Study of Stability and Vibration Reduction in Multi-Tool Ultrasonic Machining under Simultaneous Primary and Internal Resonance

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ABSTRACT
The main object of this paper is the mathematical study of the vibration behavior in ultrasonic machining (USM) described by non-linear differential equations. The ultrasonic machining (USM) consists of the tool holder and the absorbers representing the tools. This leads to four-degree-of-freedom system subject to multi-external excitation forces. The aim of this project is the reduction of the vibrations in the tool holder and have reasonable amplitudes for the tools represented by the multi-absorbers. Multiple scale perturbation method is applied to obtain the solution up to the second order approximation and to study the stability of the steady state solution near different simultaneous resonance cases. The resulting different resonance cases are reported and studied numerically. The stability of the steady state solution near the selected resonance cases is studied applying both frequency response equations and phase-plane technique. The effects of the different parameters of the system and the absorbers on the system behavior are studied numerically. Optimum working conditions for the tools were obtained. Comparison with the available published work is reported.

Keywords: Passive Vibration Control; Stability; Resonance; Ultrasonic Machining (USM)

1. Introduction
Ultrasonic machining (USM) is of particular interest for the machining of non-conductive, brittle materials such as engineering ceramics. Rupinder and Aspinwall [1,2] introduced a review for the fundamental principles of stationary ultrasonic machining, the material removal mechanisms involved and the effect of operating parameters on material removal rate, tool wear rate, and work piece surface finish of titanium and its alloys for application in manufacturing industry. The USM mechanism is dependent on vibration control of the machine head at resonance, while the tool represented by a dynamic absorber is doing the machining. Lim et al. [3] studied the behavior of the (USM) hypothesized theoretical model. The theoretical results showed that controlled variations in the softening stiffness can have a significant effect on the overall non-linear response of the system, by making the overall effect hardening, softening, or approximately linear. Experimentally, it has also been demonstrated that coupling of ultrasonic components with different non-linear characteristics can strongly influence the performance of the system. Amer [4] investigated the coupling of two non-linear oscillators of the main system and absorber representing ultrasonic cutting process subjected to parametric excitation forces. A threshold value of main system linear damping has been obtained, where vibration can be reduced dramatically. This threshold value can be used effectively for passive vibration control, if it is economical. This will be more useful than usual passive control, and active control. It can be applicable for all excitation frequencies. Asfar, Eissa, El-Bassiouny and Shitikova [5-16] showed how effective is the passive vibration control reduction at resonance. Eissa, El-Bassiouny and Jaensch [17-23] showed how effective is the active control in vibration reduction at resonance at different modes of vibration. They demonstrated the advantages of active control over the passive one. Eissa et al. [24-26] investigated saturation phenomena in non-linear oscillating systems subject to multi-parametric and/or external excitations. The system represents the vibration of a single-degree-of-freedom cantilever or the wing of an aircraft. They reported the occurrence of saturation phenomena at different parameters values. They applied saturation values of different parameters as optimum working conditions for vibration suppression of the cantilever. El Ganaini et al. [27-29] studied USM model subject to multi-external or...
both multi-external and multi-parametric and both multi-
external and tuned excitation forces. The model consists
of multi-degree-of-freedom system consisting of the tool
holder and absorbers (tools) simulating ultrasonic ma-
chining process. The advantages of using multi-tools are
to machine different materials and different shapes at the
same time. This leads to time saving and higher machin-
ing efficiency. Besides, devoting all the available energy
in the cutting process. The multiple time scale perturba-
tion technique is applied throughout to get an approxi-
mate solution up to the second order approximation. The
stability of the system is investigated applying both phase-
plane and frequency response function methods. The ef-
effects of the different parameters of the absorbers on sys-
tem behavior are studied numerically. The objective of
this work is to study the model subject to multi-external
excitation forces. The model is represented by a four-
degree-of-freedom system consisting of the main system
(machine head) and three absorbers (tools) simulating
ultrasonic machining process. The multiple time scale per-
turbation technique is applied throughout to get an ap-
proximate solution up to the second order approximation.
The stability of the system is investigated applying both
phase-plane and frequency response functions. The ef-
effects of the different parameters of the absorber on sys-
tem behavior are studied numerically. Comparison with
the available published work is reported.

2. Mathematical Modeling

The considered model is shown schematically in Figure
1, while Figure 2 illustrates the principles of USM. It
consists of the tool holder and absorbers (tools) simulating
multi-tool ultrasonic machining process represented
by a multi-degree-of-freedom non-linear system. The main
system is exited by multi-external forces as shown in the
following equations:

\[
\ddot{X}_1 + 2\alpha_1\dot{X}_1 + 2\alpha_2(\dot{X}_1 - \dot{X}_2) + 2\alpha_3(\dot{X}_1 - \dot{X}_3) \\
+ 2\alpha_4(\dot{X}_1 - \dot{X}_4) + \omega_1^2 \dot{X}_1 + \omega_2^2 (X_1 - X_2) \\
+ \varepsilon \gamma_1 (X_1 - X_4) + \varepsilon \eta_1 X_1^3 \\
+ \varepsilon \eta_3 (X_1 - X_3)^3 + \varepsilon \eta_4 (X_1 - X_4)^3 = 0
\]

(1)

\[
\ddot{X}_2 + 2\alpha_6(\dot{X}_2 - \dot{X}_1) + \omega_2^2 (X_2 - X_1) + \varepsilon \eta_2 (X_2 - X_3)^3 = 0
\]

(2)

\[
\ddot{X}_3 + 2\alpha_7(\dot{X}_3 - \dot{X}_1) + \omega_3^2 (X_3 - X_1) + \varepsilon \eta_3 (X_3 - X_3)^3 = 0
\]

(3)

\[
\ddot{X}_4 + 2\alpha_8(\dot{X}_4 - \dot{X}_1) + \omega_4^2 (X_4 - X_1) + \varepsilon \eta_4 (X_4 - X_4)^3 = 0
\]

(4)

2.1. Perturbation Analysis

Multiple scale perturbation method is conducted to obtain
an approximate solution for Equations (1)-(4). Assuming
the solution in the form:

\[
x_n(t, \varepsilon) = x_{n0}(T_0, T_1) + \varepsilon x_{n1}(T_0, T_1) + \varepsilon^2 x_{n2}(T_0, T_1) + \cdots
\]

(5)

and the time derivatives became

\[
\frac{d}{dt} = D_0 + \varepsilon D_1, \quad \frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1
\]

(6)

where $T_0 = \varepsilon t$. ($n = 0, 1$) are the fast and slow time scales
respectively.

Substituting Equations (5) and (6) into Equations (1)-(4),
and equating the coefficients of the same power of $\varepsilon$
in both sides, we obtain

\[
(D_0^2 + \omega_1^2)x_{10} = 0
\]

(7)

\[
(D_0^2 + \omega_m^2) x_{m0} = \omega_m^2 x_{10}, \quad m = (2, 3, 4)
\]

(8)
\[ (D_0^2 + \omega_0^2) x_{11} = \sum_{s=1}^{n} F_s \cos \Omega T_0 - 2(D_1 + \xi_1 + \xi_2 + \xi_3 + \xi_4) (D_0 x_{10}) + 2\xi_2 D_0 x_{20} + 2\xi_3 D_0 x_{30} + 2\xi_4 D_0 x_{40} \]
\[ - \left( \gamma_1 + \gamma_2 + \gamma_3 \right) x_{10} + \gamma_1 x_{20} + \gamma_2 x_{30} + \gamma_3 x_{40} - \eta_x x_{10}^3 - \eta_y (x_{10} - x_{20})^3 - \eta_z (x_{10} - x_{30})^3 \]
\[ \left( D_0^2 + \omega_0^2 \right) x_{21} = -2D_0 x_{12} + 2\xi_2 D_0 x_{20} - 2\xi_3 D_0 x_{30} + \omega_0^2 x_{11} + \eta_x (x_{10} - x_{20})^3 \]
\[ \left( D_0^2 + \omega_0^2 \right) x_{31} = -2D_0 x_{13} + 2\xi_3 D_0 x_{20} - 2\xi_4 D_0 x_{30} + \omega_0^2 x_{11} + \eta_x (x_{10} - x_{30})^3 \]
\[ \left( D_0^2 + \omega_0^2 \right) x_{41} = -2D_0 x_{14} + 2\xi_4 D_0 x_{20} - 2\xi_5 D_0 x_{30} + \omega_0^2 x_{11} + \eta_x (x_{10} - x_{40})^3 \]

The solution of Equation (7) can be expressed in the form
\[ x_{10} = A_m \exp(i \omega_1 T_0) + cc \]
Using Equation (13) into Equation (8) yields
\[ x_{m0} = A_m \exp(i \omega_m T_0) + E_{m-1} \exp(i \omega_1 T_0) + cc, \]
where \( m = (2, 3, 4) \),
\[ x_{11} = \sum_{s=1}^{n} Q_{s1} \exp(i \omega_s T_0) + E_4 \exp(i \omega_4 T_0) + E_5 \exp(i \omega_5 T_0) + E_6 \exp(i \omega_6 T_0) + E_7 \exp(i \omega_7 T_0) + E_8 \exp(i \omega_8 T_0) + E_9 \exp(i \omega_9 T_0) \]
+ \( E_{10} \exp(i \omega_{10} T_0) + E_{11} \exp(i \omega_{11} T_0) + E_{12} \exp(i \omega_{12} T_0) + E_{13} \exp(i \omega_{13} T_0) + E_{14} \exp(i \omega_{14} T_0) + E_{15} \exp(i \omega_{15} T_0) + E_{16} \exp(i \omega_{16} T_0) + E_{17} \exp(i \omega_{17} T_0) + E_{18} \exp(i \omega_{18} T_0) + E_{19} \exp(i \omega_{19} T_0) + E_{20} \exp(i \omega_{20} T_0) + E_{21} \exp(i \omega_{21} T_0) + E_{22} \exp(i \omega_{22} T_0) + cc \]
where \( Q_{s1} \) and \( E_s \) (s = 4, 5, …, 22) are complex functions in \( T_1 \). From Equations (13)-(15) into Equations (10)-(12) and eliminating the secular terms to obtain the solutions are given by:
\[ x_{21} = \sum_{s=1}^{n} Q_{s2} \exp(i \omega_s T_0) + E_{23} \exp(i \omega_{23} T_0) + E_{24} \exp(i \omega_{24} T_0) + E_{25} \exp(i \omega_{25} T_0) + E_{26} \exp(i \omega_{26} T_0) + E_{27} \exp(i \omega_{27} T_0) + E_{28} \exp(i \omega_{28} T_0) + E_{29} \exp(i \omega_{29} T_0) + E_{30} \exp(i \omega_{30} T_0) + E_{31} \exp(i \omega_{31} T_0) + E_{32} \exp(i \omega_{32} T_0) + E_{33} \exp(i \omega_{33} T_0) + E_{34} \exp(i \omega_{34} T_0) + E_{35} \exp(i \omega_{35} T_0) + E_{36} \exp(i \omega_{36} T_0) + E_{37} \exp(i \omega_{37} T_0) + E_{38} \exp(i \omega_{38} T_0) + E_{39} \exp(i \omega_{39} T_0) + E_{40} \exp(i \omega_{40} T_0) + E_{41} \exp(i \omega_{41} T_0) + cc \]
\[ x_{31} = \sum_{s=1}^{n} Q_{s3} \exp(i \omega_s T_0) + E_{42} \exp(i \omega_{42} T_0) + E_{43} \exp(i \omega_{43} T_0) + E_{44} \exp(i \omega_{44} T_0) + E_{45} \exp(i \omega_{45} T_0) + E_{46} \exp(i \omega_{46} T_0) + E_{47} \exp(i \omega_{47} T_0) + E_{48} \exp(i \omega_{48} T_0) + E_{49} \exp(i \omega_{49} T_0) + E_{50} \exp(i \omega_{50} T_0) + E_{51} \exp(i \omega_{51} T_0) + E_{52} \exp(i \omega_{52} T_0) + E_{53} \exp(i \omega_{53} T_0) + E_{54} \exp(i \omega_{54} T_0) + E_{55} \exp(i \omega_{55} T_0) + E_{56} \exp(i \omega_{56} T_0) + E_{57} \exp(i \omega_{57} T_0) + E_{58} \exp(i \omega_{58} T_0) + E_{59} \exp(i \omega_{59} T_0) + E_{60} \exp(i \omega_{60} T_0) + cc \]
\[ x_{1s} = \sum_{s=1}^{5} Q_{s} \exp(is \Omega T_o) + E_1 \exp(i \omega_1 T_0) + E_2 \exp(i \omega_2 T_0) + E_3 \exp(i \omega_3 T_0) + E_4 \exp(3i \omega_3 T_0) + E_5 \exp(3i \omega_3 T_0) \\
+ E_6 \exp(3i \omega_3 T_0) + E_7 \exp(3i \omega_4 T_0) + E_8 \exp(i(\omega_1 + 2\omega_2) T_0) + E_9 \exp(i(\omega_2 - 2\omega_1) T_0) + E_{10} \exp(i(2\omega_1 + \omega_2) T_0) \\
+ E_{11} \exp(i(2\omega_1 - \omega_2) T_0) + E_{12} \exp(i(\omega_1 + 2\omega_2) T_0) + E_{13} \exp(i(\omega_2 - 2\omega_1) T_0) + E_{14} \exp(i(2\omega_1 + \omega_2) T_0) \\
+ E_{15} \exp(i(2\omega_1 - \omega_2) T_0) + cc \] (18)

where \( Q_{s1}, Q_{s2}, Q_{s4} \) and \( E_s (s = 23, \ldots, 79) \) are complex functions in \( T_o \).

The reported resonance cases at this approximation order are:

1) Trivial resonance: \( \Omega \equiv \pm \omega_1 \equiv \pm \omega_2 \equiv \pm \omega_3 \equiv \pm \omega_4 = 0 \).

Primary resonance:
\[ \Omega \equiv \pm \omega_1, \Omega \equiv \pm \omega_2, \Omega \equiv \pm \omega_3, \Omega \equiv \pm \omega_4 \]

2) Sub-harmonic resonance:
\[ \Omega \equiv \pm \omega_1/2, \Omega \equiv \pm \omega_2/2, \Omega \equiv \pm \omega_3/3 \]

3) Super-harmonic resonance:
\[ \Omega \equiv \pm \omega_1/2, \Omega \equiv \pm \omega_3/2, \Omega \equiv \pm \omega_2/3 \]

4) Internal resonance: \( \omega_1 \equiv \pm \omega_2 \equiv \pm \omega_3 \equiv \pm \omega_4 \),
\[ \omega_1 \equiv \pm 3\omega_n, \omega_2 \equiv \pm 4\omega_n, \omega_i \equiv \pm 5\omega_n, \]
\[ 3\omega_1 \equiv 5\omega_n, 5\omega_1 \equiv 3\omega_n, n = (2, 3, 4). \]

5) Combined resonance:
\[ \Omega \equiv (\omega_1 + \omega_2), \Omega \equiv (\omega_1 - \omega_2), \Omega \equiv (\omega_1 + 2\omega_2), \]
\[ \Omega \equiv (\omega_1 - 2\omega_2), \Omega \equiv (2\omega_1 + \omega_2), \Omega \equiv (2\omega_1 - \omega_2) \]

6) Simultaneous or incident resonance: Any combination of the above resonance cases is considered as simultaneous resonance.

### 2.2. Numerical Results

Table 1 illustrates the selected values of the equations parameters used in resonance case calculations and its units.

Table 2 summarizes some of different resonance cases and the effectiveness of the absorbers.

Figure 2 illustrates the response for the system with absorber at the simultaneous primary resonance \( \Omega \equiv \omega_1 \equiv \omega_2 \equiv \omega_3 \equiv \omega_4 \). The effectiveness of the absorber \( E_a \) (the steady state amplitude of the main system without absorber/the steady state amplitude of main system with absorber) is about 7, which means that the maximum amplitude is reduced to about 14\% of its original value.

### 3. Results and Discussion

One of the effective resonance cases where the tool holder has low amplitude and at the same time, the absorbers have high amplitudes is studied in the next section.

#### 3.1. Stability of the System

Introducing the detuning parameters \( \sigma_1, \sigma_2, \sigma_3 \) and \( \sigma_4 \) in the primary and internal resonance to convert the small-divisor terms into the secular terms, according to:
\[ \Omega \equiv \omega_1 + \varepsilon \sigma_1, \omega_2 \equiv \omega_1 + \varepsilon \sigma_2, \]
\[ \omega_3 \equiv \omega_3 + \varepsilon \sigma_3, \omega_4 \equiv \omega_3 + \varepsilon \sigma_4 \] (19)

### Table 1. The values of the equations parameters.

<table>
<thead>
<tr>
<th>Damping coefficients values (Newton/sec/micrometer)</th>
<th>( \zeta_1 = 0.01 )</th>
<th>( \zeta_2 = 0.001 )</th>
<th>( \zeta_3 = 0.001 )</th>
<th>( \zeta_4 = 0.001 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \zeta_1 = 0.01 )</td>
<td>( \zeta_2 = 0.001 )</td>
<td>( \zeta_3 = 0.001 )</td>
<td>( \zeta_4 = 0.001 )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Non-linear parameters values (Newton/micrometer)</th>
<th>( \eta_1 = 0.01 )</th>
<th>( \eta_2 = 0.005 )</th>
<th>( \eta_3 = 0.005 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta_1 = 0.01 )</td>
<td>( \eta_2 = 0.005 )</td>
<td>( \eta_3 = 0.005 )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Natural and Excitation frequencies values (Hertz)</th>
<th>( \Omega = 1 ), ( \omega_1 = \omega_2 = \omega_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Omega = 1 )</td>
<td>( \omega_1 = \omega_2 = \omega_3 )</td>
</tr>
</tbody>
</table>

| Excitation amplitudes values (micrometer/sec²) | \( F_1 = 4 \) | \( F_2 = 2 \) |
| --- | --- |
| \( F_1 = 4 \) | \( F_2 = 2 \) |
Table 2. Summarizes the resonance cases for the tool holder and absorber.

<table>
<thead>
<tr>
<th>Cases</th>
<th>Conditions</th>
<th>$x_1/F_1$</th>
<th>$x_2/F_2$</th>
<th>$x_3/F_3$</th>
<th>$x_4/F_4$</th>
<th>Remarks*</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega \equiv \omega_1$</td>
<td>$\omega_1 \equiv \omega_2 \equiv \omega_3 \equiv \omega_4$</td>
<td>12.5%</td>
<td>85%</td>
<td>85%</td>
<td>85%</td>
<td>Limit cycle</td>
</tr>
<tr>
<td></td>
<td>$\omega_1 \equiv \omega_2/2 \equiv \omega_3 \equiv \omega_4$</td>
<td>15.5%</td>
<td>50%</td>
<td>118%</td>
<td>118%</td>
<td>Limit cycle</td>
</tr>
<tr>
<td></td>
<td>$\omega_1 \equiv \omega_2/3 \equiv \omega_3 \equiv \omega_4$</td>
<td>18.75%</td>
<td>30%</td>
<td>118%</td>
<td>118%</td>
<td>Limit cycle</td>
</tr>
<tr>
<td></td>
<td>$\omega_1 \equiv \omega_2/5 \equiv \omega_3 \equiv \omega_4$</td>
<td>20%</td>
<td>22%</td>
<td>118%</td>
<td>118%</td>
<td>Limit cycle</td>
</tr>
<tr>
<td></td>
<td>$\omega_1 \equiv \omega_2/3 \equiv \omega_3 \equiv \omega_4/5$</td>
<td>34%</td>
<td>165%</td>
<td>74%</td>
<td>43%</td>
<td>Limit cycle</td>
</tr>
<tr>
<td>$\Omega \equiv 3\omega_1$</td>
<td>$\omega_1 \equiv \omega_2 \equiv \omega_3 \equiv \omega_4$</td>
<td>3.5%</td>
<td>0.4%</td>
<td>0.4%</td>
<td>0.4%</td>
<td>Limit cycle</td>
</tr>
<tr>
<td></td>
<td>$\omega_1 \equiv \omega_2/2 \equiv \omega_3 \equiv \omega_4$</td>
<td>3.75%</td>
<td>27%</td>
<td>0.45%</td>
<td>0.45%</td>
<td>Limit cycle</td>
</tr>
<tr>
<td>$\Omega \equiv \omega_1/2$</td>
<td>$\omega_1 \equiv \omega_2 \equiv \omega_3 \equiv \omega_4$</td>
<td>39%</td>
<td>170%</td>
<td>50%</td>
<td>40%</td>
<td>Limit cycle</td>
</tr>
<tr>
<td></td>
<td>$\omega_1 \equiv \omega_2 \equiv \omega_3 \equiv \omega_4$</td>
<td>3.75%</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
<td>Limit cycle</td>
</tr>
<tr>
<td>$\Omega \equiv \omega_1/3$</td>
<td>$\omega_1 \equiv \omega_2 \equiv \omega_3 \equiv \omega_4$</td>
<td>65%</td>
<td>85%</td>
<td>85%</td>
<td>85%</td>
<td>Limit cycle</td>
</tr>
<tr>
<td></td>
<td>$\omega_1 \equiv \omega_2 \equiv \omega_3 \equiv \omega_4$</td>
<td>63%</td>
<td>67%</td>
<td>85%</td>
<td>85%</td>
<td>Limit cycle</td>
</tr>
<tr>
<td>$\Omega \equiv 3\omega_1/2$</td>
<td>$\omega_1 \equiv \omega_2 \equiv \omega_3 \equiv \omega_4$</td>
<td>37%</td>
<td>31%</td>
<td>31%</td>
<td>31%</td>
<td>Limit cycle</td>
</tr>
</tbody>
</table>

This case represents the system best case and at the same time absorber high amplitude. Substituting Equation (19) into Equations (9)-(12) and eliminating the secular terms, leads to the solvability conditions for the first order approximation noting that $A_1$, $A_2$, $A_3$, and $A_4$ are functions in $T_i$, we get

\[
\begin{align*}
2i\omega_1 \left[ D_1 + \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 \right] A_1 + (\gamma_1 + \gamma_2 + \gamma_3) A_1 + 3\eta_1 A_2^2 A_1 + \eta_2 \left( 3A_1^2 \bar{A}_1 + 6A_1 A_2 \bar{A}_2 \right) \\
+ \eta_3 \left( 3A_1^2 \bar{A}_1 + 6A_1 A_2 \bar{A}_2 \right) + \eta_4 \left( 3A_1^2 \bar{A}_1 + 6A_1 A_2 \bar{A}_2 \right) - \frac{F_1}{2} e^{i\omega_1 T} - \left[ 2i\omega_2 A_2 \zeta_2 + \gamma_1 A_2 + \eta_2 \left( 3A_1^2 \bar{A}_1 + 6A_1 A_2 \bar{A}_2 \right) \right] e^{i\omega_1 T}
\end{align*}
\]

(20)

\[
\begin{align*}
2i\omega_2 \left[ D_2 + \zeta_2 A_1 + \eta_1 \left( 3A_2^2 \bar{A}_2 + 6A_2 A_1 \bar{A}_1 \right) \right] e^{i\omega_2 T_1} - \left[ 2i\omega_2 A_2 \zeta_2 + \gamma_1 A_2 + \eta_2 \left( 3A_1^2 \bar{A}_1 + 6A_1 A_2 \bar{A}_2 \right) \right] e^{i\omega_1 T}
\end{align*}
\]

(21)

\[
\begin{align*}
2i\omega_1 \left[ D_1 + \zeta_2 A_1 + \eta_1 \left( 3A_2^2 \bar{A}_2 + 6A_2 A_1 \bar{A}_1 \right) \right] e^{i\omega_2 T_1} + \left[ 3\eta_2 A_2^2 A_1 \right] e^{-i\omega_2 T_1} + \left[ 3\eta_2 A_2^2 A_1 \right] e^{+i\omega_2 T_1} - \left[ 3\eta_2 A_2^2 A_1 \right] e^{-i\omega_2 T_1} = 0
\end{align*}
\]

(22)

\[
\begin{align*}
2i\omega_2 \left[ D_2 + \zeta_2 A_1 + \eta_1 \left( 3A_2^2 \bar{A}_2 + 6A_2 A_1 \bar{A}_1 \right) \right] e^{i\omega_2 T_1} - \left[ 3\eta_2 A_2^2 A_1 \right] e^{i\omega_2 T_1} + \left[ 3\eta_2 A_2^2 A_1 \right] e^{-i\omega_2 T_1} = 0
\end{align*}
\]

(23)

Putting $A_n = \frac{1}{2} a_n (T_i) e^{i\phi_n(T_i)}$, $n = (1, 2, 3, 4)$
where \( a_0 \) and \( \beta_0 \) are the steady state amplitudes and the phases of the motion respectively. Substituting Equation (24) into Equations (20)-(23) and separating real and imaginary part yields,

\[
\begin{align*}
    a'_0 &= \frac{F_{0}}{2\alpha_0} \sin \theta_1 - \Gamma_8 \sin \theta_2 - \Gamma_{16} \sin \theta_2 - \Gamma_{11} \sin \theta_3 - \Gamma_{13} \sin \theta_4 \\
    a_1 &= \frac{F_{0}}{2\alpha_0} \cos \theta_1 + \Gamma_8 \cos \theta_2 + \Gamma_{16} \cos \theta_2 + \Gamma_{11} \cos \theta_3 + \Gamma_{13} \cos \theta_4 \tag{25}
\end{align*}
\]

\[
\begin{align*}
    a'_2 &= -\zeta_2 a_2 + \Gamma_{15} \cos \theta_2 - \Gamma_{16} \sin \theta_2 + \Gamma_{18} \sin \theta_2 \\
    a_2 &= \Gamma_{19} - \Gamma_{15} \sin \theta_2 - \Gamma_{16} \cos \theta_2 - \Gamma_{11} \sin \theta_2 + \Gamma_{14} \cos \theta_2 \tag{26}
\end{align*}
\]

\[
\begin{align*}
    a'_3 &= -\zeta_3 a_3 + \Gamma_{20} \cos \theta_3 - \Gamma_{21} \sin \theta_3 + \Gamma_{22} \sin \theta_3 + \Gamma_{23} \cos \theta_3 \\
    a_3 &= \Gamma_{24} - \Gamma_{20} \sin \theta_3 - \Gamma_{21} \cos \theta_3 - \Gamma_{22} \sin \theta_3 + \Gamma_{23} \cos \theta_3 \tag{27}
\end{align*}
\]

\[
\begin{align*}
    a'_4 &= -\zeta_4 a_4 + \Gamma_{25} \cos \theta_4 - \Gamma_{26} \sin \theta_4 + \Gamma_{27} \sin \theta_4 + \Gamma_{28} \cos \theta_4 \\
    a_4 &= \Gamma_{29} - \Gamma_{25} \sin \theta_4 - \Gamma_{26} \cos \theta_4 - \Gamma_{27} \cos \theta_4 + \Gamma_{28} \cos \theta_4 \tag{28}
\end{align*}
\]

where: \((\Gamma_1, \Gamma_2, \cdots, \Gamma_{26})\) are defined in the appendix,

\[
\begin{align*}
    \theta_1 &= \sigma_1 \Gamma_{1} - \beta_1, \quad \theta_2 = \sigma_2 \Gamma_{1} + \beta_2 - \beta_1, \quad \theta_3 = \sigma_3 \Gamma_{1} + \beta_3 - \beta_1, \quad \theta_4 = \sigma_4 \Gamma_{1} + \beta_4 - \beta_1 \tag{33}
\end{align*}
\]

For steady state solutions, \( a'_n = \theta'_n = 0, \quad n = (1, 2, 3, 4) \).

Then from Equation (33), we get:

\[
\begin{align*}
    \beta'_1 &= \sigma_1, \quad \beta'_2 = (\sigma_1 - \sigma_2), \quad \beta'_3 = (\sigma_1 - \sigma_3), \quad \beta'_4 = (\sigma_1 - \sigma_4) \tag{34}
\end{align*}
\]

Then it follows from Equations (25)-(32) that the steady state solutions are given by:

\[
\begin{align*}
    a_1, a_3 &= \frac{F_{0}}{2\alpha_0} \cos \theta_1 + \Gamma_{14} \cos \theta_2 + \Gamma_{15} \cos \theta_2 + \Gamma_{16} \sin \theta_4 + \Gamma_{13} \cos \theta_4 \\
    + \Gamma_{18} \sin \theta_4, \quad a_2, a_4 &= \Gamma_{19} - \Gamma_{15} \sin \theta_2 - \Gamma_{16} \cos \theta_2 - \Gamma_{11} \sin \theta_2 + \Gamma_{14} \cos \theta_2 \tag{35}
\end{align*}
\]

\[
\begin{align*}
    a'_1 &= -\zeta_6 a_1 + \Gamma_{17} \cos \theta_2 + \Gamma_{18} \sin \theta_2 \\
    a_1 &= \Gamma_{20} \cos \theta_3 - \Gamma_{21} \sin \theta_3 + \Gamma_{22} \sin \theta_3 + \Gamma_{23} \cos \theta_3 \tag{36}
\end{align*}
\]

\[
\begin{align*}
    a'_2 &= -\zeta_6 a_2 + \Gamma_{15} \cos \theta_2 - \Gamma_{16} \sin \theta_2 + \Gamma_{17} \sin \theta_2 + \Gamma_{18} \sin \theta_2 = 0 \\
    a_2 &= \Gamma_{24} - \Gamma_{20} \sin \theta_3 - \Gamma_{21} \cos \theta_3 - \Gamma_{22} \sin \theta_3 + \Gamma_{23} \cos \theta_3 \tag{37}
\end{align*}
\]

\[
\begin{align*}
    a'_3 &= -\zeta_6 a_3 + \Gamma_{20} \cos \theta_4 + \Gamma_{21} \sin \theta_4 + \Gamma_{22} \sin \theta_4 + \Gamma_{23} \cos \theta_4 \\
    a_3 &= \Gamma_{25} \cos \theta_4 - \Gamma_{26} \sin \theta_4 + \Gamma_{27} \sin \theta_4 + \Gamma_{28} \cos \theta_4 \tag{38}
\end{align*}
\]

\[
\begin{align*}
    a'_4 &= -\zeta_6 a_4 + \Gamma_{25} \cos \theta_4 - \Gamma_{26} \sin \theta_4 + \Gamma_{27} \sin \theta_4 + \Gamma_{28} \cos \theta_4 \\
    a_4 &= \Gamma_{29} - \Gamma_{25} \sin \theta_4 - \Gamma_{26} \cos \theta_4 - \Gamma_{27} \cos \theta_4 + \Gamma_{28} \cos \theta_4 \tag{39}
\end{align*}
\]

From Equations (35)-(42) we have the following case:

**Table 3** gives the final results of the frequency response equations (in Table 3), where \( K_1, K_2, K_3, K_4, K_5, K_6, K_7 \) and \( K_8 \) are real functions. The stability of the linear solution of the obtained fixed points will be determined as follows. Consider \( A_n \) in the form:

\[
A_n = \frac{1}{2} p_n \cdot i q_n \cdot e^{i \omega t}, \quad n = (1, 2, 3, 4) \tag{43}
\]
Table 3. Frequency response equations.

<table>
<thead>
<tr>
<th>Case</th>
<th>Frequency response equations (FRE)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \sigma_1 + K, \sigma_1 + K = 0, \sigma_1 + K, \sigma_1 + K = 0, \sigma_1 + K, \sigma_1 + K = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>( \sigma_1 + K, \sigma_1 + K, \sigma_1 + K, \sigma_1 + K = 0 )</td>
</tr>
</tbody>
</table>

where \( p_a \) and \( q_a \) are real and \( v_1 = \sigma_1, v_2 = (\sigma_1 - \sigma_2), v_3 = (\sigma_1 - \sigma_3) \) and \( v_4 = (\sigma_1 - \sigma_4) \). Substituting Equation (43) into the linear part of Equations (20)-(23) and separating real and imaginary part yields,

\[
p_1' + \left( \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 \right) p_1 + \frac{v_1 - \left( \gamma_1 + \gamma_2 + \gamma_3 \right)}{2\omega_1} q_1 - \frac{\omega_1\zeta_2}{2\omega_1} p_2 + \frac{\gamma_1}{2\omega_1} q_2 = 0
\]

\[
q_1' + \left( \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 \right) q_1 - \frac{v_1 - \left( \gamma_1 + \gamma_2 + \gamma_3 \right)}{2\omega_1} p_1 = 0
\]

The eigenvalues of the above system of equations are given by the equation

\[
\lambda^8 + r_1\lambda^7 + r_2\lambda^6 + r_3\lambda^5 + r_4\lambda^4 + r_5\lambda^3 + r_6\lambda^2 + r_7\lambda + r_8 = 0
\]

where, \( (r_1, r_2, \ldots, r_8) \) are real functions. According to the Routh-Hurwitz criterion, the necessary and sufficient conditions for all the roots of Equation (52) to possess negative real parts if, and only if,

\[
K_y K_{10} K_{11} K_{12} K_{13} K_{14} K_{15} K_{16} \\
-K_{17} K_y -K_{12} K_{11} -K_{14} K_{13} -K_{16} -K_{15} \\
K_{18} 0 \lambda + \zeta_6 v_2 0 0 0 0 \\
0 K_{18} -v_2 \lambda + \zeta_6 0 0 0 0 \\
K_{19} 0 0 0 \lambda + \zeta_7 v_3 0 0 \\
0 K_{19} 0 0 -v_2 \lambda + \zeta_7 0 0 \\
K_{20} 0 0 0 0 \lambda + \zeta_8 v_4 \\
0 K_{20} 0 0 0 0 -v_4 \lambda + \zeta_8
\]

and all its principle minors are positive. \( K_y, \ldots, K_{20} \) are real functions.

3.2. Numerical Results

Figures 3(a) and 4(a) shows that the effects of the detuning parameters \( \sigma_1, \sigma_2, \sigma_3 \) and \( \sigma_4 \) on the steady state amplitudes of the main system \( a_t \) and absorbers \( a_2, a_3 \) and \( a_4 \) for the stability of the practical case where \( a_1 \neq 0, a_2 \neq 0, a_3 \neq 0 \) and \( a_4 \neq 0 \). For different values of the damping coefficients \( \zeta_n \), the non-linear parameters \( \eta_n \), \( (n = 1, 2, 3, 4) \) and the non-linear parameters \( \gamma_n, (m = 2, 3, 4) \) the effects are trivial as shown in Figure 3(b). From Figure 3(c) we find that the steady state amplitude of the tool holder is a monotonic increasing function in its excitation amplitude \( F_1 \) with an increase.
in the unstable region of the solution.

The steady state amplitude of the tool holder is a monotonic decreasing function in the natural frequencies \( \omega_n \), \((n = 1, 2, 3, 4)\) with a decrease in the unstable region of the solution as shown in Figure 3(d).

Now the effect of the detuning parameters \( \sigma_2, \sigma_3 \) and \( \sigma_4 \) on the steady state amplitude of the tools \( a_2, a_3 \) and \( a_4 \) is shown in Figure 4(a).

For different values of the damping coefficients \( \zeta_j \), \((i = 6, 7, 8)\), the effects on the steady state amplitudes of the tools are trivial as shown in Figure 4(b). For different values of the non-linear parameters \( \eta_s \), \((s = 5, 6, 7)\), the steady state amplitude of the tools are monotonic increasing as shown in Figure 4(c). Figure 4(d) shows that the steady state amplitude of the tools is a monotonic decreasing function in the natural frequencies \( \omega_m \), \((m = 2, 3, 4)\) and the region of unstable solution is decreasing. For all figures no jump phenomena was observed.

4. Conclusions

The vibrations of a four-degree-of-freedom non-linear mechanical system and absorbers are investigated. The physical motivation for the system stems from applications in ultrasonic machining in which an exciter (machine head) drives tuned blades (absorbers) having both linear and cubic non-linearities. In the present work, we considered multi-tools which allow the machining of different materials and different shapes in different or one workpiece the vibration of ultrasonic machine head can be controlled via non-linear absorbers. Multiple time scale perturbation technique is applied to determine semi-closed form solutions for the coupled deferential equations describing the system up to the second order approximations. To study the stability of the system, both the frequency response equations and the phase-plane technique are applied. From the above study the following may be concluded.

1) Optimum working conditions at \( \Omega \equiv \omega_1 \), \( \omega_1 \equiv \omega_2 \equiv \omega_3 \equiv \omega_4 \), where the vibration of the tool holder is suppressed to about 12.5% of the original amplitude, and the three tools have reasonable amplitudes.

2) For different values of the damping coefficients \( \zeta_n \), the non-linear parameters \( \eta_n \), \((n = 1,2,3,4)\) and the non-linear parameters \( \gamma_m \), \((m = 2, 3, 4)\) we find the effects of these parameters on the steady state amplitude of the tool holder are trivial and same effects have been
obtained for the damping coefficients $\zeta_i$, $(i = 6, 7, 8)$ on the steady state amplitude of the three tools.

3) The steady state amplitude of the tool holder is a monotonic decreasing function in the natural frequencies $\omega_n$, $(n = 1, 2, 3, 4)$ with decreasing in the region of unstable solution.

4) The steady state amplitude of the tool holder is a monotonic increasing function in its excitation amplitude $F_i$ with increasing in the region of unstable solution.

5) The steady state amplitude of the tools is a monotonic decreasing function in the non-linear parameters $\eta_s$, $(s = 5, 6, 7)$ and the natural frequencies $\omega_m$, $(m = 2, 3, 4)$ and the region of unstable solution is decreasing.

6) To make use of machine capability, multi-tools are used to save both time and power.

7) The reported results are in a good agreement with References [3,4] regarding the amplitude reduction.

REFERENCES


Nomenclature

\(c_n\), \((n = 1, 2, 3, 4)\). The damping coefficients of the system and the absorber.

\(k_m\), \((m = 1, 2, 3, 4)\). The stiffness of the system and the absorbers.

\(h_m\), \((m = 1, 2, 3, 4)\). The non-linear parameters of the system and the absorber.

\(F_j, \Omega_j\) \((j = 1, 2, 3)\). The excitation amplitudes and frequencies.

\(m, m,m,m\). The masses of the system and the absorber.

\(\zeta_n = c_n/2m\), \((n = 1, 2, 3, 4)\). The linear damping factors of the system.

\(\zeta_s = c_s/m\). The quadratic damping factors of the system.

\(\eta_m = h_m/m\), \((m = 1, 2, 3, 4)\). The coupling non-linear parameters of the system.

\(\eta_i = h_i/m_1, \eta_i = h_i/m_4\). The non-linear parameters of the absorbers.

\(\omega_s = k_i/m_i\), \((s = 1, 2, 3, 4)\). The natural frequencies of the system and absorbers.

\(\gamma_i = k_i/m_i, \gamma_i = k_i/m_4\). The stiffness of the system.

\(x_i, i = (1, 2, 3, 4)\). Displacement of both system and absorber.

Appendix

\[
\begin{align*}
\Gamma_1 &= (\xi_1 + \xi_2 + \xi_3 + \xi_4), \\
\Gamma_2 &= \frac{\gamma_2a_2}{\omega_2}, \\
\Gamma_3 &= \frac{\gamma_2a_2^3}{2\omega_2} + \frac{3\eta_2a_2^3}{8\omega_2} + \frac{3\eta_2a_2^3}{4\omega_2}, \\
\Gamma_4 &= \frac{\gamma_2a_2}{\omega_2}, \\
\Gamma_5 &= \frac{\gamma_2a_2^3}{2\omega_2} + \frac{3\eta_2a_2^3}{8\omega_2} + \frac{3\eta_2a_2^3}{4\omega_2}, \\
\Gamma_6 &= \frac{\gamma_2a_2^3}{\omega_2}, \\
\Gamma_7 &= \frac{\gamma_2a_2^3}{2\omega_2} + \frac{3\eta_2a_2^3}{8\omega_2} + \frac{3\eta_2a_2^3}{4\omega_2}, \\
\Gamma_8 &= \frac{\gamma_2a_2^3}{\omega_2}, \\
\Gamma_9 &= \frac{\gamma_2a_2^3}{\omega_2}, \\
\Gamma_{10} &= \frac{\gamma_2a_2^3}{2\omega_2} + \frac{3\eta_2a_2^3}{8\omega_2} + \frac{3\eta_2a_2^3}{4\omega_2}, \\
\Gamma_{11} &= \frac{\gamma_2a_2^3}{\omega_2}, \\
\Gamma_{12} &= \frac{\gamma_2a_2^3}{\omega_2}, \\
\Gamma_{13} &= \frac{\gamma_2a_2^3}{\omega_2}, \\
\Gamma_{14} &= \frac{\gamma_2a_2^3}{2\omega_2} + \frac{3\eta_2a_2^3}{8\omega_2} + \frac{3\eta_2a_2^3}{4\omega_2}, \\
\Gamma_{15} &= \frac{\gamma_2a_2^3}{\omega_2}, \\
\Gamma_{16} &= \frac{\gamma_2a_2^3}{\omega_2}, \\
\Gamma_{17} &= \frac{\gamma_2a_2^3}{\omega_2}, \\
\Gamma_{18} &= \frac{\gamma_2a_2^3}{\omega_2}, \\
\Gamma_{19} &= \frac{\gamma_2a_2^3}{\omega_2}, \\
\Gamma_{20} &= \frac{\gamma_2a_2^3}{\omega_2}, \\
\Gamma_{21} &= \frac{\gamma_2a_2^3}{\omega_2}, \\
\Gamma_{22} &= \frac{\gamma_2a_2^3}{\omega_2}, \\
\Gamma_{23} &= \frac{\gamma_2a_2^3}{\omega_2}, \\
\Gamma_{24} &= \frac{\gamma_2a_2^3}{\omega_2}, \\
\Gamma_{25} &= \frac{\gamma_2a_2^3}{\omega_2}, \\
\Gamma_{26} &= \frac{\gamma_2a_2^3}{\omega_2}, \\
\Gamma_{27} &= \frac{\gamma_2a_2^3}{\omega_2}, \\
\Gamma_{28} &= \frac{\gamma_2a_2^3}{\omega_2}, \\
\Gamma_{29} &= \frac{\gamma_2a_2^3}{\omega_2}. 
\end{align*}
\]