On a Population Model of Systems

\[
\begin{align*}
\begin{cases}
    x_{n+1} &= \alpha x_n e^{-y_n} + \beta \\
y_{n+1} &= \alpha x_n \left(1 - e^{-y_n}\right)
\end{cases}
\end{align*}
\]

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ABSTRACT

In this paper, we investigate the global character of all positive solutions of a population model of systems. Some interesting convergence properties of the solution are given, and lastly, we obtain that the solution is permanent under some conditions.

Keywords: Population Model; Global Attractor; Difference Equations

1. Introduction

In the recent monograph [1, p.129], Kulenovic and Glass give an open problem as follows:

Open problem 6.10.16 (A population model).
Assume that \( \alpha \in (0,1) \) and \( \beta \in (1,\infty) \). Investigate the global character of all positive solutions of the systems:

\[
\begin{align*}
\begin{cases}
    x_{n+1} &= \alpha x_n e^{-y_n} + \beta \\
y_{n+1} &= \alpha x_n \left(1 - e^{-y_n}\right)
\end{cases}
\end{align*}
\]

where \( n = 0,1,\cdots \), which may be viewed as a population model.

To this end, we consider Equation (1) and obtain some interesting results about the positive solutions of Equation (1).

2. Basic Lemma

Lemma 1 Assume that \( \alpha \in (0,1) \), \( \beta \in (1,\infty) \). Then the following statements are true:

1) If \( 1 < \beta \leq \frac{1-\alpha}{\alpha} \), then Equation (1) has a unique non-negative equilibrium solution as follows:

\[
(x_i, y_i) = \left(\frac{\beta}{1-\alpha}, 0\right)
\]

2) If \( \beta > \frac{1-\alpha}{\alpha} \), then Equation (1) has two no-negative equilibrium solutions as follows:

\[
(x_i, y_i) = \left(\frac{\beta}{1-\alpha}, 0\right) \quad \text{or} \quad (x_2, y_2)
\]

where \( 0 < y_2 < \beta \), \( \beta < x_2 < \frac{\beta}{1-\alpha} \) such that

\[
1-e^{-y_2} = \frac{1-\alpha}{\alpha} \left(\frac{y_2}{\beta-y_2}\right)
\]

\[
x_2 = \frac{1}{1-\alpha} \left(\beta-y_2\right)
\]

Proof: The equilibrium equations about Equation (1) can be written as follows:

\[
\begin{align*}
\begin{cases}
    x &= \alpha x e^{-y} + \beta \\
y &= \alpha x \left(1 - e^{-y}\right)
\end{cases}
\end{align*}
\]

It is easy to see that \( x_i = \frac{\beta}{1-\alpha} \), \( y_i = 0 \) is a group solutions of Equation (3).

By (3) we obtain

\[
\begin{align*}
\bar{x} + \bar{y} &= \alpha \bar{x} + \beta \\
\bar{x} &= \frac{1}{1-\alpha} \left(\beta-\bar{y}\right)
\end{align*}
\]

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Thus
\[ y = \frac{\alpha}{1-\alpha}(\beta - y)(1-e^{-y}) \quad (5) \]

Noting that (3) and (4) we get:
\[ 0 < \bar{\gamma}_2 < \beta \quad \text{and} \quad \beta < \bar{\gamma}_2 < \frac{\beta}{1-\alpha} \]

Changing (5) to (6)
\[ 1-e^{-\gamma} = \frac{1-\alpha}{\alpha} \left( \frac{\bar{\gamma}}{\beta - \bar{\gamma}} \right) \quad (6) \]

Set
\[ f(x) = \frac{\alpha}{1-\alpha}[(\beta-x)(1-e^{-x})-x], \]
for \( 0 < \alpha < \beta, 0 < x < \beta \)

Observing that
\[ f(0) = 0, \quad f(\beta) = -\beta \]
\[ f'(x) = \frac{\alpha}{1-\alpha}[-1+(\beta+1-x)e^{-x}] - 1 \]
\[ f''(x) = \frac{\alpha}{1-\alpha}[-e^{-x}-(\beta+1-x)e^{x}] < 0 \]

So, by the convex functions properties, if \( \lim_{x \to 0^+} f'(x) > 0 \), then we can obtain Equation (6) has a unique positive solution \( \bar{\gamma}_2 \).

In fact, by the continuous of \( f \), we can get
\[ \lim_{x \to 0^+} f'(x) = f'(0) = \frac{\alpha}{1-\alpha}\beta - 1 > 0. \]

Hence, we complete the proof.

3. Main Results

**Theorem 3.1** Assume that \( \alpha \in (0,1) \) and \( \beta \in (1,\infty) \). Then every positive solutions \( \{x_n\}_{n=0}^{\infty} \) and \( \{y_n\}_{n=0}^{\infty} \) of Equation (1) have the following properties:

1) \( \limsup_{n \to \infty} x_n \leq \frac{\beta}{1-\alpha} \liminf_{n \to \infty} x_n > \beta ; \)

2) \( \limsup_{n \to \infty} y_n < \frac{\alpha\beta}{1-\alpha} \liminf_{n \to \infty} y_n \geq 0 . \)

**Proof:** By Equation (1) we have
\[ \beta < x_{n+1} \leq \alpha x_n + \beta \leq \alpha[ax_{n-1}] + \beta \]
\[ \leq \cdots \leq \beta + \alpha \beta + \cdots + \alpha^{n-2} \beta + \alpha^{n-1} x_0 \]
\[ y_{n+1} < \alpha x_n \]

Thus \( \sup\{y_n\} < \frac{\alpha\beta}{1-\alpha}, \liminf\{y_n\} \geq 0 . \)

This completes the proof.

**Theorem 3.2** Assume that \( \alpha \in (0,\infty) \), \( \beta \in (1,\infty) \) and \( \beta \leq \frac{1-\alpha}{\alpha} \). Then every positive solutions of Equation (1) converges to the unique non-negative equilibrium solution \( \left( \frac{\beta}{1-\alpha}, 0 \right) . \)

**Proof:** By Theorem 3.1, we have that there exists a nature number \( n_0 \) such that \( x_n = \frac{\beta}{1-\alpha} \) for \( n > n_0 \).

Hence, by Equation (1) we get
\[ y_{n+1} \leq \alpha x_n \left(1-e^{-y_n}\right) \leq \alpha x_n y_n \leq \frac{\alpha\beta}{1-\alpha} y_n \leq y_n \]

Thus \( \{y_n\}_{n=n_0+1}^{\infty} \) is decreasing.

Suppose that
\[ \lim_{n \to \infty} y_{n+1} = l_0 > 0 \quad (7) \]

Then by Equation (1) we have
\[ x_{n+1} \leq \alpha e^{-b} x_n + \beta \quad \text{for} \quad n \geq n_0 + 1 \]

By induction we obtain
\[ x_{n+1} \leq \alpha e^{-b} x_n + \beta \leq \alpha e^{-b} \left[ \alpha e^{-b} + \beta \right] + \beta \]
\[ \leq \cdots \leq \left( \alpha e^{-b} \right)^{n-n_0+1} \beta + \cdots + \beta + \left( \alpha e^{-b} \right)^{n-n_0} x_{n_0+1} \]

Thus \( \sup\{x_n\} \leq \frac{\beta}{1-\alpha} e^{-b} \). Hence there exists a \( n_0' \in N^+ \) such that \( x_n \leq \frac{\beta}{1-\alpha} e^{-b} \) for \( n > n_0' \).

Noting that Equation (1)
\[ y_{n+1} = \alpha x_n \left(1-e^{-y_n}\right) \leq \frac{\alpha\beta}{1-\alpha} y_n \quad \text{for} \quad n > n_0' \]

By induction,
\[ y_{n+1} \leq \left[ \frac{\alpha\beta}{1-\alpha} e^{-b} \right]^{n-n_0+1} y_{n_0+1} \]

It is to see that \( \lim y_n = 0 \). This is a contradiction with (7), then \( \lim y_n = 0 \).

Noting that Equation (1) we have
\[ x_{n+1} + y_{n+1} = \alpha x_n + \beta \]

i.e.
\[ x_{n+1} = \alpha x_n + \beta - y_{n+1} \]

Let \( \sup \{ y_{n+1} \} = \mu_1 \), \( \inf \{ y_{n+1} \} = \lambda_1 \). Then
\[
\alpha x_n + \beta - \mu_1 \leq x_{n+1} < \alpha x_n + \beta - \lambda_1
\]

By induction we obtain
\[
\frac{\beta - \mu_1}{1 - \alpha} \left(1 - \alpha^{n+1}\right) + \alpha^{n+1} x_0 \leq x_{n+1} \\
\leq \frac{\beta - \lambda_1}{1 - \alpha} \left(1 - \alpha^{n+1}\right) + \alpha^{n+1} x_0
\]

as \( 0 < \alpha < 1 \), then
\[
\lim_{n \to \infty} \sup \{ x_n \} \leq \frac{\beta - \lambda_1}{1 - \alpha} \quad \text{and} \quad \lim_{n \to \infty} \inf \{ x_n \} \geq \frac{\beta - \mu_1}{1 - \alpha} \quad (8)
\]

Because \( \lim_{n \to \infty} y_n = 0 \), we obtain that \( \lambda_1 = \mu_1 = 0 \).

Hence
\[
\lim_{n \to \infty} \sup \{ x_n \} \leq \frac{\beta}{1 - \alpha} \quad \text{and} \quad \lim_{n \to \infty} \inf \{ x_n \} \geq \frac{\beta}{1 - \alpha} \quad (9)
\]

By (9) we get \( \lim_{n \to \infty} x_n = \frac{\beta}{1 - \alpha} \).

We complete the proof.

**Theorem 3.3** Assume that \( \alpha \in (0, \infty) \), \( \beta \in (1, \infty) \) and \( \alpha \beta > 1 \). Then Equation (1) is permanent.

**Proof:** By Equation (1) we obtain
\[
y_{n+1} > \alpha x_n \left(1 - e^{-y_n}\right) > \alpha \beta \left(1 - e^{-y_n}\right)
\]
\[
= \alpha \beta \left[y_n - \frac{y_n^2}{2!} + \frac{y_n^3}{3!} - \frac{y_n^4}{4!} + \cdots\right]
\]
\[
= \alpha \beta \left[1 - \frac{y_n}{2!} + \frac{y_n^2}{3!} - \frac{y_n^3}{4!} + \cdots\right] y_n
\]

There exists two positive constants \( \delta_1 \) and \( \delta_2 \) such that
\[
y_{n+1} \geq y_n \quad \text{for} \quad \delta_2 < y_n < \delta_1 < 1
\]

Hence \( \lim_{n \to \infty} \inf \{ y_n \} > 0 \).

Using Theorem 3.1, we complete the proof.

**REFERENCES**