb{R}^3 \) respectively, because our technique makes use of some Sobolev type inequalities available in \( \mathbb{R}^2 \) and in \( \mathbb{R}^3 \) only. For convenience we include the proof of one of these inequalities in Section 4.

2. Conditions for Blow-Up in Finite Time \( t^* \)

Let \( \lambda_1 \) be the first eigenvalue and \( \phi_1 \) be the associated eigenfunction of the Dirichlet-Laplace operator defined as

\[
\Delta \phi_1 + \lambda_1 \phi_1 = 0, \phi_1 > 0, x \in \Omega, \quad \phi_1 = 0, x \in \partial \Omega,
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N, N \geq 2 \). The initial data \((u_0,v_0)\) as well as the data \( k_1(t), k_2(t)\), \( f_1(t), f_2(t)\) are assumed nonnegative, so that the solution \((u,v)\) of (1.1) will be nonnegative by the maximum principle. More specific assumptions on the data will be made later.

Copyright © 2012 SciRes.
We then compute
\[
\psi'(t) = \int_\Omega [\Delta u + k_1 f_1(u)] \phi \, dx \geq -\lambda \psi(t) + k_1(t) \int_\Omega \psi \phi \, dx
\] (2.7)
Making use of Hölder’s inequality, we have
\[
\psi(t) = \int_\Omega \psi \phi \, dx \leq \left( \int_\Omega \psi^p \phi \, dx \right)^{1/p} \left( \int_\Omega \phi \, dx \right)^{1/q}.
\] (2.8)
Combining (2.7) and (2.8), we obtain
\[
\psi'(t) \geq -\lambda \psi(t) + k_1(t) \left( \psi(t) \right)^p.
\] (2.9)
A similar computation leads to
\[
\chi'(t) \geq -\lambda \chi(t) + k_1(t) \left( \chi(t) \right)^p.
\] (2.10)
Adding (2.9) and (2.10), we obtain
\[
\theta'(t) = \psi'(t) + \chi'(t) \geq -\lambda \theta(t) + \left( \theta(t) \right)^p + K \psi^p + \chi^p,
\] (2.11)
where \( K \) is defined in (2.6). We first investigate the particular case \( p = q \). Making use of Hölder’s inequality, we have
\[
\psi^p + \chi^p \geq 2^{1-q} \left( \psi + \chi \right)^p = 2^{1-q} \left( \theta(t) \right)^p.
\] (2.12)
Inserted in (2.11), we obtain the first order differential inequality
\[
\theta'(t) \geq -\lambda \theta + 2^{1-q} K \theta^p, t \in (0, t'),
\] (2.13)
Integrating (2.13) from 0 to \( t' \), we obtain the inequality
\[
\left( \theta(t) \right)^{1-q} \leq e^{(q-1)\lambda t} \left[ \left( \theta(0) \right)^{1-q} - \frac{2^{1-q} \lambda K}{\lambda_1} \right] + \frac{2^{1-q} \lambda K}{\lambda_1},
\] (2.14)
\[
=: e(t).
\]
Suppose that the data satisfy the condition
\[
\theta(0) > 2 \left( \frac{\lambda_1}{K} \right)^{1/q-1}.
\] (2.15)
Then \( e(t) \) vanishes at some time \( t_0 > 0 \), and \( \theta(t) \) must blow up at some time \( t' = t_0 \). We obtain
\[
t' \leq t_0 := \frac{1}{q-1} \log \left( 1 - \frac{2^{q-1} \lambda_1}{K \left( \theta(0) \right)^{1/q-1}} \right).
\] (2.16)
In the general case, we suppose without loss of generality that \( p > q \), and make use of the inequality
\[
\chi^p = \left( c \chi \right)^{q/p} \left( c^{q-p} \right)^{p/q} \leq \frac{q}{p} \chi^q + \frac{p-q}{p} c^{q-p},
\] (2.17)
valid for arbitrary \( c > 0 \). Choosing \( c := \frac{p}{q} \), we obtain
\[
\chi^p \leq \chi^q + Q,
\] (2.18)
with
\[
Q := \frac{p-q}{p} \left( \frac{q}{p} \right)^{p/q} > 0.
\] (2.19)
Inserted in (2.12), we obtain the first order differential inequality
\[
\theta'(t) \geq 2^{1-q} K \theta^q - \lambda \theta - K =: \Theta(\theta).
\] (2.20)
Suppose that the initial data are so large that \( \Theta(\theta(0)) > 0 \). Then \( \theta(t) \) is increasing for \( t \) small. Since \( \Theta(\theta) \) is increasing in \( \theta \) from its negative minimum, it follows then that \( \Theta(\theta(t)) \) is increasing for \( t > 0 \). This shows that \( \theta(t) \) remains positive, so that \( \theta(t) \) blows up at time \( t' \). Integrating (2.20) leads to the following upper bound for \( t' \nabla
t' = \int_0^t \frac{d\theta}{\Theta(\theta)}.
\] (2.21)
These results are summarized in the following.

**Theorem 1**

1) Assume (2.5) with \( p = q > 1 \), (2.6), and (2.15). Then \( \theta(t) \) defined in (2.5) blows up at finite time \( t' \) bounded above by (2.16).

2) Assume (2.5) with \( p > q > 1 \), (2.6), and \( \Theta(\theta(0)) > 0 \) with \( \Theta(\theta) \) defined in (2.20). Then \( \theta(t) \) blows up at finite time \( t' \) bounded above by (2.21).

To conclude this section, we note that if the condition (2.6) is replaced by
\[
\min_{t \geq 1} \left\{ k_1(t), k_2(t) \right\} =: K > 0,
\] (2.22)
then we have to replace the initial data \( \theta(0) \) by \( \theta(\tau) \) in Theorem 1. Clearly we may use a lower bound for \( \theta(\tau) \). For instance we may integrate the differential inequality
\[
\theta' \geq -\lambda \theta
\] (2.23)
that follows from (2.11), leading to the lower bound
\[
\theta(\tau) \geq e^{-\lambda \tau} \theta(0).
\] (2.24)

3. Lower Bounds for \( t' \star \)

In this section we assume that the data \( f_1, f_2, \) satisfy the conditions
\[
0 \leq f_1(s) \leq s^q, p > 1, 0 \leq f_2(s) \leq s^q, q > 1, s > 0,
\] (3.1)
and that the data \( k_1(t), k_2(t) \) are nonnegative for all \( t > 0 \). Moreover the solution is assumed to blow up in the sense that \( \Phi(t) \rightarrow \infty \) as \( t \rightarrow t' \), where \( \Phi(t) \) is defined as
\[
\Phi(t) := M_1^t U(t) + M_2^t V(t),
\] (3.2)
with
\[ U(t) := \int_\Omega u^{q+1} \, dx, \quad M_1 := \int_\Omega u^{q+1} \, dx, \]  
(3.3)
\[ V(t) := \int_\Omega v^{q+1} \, dx, \quad M_2 := \int_\Omega v^{q+1} \, dx. \]  
(3.4)
Differentiating (3.3) and making use of (1.1), (3.1), we obtain
\[ U'(t) \leq 2q \int_\Omega u^{q+1} \left[ \Delta u + k_1(t) \nu^\rho \right] \, dx, \]
\[ = 2qk_1(t) \int_\Omega u^{q+1} \rho \, dx - 2q(2q-1)J(t), \]  
(3.5)
with
\[ J(t) := \int_\Omega u^{2(q-1)} \| \nabla u \|^2 \, dx. \]  
(3.6)
Making use of Schwarz and Hölder’s inequalities we have
\[ \int_\Omega u^{q+1} \rho \, dx \leq \left( \int_\Omega u^{q+1} \, dx \right)^{\frac{q}{q+1}} \left( \int_\Omega \rho^2 \, dx \right)^{\frac{1}{q+1}} \]
(3.7)
\[ \leq \left( \int_\Omega u^{q+1} \, dx \right)^{\frac{q}{q+1}} \left( J(t) \right)^{\frac{1}{q+1}} \left( \int_\Omega \rho^2 \, dx \right)^{\frac{1}{q+1}}. \]  
(3.8)
In \( \mathbb{R}^2 \) we make use of the following Sobolev type inequality
\[ \int_\Omega u^{q+1} \rho \, dx \leq \frac{2^q}{2} \int_\Omega u^{q+1} \| \nabla u \|^2 \int_\Omega u^{q+1} \, dx, \]  
(3.9)
derived in the last section of the paper. Combining (3.7) and (3.8), we obtain
\[ \int_\Omega u^{q+1} \rho \, dx \leq \left( \frac{q}{2} \right)^{\frac{q}{q+1}} \left( J(t) \right)^{\frac{1}{q+1}} \left( \int_\Omega u^{q+1} \, dx \right)^{\frac{q}{q+1}} \]  
(3.10)
\[ \leq \frac{1}{2^q} \left( J(t) \right)^{\frac{1}{q+1}} M_1^{1/2} M_2^{1/2} \Phi(t), \]
where we have used the arithmetic-geometric mean inequality. Making use of the inequality
\[ a^r b^{1-r} \leq ra + (1-r)b, \]
\[ r \in (0, 1), a > 0, b > 0, \]  
(3.11)
we have
\[ \left( J(t) \right)^{\frac{1}{q+1}} \Phi \leq \left( \frac{q-1}{2q} \right)^{\frac{1}{q+1}} \left( \frac{q+1}{2q} \right)^{\frac{1}{q+1}} \left( \frac{q+1}{2q} \right)^{\frac{2q}{q+1}}, \]  
(3.12)
valid for arbitrary \( c > 0 \) to be chosen later. Inserted in (3.9) and (3.5), we obtain
\[ U'(t) \leq \left( \frac{q-1}{2q} \right)^{\frac{1}{q+1}} \left( \frac{q+1}{2q} \right)^{\frac{1}{q+1}} \left( \frac{q+1}{2q} \right)^{\frac{2q}{q+1}} \left( \frac{q+1}{2q} \right)^{\frac{2q}{q+1}}, \]  
(3.13)
where we have used the arithmetic-geometric mean inequality. Making use of (3.4), we obtain
\[ \left( J(t) \right)^{\frac{1}{q+1}} \Phi \leq \left( \frac{q-1}{2q} \right)^{\frac{1}{q+1}} \left( \frac{q+1}{2q} \right)^{\frac{1}{q+1}} \left( \frac{q+1}{2q} \right)^{\frac{2q}{q+1}}, \]  
(3.14)
\[ \leq \left( \frac{q+1}{2q} \right)^{\frac{2q}{q+1}} \left( \frac{q}{2} \right)^{\frac{2q}{q+1}} \left( \frac{q+1}{2q} \right)^{\frac{2q}{q+1}}, \]  
(3.15)
\[ \leq \left( \frac{q+1}{2q} \right)^{\frac{2q}{q+1}} \left( \frac{q}{2} \right)^{\frac{2q}{q+1}} \left( \frac{q+1}{2q} \right)^{\frac{2q}{q+1}}, \]  
(3.16)
\[ \frac{1}{2} \left( \frac{q+1}{2q} \right)^{\frac{2q}{q+1}} \left( \frac{q}{2} \right)^{\frac{2q}{q+1}} \left( \frac{q+1}{2q} \right)^{\frac{2q}{q+1}}, \]  
(3.17)
\[ \frac{1}{2} \left( \frac{q+1}{2q} \right)^{\frac{2q}{q+1}} \left( \frac{q}{2} \right)^{\frac{2q}{q+1}} \left( \frac{q+1}{2q} \right)^{\frac{2q}{q+1}}, \]  
(3.18)
\[ \frac{1}{2} \left( \frac{q+1}{2q} \right)^{\frac{2q}{q+1}} \left( \frac{q}{2} \right)^{\frac{2q}{q+1}} \left( \frac{q+1}{2q} \right)^{\frac{2q}{q+1}}, \]  
(3.19)
\[ \frac{1}{2} \left( \frac{q+1}{2q} \right)^{\frac{2q}{q+1}} \left( \frac{q}{2} \right)^{\frac{2q}{q+1}} \left( \frac{q+1}{2q} \right)^{\frac{2q}{q+1}}, \]  
(3.20)

Moreover we make use of (3.10) to write
\[
\left(M_1^1U(t)\right)^{\frac{q+1}{4q}} J^{\frac{3(q-1)}{4q}}
\]
\[
\leq \left( c^{-1} J + \frac{q+3}{4q} \left(M_1^1 U\right)^{\frac{q+1}{4q}} \right) \left( t \right)
\]
\[
\leq \frac{3(q-1)}{4q} + \frac{q+3}{c^q} \left(M_1^1 U\right)^{\frac{q+1}{q+3}},
\]
with arbitrary \( c > 0 \) to be chosen later. Combining (3.5), (3.19) and (3.21), we obtain
\[
U'(t)
\]
\[
\leq \left[ \frac{3(q-1)}{2} C(q) M_1^{\frac{q+1}{4q}} M_2^{\frac{1}{2q}} \left(M_2^1 V\right)^{\frac{1}{2q}} k_1(t) c^{-1}
\]
\[
-2q(2q-1) \right] J(t)
\]
\[
+ \frac{q+3}{2} C(q) M_1^{\frac{q+1}{4q}} M_2^{\frac{1}{2q}} \left(M_2^1 V\right)^{\frac{1}{2q}} c^{q+3} \left(M_1^1 U\right)^{\frac{q+1}{q+3}} k_1(t).
\]

We now select \( c \) such that the quantity \( \left\{ \right\} \) in (3.22) vanishes. We are then led to the inequality
\[
U'(t) \leq
\]
\[
A(q) M_1^{\frac{q+1}{4q}} M_2^{\frac{2q}{3q+1}} k_1(t) M_1^{\frac{q+1}{3q+1}} M_2^{\frac{1}{3q+1}} \left(M_2^1 V\right)^{\frac{2q}{3q+1}},
\]
with
\[
A(q) := \frac{q+3}{2} \left(C(q)\right)^{\frac{4q}{q+1}} \left(\frac{3(q-1)}{4q} \right)^{\frac{3(q-1)}{q+3}}.
\]

Finally we make use of (3.10) to write
\[
\left(M_1^1 U\right)^{\frac{q+1}{q+3}} \left(M_2^1 V\right)^{\frac{2q}{3q+1}}
\]
\[
= \left(M_1^1 U\right)^{\frac{q+1}{q+3}} \left(M_2^1 V\right)^{\frac{2q}{3q+1}} \left(M_2^1 V\right)^{\frac{2q}{3q+1}}
\]
\[
\leq \left\{ \frac{q+1}{3q+1} c \left(M_1^1 U\right) + \frac{2q}{3q+1} c c^{q+3} \right\}^{\frac{3q+1}{q+3}}
\]
and select \( c \) to satisfy \( (q+1)c = 2q c^{-(q+1)/2q} \), leading to
\[
\left(M_1^1 U\right)^{\frac{q+1}{q+3}} \left(M_2^1 V\right)^{\frac{2q}{3q+1}} \leq \left( \frac{q+1}{3q+1} \right)^{\frac{3q+1}{q+3}} \Phi^{q+3}.
\]

Inserted in (3.23), we obtain
\[
M_1^1 U'(t) \leq \Gamma(q) M_1^{\frac{2q}{q+3}} M_2^{\frac{2q}{q+3}} \left(k_1(t)\right)^{\frac{4q}{q+1}} \Phi^{q+3},
\]
with
\[
\Gamma(q) := \frac{q+3}{2} \left(\frac{3(q-1)}{4q} \right)^{\frac{3(q-1)}{q+3}} \left(\frac{q+1}{3q+1}\right)^{\frac{3q+1}{q+1}}
\]
\[
\times \left(\frac{2q}{q+1} \right)^{\frac{3q+1}{q+3}} \left(C(q)\right)^{\frac{q+1}{q+3}}.
\]

A similar computation leads to
\[
M_1^1 V'(t) \leq \Gamma(p) M_1^{\frac{p+1}{p+3}} M_2^{\frac{2p}{p+3}} \left(k_2(t)\right)^{\frac{4p}{p+1}} \Phi^{p+3}.
\]

If we suppose that
\[
\Phi(t) \to \infty \text{ as } t \to t',
\]
then there exists \( t_0 \geq 0 \) such that \( \Phi(t) \geq 1 \), \( \forall t \geq t_1 \) and we have
\[
\Phi'(t) = M_1^1 U'' + M_1^1 V'' \leq \left\{ \frac{k(t)}{\Phi^{q+3}} \right\} \text{ if } \Omega \subset \mathbb{R}^2
\]
\[
\tilde{k}(t) \Phi^{q+3}, \text{ if } \Omega \subset \mathbb{R}^3
\]
valid for \( t \geq t_0 \), with
\[
\sigma := \max \{ p, q \},
\]
\[
k(t) := F(p) M_1^{\frac{p+1}{p+3}} M_2^{\frac{2p}{p+3}} \left(k_1(t)\right)^{\frac{4p}{p+1}}
\]
\[
+ F(p) M_1^{\frac{p+1}{p+3}} M_2^{\frac{2p}{p+3}} \left(k_2(t)\right)^{\frac{4p}{p+1}},
\]
\[
\tilde{k}(t) := \Gamma(q) M_1^{\frac{q+1}{q+3}} M_2^{\frac{2q}{q+3}} \left(k_1(t)\right)^{\frac{4q}{q+1}}
\]
\[
+ \Gamma(p) M_1^{\frac{q+1}{q+3}} M_2^{\frac{2q}{q+3}} \left(k_2(t)\right)^{\frac{4q}{q+1}}.
\]

Integrating (3.31), we obtain in the two-dimensional case
\[
\frac{\sigma_+ + 1}{\sigma-1} = \int_0^{t'} \Phi^{-2q(\sigma+1)} d\Phi \leq \int_0^{t'} k(t) dt
\]
\[
\leq \int_0^{t'} k(t) dt =: K(t'),
\]
from which we obtain a lower bound for \( t' \) of the form
\[
t' \geq K^{-1} \left(\frac{\sigma_+ + 1}{\sigma-1}\right),
\]
where \( K^{-1} \) is the inverse function of \( K \). In the three-dimensional case, we obtain
\[
\frac{\sigma_+ + 3}{2(\sigma-1)} \leq \int_{t_1}^{t'} \tilde{k}(t) dt \leq \int_{t_0}^{t'} \tilde{k}(t) dt =: \tilde{K}(t'),
\]
from which we obtain a lower bound for \( t' \) of the form
\[
t' \geq \tilde{K}^{-1} \frac{\sigma_+ + 3}{2(\sigma-1)}.
\]

These results are summarized in the following
Theorem 2

Under the assumption (3.30), a lower bound for the blow-up time \( t^* \) of the solution \((u,v)\) of (1.1) is given by (3.36) in the two-dimensional case and by (3.38) in the three-dimensional case.

In the particular case in which \( k_i(t) \) and \( k_z(t) \) are constant, we have

\[
\begin{align*}
t^* & \geq \sigma + 1 \left\{ F(q)M_{t}^{1 - \frac{q}{2}}M_{2}^{\frac{4}{q} - \frac{q}{2}} + F(p)M_{t}^{p - 1}M_{2}^{\frac{2p}{p^2} - \frac{4p}{p^2}} \right\}^{-1} \\
& \quad + \left\{ \Gamma(q)M_{t}^{2 - \frac{4q}{q^2}}M_{2}^{\frac{2q}{q^2} - \frac{4q}{q^2}} + \Gamma(p)M_{t}^{p - 1}M_{2}^{\frac{2p}{p^2} - \frac{4p}{p^2}} \right\}^{-1} 
\end{align*}
\]  

(3.39)

in the two-dimensional case and

\[
\begin{align*}
t^* & \geq \frac{\sigma + 3}{2(\sigma - 1)} \left\{ \Gamma(q)M_{t}^{2 - \frac{4q}{q^2}}M_{2}^{\frac{2q}{q^2} - \frac{4q}{q^2}} + \Gamma(p)M_{t}^{p - 1}M_{2}^{\frac{2p}{p^2} - \frac{4p}{p^2}} \right\}^{-1} \\
& \quad + \left\{ \Gamma(q)M_{t}^{2 - \frac{4q}{q^2}}M_{2}^{\frac{2q}{q^2} - \frac{4q}{q^2}} + \Gamma(p)M_{t}^{p - 1}M_{2}^{\frac{2p}{p^2} - \frac{4p}{p^2}} \right\}^{-1} 
\end{align*}
\]  

(3.40)

in the three-dimensional case.

Theorem 2 could easily be extended to systems of \( n \) parabolic equations of the form

\[
\frac{\partial u_j}{\partial t} = \Delta u_j + k_i(t) f_j(u_j), \ j \neq i = 1, \ldots, n. 
\]

(3.41)

4. Sobolev Type Inequality in \( \mathbb{R}^2 \)

The Sobolev type inequality (3.8) in \( \mathbb{R}^2 \) may be known, but for the convenience of the reader we present a proof here.

**Lemma 1**

Let \( u(x,y) \) be a nonnegative piecewise \( C^1 \)-function defined in a bounded domain \( \Omega \) that vanishes on the boundary \( \partial \Omega \). Let \( q \) be any constant \( \geq 1 \). Then we have the following Sobolev type inequality

\[
\int_\Omega u^{2q} \, dx dy \leq \frac{q^2}{2} \int_\Omega u^{2q(\Omega)} \, dx dy, 
\]

(4.1)

valid for \( \Omega \subset \mathbb{R}^2 \).

For the proof of (4.1), we follow the argument of Payne in [21]. We note that (4.1) is equivalent to

\[
\int_\Omega \tilde{u}^{2q} \, dx dy \leq \frac{q^2}{2} \int_\Omega \tilde{u}^{2q(\Omega)} \, dx dy, 
\]

(4.2)

where \( \tilde{\Omega} \) is the convex hull of \( \Omega \), and \( \tilde{u} := u(x,y) \in \tilde{\Omega}, \tilde{u} = 0, (x,y) \in \tilde{\Omega} \setminus \Omega \). It is therefore sufficient to establish (4.1) for \( \tilde{\Omega} \) convex. For the proof, let \( P := (\bar{x}, \bar{y}) \) be an arbitrary point in \( \tilde{\Omega} \subset \mathbb{R}^2 \). Let \( P := (x_1, y_1) \in \tilde{\Omega}, Q := (x_2, y_2) \in \tilde{\Omega}, k = 1, 2 \) be two pairs of boundary points associated to \( P \) with \( x_1 \leq x_2, y_1 \leq y_2 \). Since \( u \) vanishes on \( \partial \Omega \), we have for any constant \( q \geq 1 \)

\[
u^{2q} (P) = 2q \int_{\Omega} \nu^{2q-1}(u) \, dx = -2q \int_{\Omega} \nu^{2q-1}u \, dx, 
\]

(4.3)

from which we obtain

\[
u^{2q} (P) \leq q \int_{\Omega} \nu^{2q-1} |u| \, dx. 
\]

(4.4)

Similarly we have

\[
u^{2q} (P) \leq q \int_{\Omega} \nu^{2q-1} |u| \, dy. 
\]

(4.5)

Multiplying (4.4) by (4.5) and integrating over \( \Omega \) leads to

\[
\int_{\Omega} u^{2q} \, dx dy \leq q \int_{\Omega} u^{2q-1} |u| \, dx dy \int_{\Omega} u^{2q-1} |u| \, dx dy \\
\leq q \left( \int_{\Omega} u^{2q(|\Omega)|} \, dx dy \int_{\Omega} u^{2q(|\Omega)|} \, dx dy \right)^{1/2} \int_{\Omega} u^{2q} \, dx dy \\
\leq \frac{1}{2} q \int_{\Omega} u^{2q(|\Omega)|} |u| \, dx dy \int_{\Omega} u^{2q} \, dx dy, 
\]

(4.6)

which is the desired inequality (4.1). We note that we have used the Schwarz and the arithmetic-geometric mean inequalities in the two last steps of (4.6).

**REFERENCES**


doi:10.1016/j.jmaa.2007.01.083


