Common Fixed Point Theorem in Intuitionistic Fuzzy Metric Space Using R-Weakly Commuting Mappings

Saurabh Manro¹, Satwinder Singh Bhatia¹, Sanjay Kumar²
¹School of Mathematics and Computer Applications, Thapar University, Patiala, India
²Deenbandhu Chhotu Ram, University of Science and Technology, Murthal, India
Email: sauravmanro@hotmail.com

Received August 22, 2011; revised October 10, 2011; accepted October 18, 2011

ABSTRACT
In this paper, we prove a common fixed point theorem in intuitionistic fuzzy metric space by using pointwise R-weak commutativity and reciprocal continuity of mappings satisfying contractive conditions.

Keywords: Intuitionistic Fuzzy Metric Space; Reciprocal Continuity; R-Weakly Commuting Mappings; Common Fixed Point Theorem

1. Introduction

The aim of this paper is to prove a common fixed point theorem in intuitionistic fuzzy metric space by using pointwise R-weak commutativity [5] and reciprocal continuity [9] of mappings satisfying contractive conditions.

2. Preliminaries
Definition 2.1 [13]. A binary operation \( * : [0,1] \times [0,1] \rightarrow [0,1] \) is continuous t-norm if \( * \) satisfies the following conditions:
1) * is commutative and associative;
2) * is continuous;
3) \( a * 1 = a \) for all \( a \in [0,1] \);
4) \( a * b \leq c * d \) whenever \( a \leq c \) and \( b \leq d \) for all \( a, b, c, d \in [0,1] \).

Definition 2.2 [13]. A binary operation \( \diamond : [0,1] \times [0,1] \rightarrow [0,1] \) is continuous t-conorm if \( \diamond \) satisfies the following conditions:
1) \( \diamond \) is commutative and associative;
2) \( \diamond \) is continuous;
3) \( a \diamond 0 = a \) for all \( a \in [0,1] \);
4) \( a \diamond b \leq c \diamond d \) whenever \( a \leq c \) and \( b \leq d \) for all \( a, b, c, d \in [0,1] \).

Alaca et al. [3] defined the notion of intuitionistic fuzzy metric space as:

Definition 2.3 [3]. A 5-tuple \( (X, M, N, *, \diamond) \) is said to be an intuitionistic fuzzy metric space if \( X \) is an arbitrary set, * is a continuous t-norm, \( \diamond \) is a continuous t-conorm and \( M, N \) are fuzzy sets on \( X \times (0, \infty) \) satisfying the conditions:
1) \( M(x, y, t) + N(x, y, t) \leq 1 \) for all \( x, y \in X \) and \( t > 0 \);
2) \( M(x, y, 0) = 0 \) for all \( x, y \in X \);
3) \( M(x, y, t) = 1 \) for all \( x, y \in X \) and \( t > 0 \) if and only if \( x = y \);
4) \( M(x, y, t) = M(y, x, t) \) for all \( x, y \in X \) and \( t > 0 \);
5) \( M(x, y, t) * M(y, z, s) \leq M(x, z, t + s) \) for all \( x, y, z \in X \) and \( s, t > 0 \);
6) \( M(x, y, .) : [0, \infty) \rightarrow [0,1] \) is left continuous, for all \( x, y \in X \);
7) \( \lim_{t \rightarrow \infty} M(x, y, t) = 1 \) for all \( x, y \in X \) and \( t > 0 \);
8) \( N(x, y, 0) = 1 \) for all \( x, y \in X \);
9) \( N(x, y, t) = 0 \) for all \( x, y \in X \) and \( t > 0 \) if and only if \( x = y \);
10) \( N(x, y, t) = N(y, x, t) \) for all \( x, y \in X \) and \( t > 0 \);
11) \( N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s) \) for all \( x, y, z \in X \) and \( s, t > 0 \);
12) \( N(x, y, .) : [0, \infty) \rightarrow [0,1] \) is right continuous, for
all \( x, y \in X \);

13) \( \lim_{t \to \infty} N(x, y, t) = 0 \) for all \( x, y \in X \).

The functions \( M(x, y, t) \) and \( N(x, y, t) \) denote the degree of nearness and the degree of non-nearness between \( x \) and \( y \) w.r.t. \( t \) respectively.

**Remark 2.1** [12]. Every fuzzy metric space \((X, M, *)\) is an intuitionistic fuzzy metric space of the form \((X, M, 1 - M, *, \emptyset)\) such that \( t \)-norm * and \( t \)-conorm \( \emptyset \) are associated as \( x \odot y = 1 - ((1-x)(1-y)) \) for all \( x, y \in X \).

**Definition 2.2** [12]. In intuitionistic fuzzy metric space \((X, M, N, *, \emptyset)\), \( M(x, y, *) \) is non-decreasing and \( N(x, y, \emptyset) \) is non-increasing for all \( x, y \in X \).

**Definition 2.4** [3]. Let \((X, M, N, *, \emptyset)\) be an intuitionistic fuzzy metric space. Then

1) A sequence \( \{x_n\} \) in \( X \) is said to be a Cauchy sequence if, for all \( t > 0 \) and \( p > 0 \),
\[
\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1
\]
and
\[
\lim_{n \to \infty} N(x_{n+p}, x_n, t) = 0.
\]

2) A sequence \( \{x_n\} \) in \( X \) is said to be convergent to a point \( x \in X \) if, for all \( t > 0 \),
\[
\lim_{n \to \infty} M(x_n, x, t) = 1
\]
and
\[
\lim_{n \to \infty} N(x_n, x, t) = 0.
\]

**Definition 2.5** [3]. An intuitionistic fuzzy metric space \((X, M, N, *, \emptyset)\) is said to be complete if and only if every Cauchy sequence in \( X \) is convergent.

**Example 2.1** [3]. Let \( X = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\} \) and let * be the continuous \( t \)-norm and \( \emptyset \) be the continuous \( t \)-conorm defined by \( a * b = ab \) and \( a \bullet b = \min\{1, a + b\} \) respectively, for all \( a, b \in [0,1] \). For each \( t \in (0, \infty) \) and \( x, y \in X \), define \( M \) and \( N \) by
\[
M(x, y, t) = \begin{cases} 
\frac{t}{t + |x - y|}, & t > 0 \\
0, & t = 0
\end{cases}
\]
and
\[
N(x, y, t) = \begin{cases} 
\frac{|x - y|}{t + |x - y|}, & t > 0 \\
1, & t = 0
\end{cases}
\]
Clearly, \((X, M, N, *, \emptyset)\) is complete intuitionistic fuzzy metric space.

**Definition 2.6** [3]. A pair of self mappings \((A, S)\) of an intuitionistic fuzzy metric space \((X, M, N, *, \emptyset)\) is said to be commuting if \( M(ASx, SAx, t) = 1 \) and \( N(ASx, SAx, t) = 0 \) for all \( x \in X \).

**Definition 2.7** [3]. A pair of self mappings \((A, S)\) of an intuitionistic fuzzy metric space \((X, M, N, *, \emptyset)\) is said to be weakly commuting if
\[
M(ASx, SAx, t) \geq M(Ax, Sx, t) \quad \text{and} \quad N(ASx, SAx, t) \leq N(Ax, Sx, t)
\]
for all \( x \in X \) and \( t > 0 \).

**Definition 2.8** [12]. A pair of self mappings \((A, S)\) of an intuitionistic fuzzy metric space \((X, M, N, *, \emptyset)\) is said to be compatible if \( \lim_{n \to \infty} M(ASx_n, SAx_n, t) = 1 \), and \( \lim_{n \to \infty} N(ASx_n, SAx_n, t) = 0 \) for all \( t > 0 \), whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = u \) for some \( u \in X \).

**Definition 2.9** [5]. A pair of self mappings \((A, S)\) of an intuitionistic fuzzy metric space \((X, M, N, *, \emptyset)\) is said to be pointwise \( R \)-weakly commuting, if given \( x \in X \), there exist \( R > 0 \) such that for all \( t > 0 \)
\[
M(ASx, SAx, t) \geq M(Ax, Sx, \frac{t}{R})
\]
and
\[
N(ASx, SAx, t) \leq N(Ax, Sx, \frac{t}{R})
\]
Clearly, every pair of weakly commuting mappings is pointwise \( R \)-weakly commuting with \( R = 1 \).

**Definition 2.10** [9]. Two mappings \( A \) and \( S \) of an intuitionistic fuzzy metric space \((X, M, N, *, \emptyset)\) are called reciprocally continuous if \( ASu_n \to Az \), \( SAu_n \to Sz \), whenever \( \{u_n\} \) is a sequence such that \( Au_n \to z \), \( Su_n \to z \) for some \( z \) in \( X \).

If \( A \) and \( S \) are both continuous, then they are obviously reciprocally continuous but converse is not true.

### 3. Lemmas

The proof of our result is based upon the following lemmas of which the first two are due to Alaca et al. [12]:

**Lemma 3.1** [12]. Let \( \{u_n\} \) is a sequence in an intuitionistic fuzzy metric space \((X, M, N, *, \emptyset)\). If there exists a constant \( k \in (0,1) \) such that
\[
M(u_n, u_{n+1}, kt) \geq M(u_{n-1}, u_n, t),
\]
\[
N(u_n, u_{n+1}, kt) \leq N(u_{n-1}, u_n, t)
\]
for all \( n = 0, 1, 2, \ldots \). Then \( \{u_n\} \) is a Cauchy sequence in \( X \).

**Lemma 3.2** [12]. Let \((X, M, N, *, \emptyset)\) be intuitionistic fuzzy metric space and for all \( x, y \in X \), \( t > 0 \) and if
for a number \( k \in (0, 1) \), \( M(x, y, t) \geq M(x, y, t) \) and 
\( N(x, y, t) \leq N(x, y, t) \). Then \( x = y \).

**Lemma 3.3.** Let \((X, M, N, *, \odot)\) be a complete intuitionistic fuzzy metric space with continuous \( t\)-norm * and continuous \( t\)-conorm \( \odot \) defined by \( t \ast t \geq t \) and 
\((1-t) \odot (1-t) \leq (1-t) \) for all \( t \in [0, 1] \). Further, let \((A, S) \) and \((B, T) \) be pointwise \( R \)-weakly commuting pairs of self mappings of \( X \) satisfying:

\[ (3.1) \quad A(X) \subseteq T(X), B(X) \subseteq S(X), \]
\[ (3.2) \quad \exists \text{ a constant } k \in (0, 1) \text{ such that } \]
\[ M(Ax, By, kt) \geq M(Ty, By, t) \ast M(Sx, Ax, t) \]
\[ \ast M(Sx, By, \alpha t) \ast M(Ty, Ax, (2-\alpha)t) \ast M(Ty, Sx, t) \]
\[ N(Ax, By, kt) \leq N(Ty, By, t) \circ N(Sx, Ax, t) \]
\[ \circ N(Sx, By, \alpha t) \circ N(Ty, Ax, (2-\alpha)t) \circ N(Ty, Sx, t) \]

for all \( x, y \in X \), \( t > 0 \) and \( \alpha \in (0, 2) \). Then the continuity of one of the mappings in compatible pair \((A, S)\) or \((B, T)\) on \((X, M, N, *, \odot)\) implies their reciprocal continuity.

**Proof.** First, assume that \( A \) and \( S \) are compatible and \( S \) is continuous. We show that \( MSu \rightarrow Sz \) and \( SUu \rightarrow Sz \) for some \( z \in X \) as \( n \rightarrow \infty \).

Since \( S \) is continuous, we have \( SUu \rightarrow Sz \) and \( SSu \rightarrow Sz \) as \( n \rightarrow \infty \) and since \((A, S)\) is compatible, we have:

\[ \lim_{n \rightarrow \infty} M(SSu_n, SAu_n, t) = 1, \lim_{n \rightarrow \infty} N(SSu_n, SAu_n, t) = 0 \]
\[ \Rightarrow \lim_{n \rightarrow \infty} M(SSu_n, Sz, t) = 1, \lim_{n \rightarrow \infty} N(SSu_n, Sz, t) = 0 \]

That is, \( SSu_n \rightarrow Sz \) as \( n \rightarrow \infty \). By (3.1), for each \( n \), there exists \( v_n \in X \) such that \( SSu_n = Tv_n \). Thus, we have \( Ssu_n \rightarrow Sz \), \( SAu_n \rightarrow Sz \), \( SSu_n \rightarrow Sz \) and \( Tu_n \rightarrow Sz \) as \( n \rightarrow \infty \) whenever \( SSu_n = Tv_n \).

Now we claim that \( BV_n \rightarrow Sz \) as \( n \rightarrow \infty \).

Suppose not, then taking \( \alpha = 1 \) in (3.2), we have:

\[ M(SSu_n, BV_n, kt) \geq M(Tv_n, BV_n, t) \ast M(SSu_n, ASu_n, t) \]
\[ \ast M(SSu_n, BV_n, \alpha t) \ast M(Tv_n, ASu_n, (2-\alpha)t) \ast M(Tv_n, SSu_n, t) \]
\[ N(SSu_n, BV_n, kt) \leq N(Tv_n, BV_n, t) \circ N(SSu_n, ASu_n, t) \]
\[ \circ N(SSu_n, BV_n, \alpha t) \circ N(Tv_n, ASu_n, (2-\alpha)t) \circ N(Tv_n, SSu_n, t) \]

Taking \( n \rightarrow \infty \), we get:

\[ M(Sz, BV_n, kt) \geq M(Sz, BV_n, t) \ast M(Sz, Sz, t) \]
\[ \ast M(Sz, BV_n, t) \ast M(Sz, Sz, t) \ast M(Sz, Sz, t) \]

Therefore, by use of Lemma 3.2, we have \( AZ = Sz \). Hence, \( Ssu_n \rightarrow Sz \), \( SAu_n \rightarrow Sz = AZ \) as \( n \rightarrow \infty \).

This proves that \( A \) and \( S \) are reciprocally continuous on \( X \). Similarly, it can be proved that \( B \) and \( T \) are reciprocally continuous if the pair \((B, T)\) is assumed to be compatible and \( T \) is continuous.

**4. Main Result**

The main result of this paper is the following theorem:

**Theorem 4.1.** Let \((X, M, N, *, \odot)\) be a complete intuitionistic fuzzy metric space with continuous \( t\)-norm * and continuous \( t\)-conorm \( \odot \) defined by \( t \ast t \geq t \) and 
\((1-t) \odot (1-t) \leq (1-t) \) for all \( t \in [0, 1] \).

Further, let \((A, S)\) and \((B, T)\) be pointwise \( R \)-weakly commuting pairs of self mappings of \( X \) satisfying (3.1), (3.2). If one of the mappings in compatible pair \((A, S)\) or \((B, T)\) is continuous, then \( A, B, S \) and \( T \) have a unique common fixed point.

**Proof.** Let \( x_n \in X \). By (3.1), we define the sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that for all \( n = 0, 1, 2, \ldots \)

\[ y_{2n} = Ax_{2n} = Tx_{2n+1}, \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}. \]

We show that \( \{y_n\} \) is a Cauchy sequence in \( X \). By (3.2) take \( \alpha = 1 - \beta, \beta \in (0, 1) \), we have
\[ M(y_{2n+1}, y_{2n+2}, t) = M(Bx_{2n+1}, Ax_{2n+2}, t) = M(Ax_{2n+1}, Bx_{2n+1}, t) \geq M(Tx_{2n+1}, Bx_{2n+1}, t) \]

\[ \Rightarrow M(Sx_{2n+2}, Ax_{2n+2}, t) \cdot M(Sx_{2n+1}, Bx_{2n+1}, (1-\beta)t) \cdot M(Tx_{2n+1}, Ax_{2n+2}, (1+\beta)t) \cdot M(Tx_{2n+1}, Sx_{2n+2}, t) \]

\[ = M(y_{2n}, y_{2n+1}, t) \cdot M(y_{2n+1}, y_{2n+2}, t) \cdot M(y_{2n+1}, y_{2n+1}, (1-\beta)t) \cdot M(y_{2n+1}, y_{2n+1}, (1+\beta)t) \cdot M(y_{2n+1}, y_{2n+1}, t) \]

\[ \geq M(y_{2n+1}, y_{2n+1}, t) \cdot M(y_{2n+1}, y_{2n+2}, t) \cdot M(y_{2n+1}, y_{2n+2}, \beta t) \]

Now, taking \( \beta \to 1 \), we have

\[ M(y_{2n+1}, y_{2n+2}, t) \geq M(y_{2n+1}, y_{2n+2}, t) \cdot M(y_{2n+1}, y_{2n+2}, t) \cdot M(y_{2n+1}, y_{2n+2}, t) \]

\[ M(y_{2n+1}, y_{2n+2}, t) \geq M(y_{2n+1}, y_{2n+1}, t) \]

\[ M(y_{2n+1}, y_{2n+2}, t) \geq M(y_{2n+1}, y_{2n+1}, t) \]

Similarly, we can show that

\[ M(y_{2n+2}, y_{2n+3}, t) \geq M(y_{2n+1}, y_{2n+2}, t) \]

Also,

\[ N(y_{2n+1}, y_{2n+2}, t) = N(Bx_{2n+1}, Ax_{2n+2}, t) = N(Ax_{2n+1}, Bx_{2n+1}, t) \leq N(Tx_{2n+1}, Bx_{2n+1}, t) \cdot N(Sx_{2n+2}, Ax_{2n+2}, t) \]

\[ \Rightarrow N(Sx_{2n+2}, Bx_{2n+1}, (1-\beta)t) \cdot N(Tx_{2n+1}, Ax_{2n+2}, (1+\beta)t) \cdot N(Tx_{2n+1}, Sx_{2n+2}, t) \]

\[ = N(y_{2n}, y_{2n+1}, t) \cdot N(y_{2n+1}, y_{2n+2}, t) \cdot N(y_{2n+1}, y_{2n+1}, (1-\beta)t) \cdot N(y_{2n+1}, y_{2n+1}, (1+\beta)t) \cdot N(y_{2n+1}, y_{2n+1}, t) \]

\[ \leq N(y_{2n}, y_{2n+1}, t) \cdot N(y_{2n+1}, y_{2n+2}, t) \cdot N(y_{2n+1}, y_{2n+1}, t) \]

Taking \( \beta \to 1 \), we get

\[ N(y_{2n+1}, y_{2n+2}, t) \leq N(y_{2n+1}, y_{2n+1}, t) \cdot N(y_{2n+1}, y_{2n+2}, t) \cdot N(y_{2n+1}, y_{2n+2}, t) \]

\[ N(y_{2n+1}, y_{2n+2}, t) \leq N(y_{2n+1}, y_{2n+1}, t) \]

Similarly, it can be shown that

\[ N(y_{2n+2}, y_{2n+3}, t) \leq N(y_{2n+1}, y_{2n+2}, t) \]

Therefore, for any \( n \) and \( t \), we have

\[ M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, t) \]

\[ N(y_n, y_{n+1}, t) \leq N(y_{n-1}, y_n, t) \]

Hence, by Lemma 3.1, \( \{y_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, so \( \{y_n\} \) converges to \( z \) in \( X \). Its subsequences \( \{Ax_n\}, \{Tx_{2n+1}\}, \{Bx_{2n+1}\} \) and \( \{Sx_{2n+2}\} \) also converge to \( z \).

Now, suppose that \( (A, S) \) is a compatible pair and \( S \) is continuous. Then by Lemma 3.2, \( A \) and \( S \) are reciprocally continuous, then \( SAx_n \to Sz \), \( ASx_n \to Az \) as \( n \to \infty \).

As \( (A, S) \) is a compatible pair. This implies

\[ \lim_{n \to \infty} M(Ax_n, SaX_n, t) = 1, \lim_{n \to \infty} N(Ax_n, S4X_n, t) = 0; \]

This gives \( M(Az, Sz, t) = 1, N(Az, Sz, t) = 0 \) as \( n \to \infty \).

Hence, \( Sz = Az \).

Since \( A(X) \subseteq T(X) \), therefore there exists a point \( p \in X \) such that \( Sz = Az = Tp \).

Now, again by taking \( \alpha = 1 \) in (3.2), we have

\[ M(Az, Bp, t) \geq M(Tp, Bp, t) \cdot M(Sz, Az, t) \]

\[ \Rightarrow M(Az, Bp, t) \cdot M(Sz, Az, t) \cdot M(Tp, Sz, t) \]

\[ = M(Az, Bp, t) \cdot M(Tp, Az, t) \cdot M(Az, Az, t) \]

\[ \Rightarrow M(Az, Bp, t) \cdot M(Az, Az, t) \]

\[ N(Az, Bp, t) \leq N(Tp, Az, t) \]

\[ \Rightarrow N(Az, Bp, t) \leq N(Tp, Sz, t) \]

\[ N(Az, Bp, t) \leq N(Az, Az, t) \]

\[ M(Az, Bp, t) \geq M(Az, Bp, t) \cdot M(Az, Az, t) \]

\[ N(Az, Bp, t) \leq N(Az, Bp, t) \]

Thus, by Lemma 3.2, we have \( Az = Bp \).
Thus, \( Az = Bp = Sz = Tp \).
Since, \( A \) and \( S \) are pointwise \( R \)-weakly commuting mappings, therefore there exists \( R > 0 \), such that
\[
M\left( Asz, SAz, t \right) \geq M\left( Az, Sz, \frac{t}{R} \right) = 1
\]
and
\[
N\left( Asz, SAz, t \right) \leq N\left( Az, Sz, \frac{t}{R} \right) = 0
\]
Hence, \( Asz = Saz \) and \( Asz = Saz = AAz = SSz \).
Similarly, \( B \) and \( T \) are pointwise \( R \)-weakly commuting mappings, we have \( BBp = BTP = TBP = TTP \).
Again, by taking \( \alpha = 1 \) in (3.2),
\[
M\left( AAZ, Bp, kt \right) \geq M\left( Tp, Bp, t \right) * M\left( Saz, AAZ, t \right)
\]
\[
* M\left( Saz, Bp, t \right) * M\left( Tp, AAZ, t \right) * M\left( Tp, Saz, t \right)
\]
\[
M\left( AAZ, Az, kt \right) \geq M\left( Tp, Tp, t \right) * M\left( AAZ, AAZ, t \right)
\]
\[
* M\left( AAZ, Az, t \right) * M\left( AAZ, Az, t \right) * M\left( AAZ, AAZ, t \right)
\]
and
\[
N\left( AAZ, Bp, kt \right) \leq N\left( Tp, Bp, t \right) \triangleright N\left( Saz, AAZ, t \right)
\]
\[
\triangleright N\left( Saz, Bp, t \right) \triangleright N\left( Tp, AAZ, t \right) \triangleright N\left( Tp, Saz, t \right)
\]
\[
N\left( AAZ, Az, kt \right) \leq N\left( Tp, Tp, t \right) \triangleright N\left( AAZ, AAZ, t \right)
\]
\[
\triangleright N\left( AAZ, Az, t \right) \triangleright N\left( AAZ, Az, t \right) \triangleright N\left( AAZ, AAZ, t \right)
\]
\[
M\left( AAZ, Az, kt \right) \geq M\left( AAZ, Az, t \right),
\]
\[
N\left( AAZ, Az, kt \right) \leq N\left( AAZ, Az, t \right)
\]
By Lemma 3.2, we have \( Saz = AAZ = Az \). Hence \( Az \) is common fixed point of \( A \) and \( S \). Similarly by (3.2), \( Bp = Az \) is a common fixed point of \( B \) and \( T \). Hence, \( Az \) is a common fixed point of \( A, B, S \) and \( T \).

**Uniqueness:** Suppose that \( Ap \neq Az \) is another common fixed point of \( A, B, S \) and \( T \).
Then by (3.2), take \( \alpha = 1 \)
\[
M\left( AAZ, BAp, kt \right) \geq M\left( TAp, BAp, t \right) * M\left( SAz, AAZ, t \right)
\]
\[
* M\left( SAz, BAp, t \right) * M\left( TAp, AAZ, t \right) * M\left( TAp, SAz, t \right)
\]
\[
M\left( Az, Ap, kt \right) \geq M\left( Ap, Ap, t \right) * M\left( Az, Az, t \right)
\]
\[
* M\left( Az, Ap, t \right) * M\left( Ap, Az, t \right) * M\left( Ap, Az, t \right)
\]
and
\[
N\left( AAZ, BAp, kt \right) \leq N\left( TAp, BAp, t \right) \triangleright N\left( SAz, AAZ, t \right)
\]
\[
\triangleright N\left( SAz, BAp, t \right) \triangleright N\left( TAp, AAZ, t \right) \triangleright N\left( TAp, SAz, t \right)
\]
\[
N\left( Az, Ap, kt \right) \leq N\left( Ap, Ap, t \right) \triangleright N\left( Az, Az, t \right)
\]
\[
\triangleright N\left( Az, Ap, t \right) \triangleright N\left( Ap, Az, t \right) \triangleright N\left( Ap, Ap, t \right)
\]
This gives
\[
M\left( Az, Ap, kt \right) \geq M\left( Az, Ap, t \right), \quad and
\]
\[
N\left( Az, Ap, kt \right) \leq N\left( Az, Ap, t \right)
\]
By Lemma 3.2, \( Ap = Az \). Thus, uniqueness follows.
Taking \( S = T = I_x \) in above theorem, we get following result:

**Corollary 4.1.** Let \( \left( X, M, N, *, \Diamond \right) \) be a complete intuitionistic fuzzy metric space with continuous \( t \)-norm \( * \) and continuous \( t \)-conorm \( \Diamond \) defined by \( t * t \geq t \) and \( (1-t) \Diamond (1-t) \leq (1-t) \) for all \( t \in [0,1] \). Further, let \( A \) and \( B \) are reciprocally continuous mappings on \( X \) satisfying
\[
M\left( Ax, By, kt \right) \geq M\left( y, By, t \right) * M\left( x, Ax, t \right)
\]
\[
* M\left( x, By, \alpha t \right) * M\left( y, Ax, (2-\alpha) t \right) * M\left( y, x, t \right)
\]
\[
N\left( Ax, By, kt \right) \leq N\left( y, By, t \right) \Diamond N\left( x, Ax, t \right)
\]
\[
\Diamond N\left( x, By, \alpha t \right) \Diamond N\left( y, Ax, (2-\alpha) t \right) \Diamond N\left( y, x, t \right)
\]
for all \( u, v \in X \), \( \alpha \in (0,2) \) then pair \( A \) and \( B \) has a unique common fixed point.

We give now example to illustrate the above theorem:

**Example 4.1.** Let \( X = [0, \infty) \) and let \( M \) and \( N \) be defined by \( M\left( u, v, t \right) = \frac{t}{t + |u - v|} \) and \( N\left( u, v, t \right) = \frac{|u - v|}{t + |u - v|} \).

Then \( \left( X, M, N, *, \Diamond \right) \) is complete intuitionistic fuzzy metric space. Let \( A, B, S \) and \( T \) be self maps on \( X \) defined as:
\[
Ax = Bx = \frac{3x}{4} \quad and \quad Sx = Tx = 2x \quad for \quad all \quad x \in X .
\]

Clearly,
1) either of pair \( \left( A, S \right) \) or \( \left( B, T \right) \) be continuous self-mappings on \( X \);
2) \( A\left( X \right) \subseteq T\left( X \right) \), \( B\left( X \right) \subseteq S\left( X \right) \);
3) \( \left\{ A, S \right\} \) and \( \left\{ B, T \right\} \) are \( R \)-weakly commuting pairs as both pairs commute at coincidence points;
4) \( \left\{ A, S \right\} \) and \( \left\{ B, T \right\} \) satisfies inequality (3.2), for all \( x, y \in X \), where \( k \in (0,1) \).

Hence, all conditions of Theorem 4.1 are satisfied and \( x = 0 \) is a unique common fixed point of \( A, B, S \) and \( T \).

5. Acknowledgements

We would like to thank the referee for the critical comments and suggestions for the improvement of my paper.

**REFERENCES**


