No Degeneracy of the Ground State for the Impact Parameter Model

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Received May 25, 2011; revised July 6, 2011; accepted July 13, 2011

Abstract

A time dependent Hamiltonian associated to the impact parameter model for the scattering of a light particle and two heavy ones is considered. Existence and non degeneracy of the ground state is shown.

Keywords: Impact Parameter Model, Non Degeneracy of the Ground State

1. Introduction

In \cite{1,2}, the impact parameter model for the scattering of two heavy particles and a light one is studied, where it is assumed that the heavy particles are infinitely massive and that their motion along a classical trajectory is not affected by the light particle. Also, rigorous proof from first principles of the validity of Massey’s criterion is given \cite{1,3}.

The above mentioned results were proved for a simple Hamiltonian, by means of an adiabatic argumentation. Now we study a more complicated one than in \cite{1}, where a precise knowledge of the discrete spectrum of the corresponding Hamiltonian was needed.

A physical ground state is a state of minimal energy, and therefore it has a relevant role in quantum theories. See for instance \cite{4-17}.

In this work we prove non degeneracy of the ground state for the Hamiltonian

\[ H(t) = -\frac{1}{2} \Delta - \lambda_1 V_1 - \mu_1 V_2 - \lambda_2 V_{1,\rho} - \mu_2 V_{2,\rho}, \]

defined as an operator in the Hilbert space $L^2(\mathbb{R}^n)$ of all complex valued Lebesgue measurable square integrable functions on $\mathbb{R}^n$, with domain $H^2(\mathbb{R}^n)$, the Sobolev space of order two \cite{18}. $\Delta$ is the Laplace operator \cite{11}.

with derivatives in the distribution sense, and, $\lambda_1, \lambda_2, \mu_1, \mu_2$ are positive constants. Also, for $k=1,2$, we will take the potentials $V_k$ of rank one:

\[ V_k \phi = (g_k, \phi) g_k, \quad \forall \phi \in L^2(\mathbb{R}^n), \]

with $g_1, g_2$ fixed elements in $L^2(\mathbb{R}^n)$. Here $(\cdot, \cdot)$ denotes the scalar product in $L^2(\mathbb{R}^n)$, antilinear on the factor on the left. Moreover,

\[ V_{x,\rho} \phi = (g_{x,\rho}, \phi) g_{x,\rho}, \quad g_{x,\rho} (x) := g_x (x - \rho(t)), \]

$\rho(t)$ being a continuous function on $\mathbb{R}$ with values in $\mathbb{R}^n$ satisfying $\rho(0) = 0 \in \mathbb{R}^n$ and

\[ \lim_{\|x\| \to \infty} (\rho(t)) = \infty. \]

We denote by $^\wedge$ the Fourier transform \cite{19}, as an unitary operator in $L^2(\mathbb{R}^n)$:

\[ \hat{g} (p) = \lim_{K \to \infty} \int_{|x| \leq K} e^{ipx} g(x) dx, \quad g \in L^2(\mathbb{R}^n), \]

where the limit is taken in the $L^2$-norm.

2. Main Theorem

From Weyl’s theorem \cite{16}, one knows that for each $t \in \mathbb{R}$, $H(t)$ is a self-adjoint operator with discrete spectrum in $(-\infty, 0)$. The eigenvector corresponding to the infimum of the spectrum of $H(t)$ is called the
ground state for $H(t)$. The following theorem was proved in [20].

**Theorem 2.1.** For $i = 1, 2$, let $g_i \in L^2(\mathbb{R}^n)$ and $\dot{g}_i$ nonnegative functions obeying $|p|\dot{g}_i \in L^2(\mathbb{R})$. Moreover, we suppose the constants $\lambda_i, \mu_i$ in Equation (1) satisfy

$$\lambda_i > \mu_i + \mu_2 > \mu_1 > \lambda_2 > \mu_2 > 0.$$ 

such that $0 < E_0(2) < E_1$ and $0 < E_{\lambda_1} < E_{\lambda_2}$. Here $-E_1, -E_0(2), -E_{\lambda_2},$ and $-E_{\mu_2}$ are the ground state eigenvalues associated to

$$\frac{1}{2} \Delta - \lambda_i V_1, - \frac{1}{2} \Delta - (\mu_i + \mu_2) V_2,$$

respectively. Then the following statements are valid:

1) The eigenvalue $-E_0$, corresponding to the ground state for the operator

$$H(0) = - \frac{1}{2} \Delta - (\lambda_1 + \lambda_2) V_1 - (\mu_1 + \mu_2) V_2,$$

and the eigenvalue $-E_\alpha$, corresponding to the ground state for the operator

$$H(\pm \alpha) = - \frac{1}{2} \Delta - \lambda_i V_1 - \mu V_2,$$

are strictly negative and the inequality $-E_0 < -E_\alpha$ holds.

2) The eigenvalue $-E(t)$, corresponding to the ground state for $H(t)$ for all $t \in \mathbb{R}$ lies in the interval $[-E_0, -E_\alpha]$. 

3) In the interval $(-E_\alpha, -E_1)$ there are no eigenvalues of $H(t)$ for every $t \in \mathbb{R}$.

We mention that for a given function $0 \neq g \in L^2(\mathbb{R}^n)$, one can find a sufficiently large positive constant $\alpha_0$ such that the operator

$$- \frac{1}{2} \Delta - \alpha (g, \cdot) g$$

has a (unique) negative eigenvalue $-E_\alpha$ for $\alpha \geq \alpha_0$. In fact, $-E_\alpha$ is a negative eigenvalue iff [1]

$$\frac{1}{\alpha} \left( \frac{\dot{g}}{|p|^2 + E} \right)^{1/2} = 0,$$

where we denote $p^2 := |p|^2$. Note also that for a given $g$ the right hand side of (5) is a monotone decreasing function of $E$. Therefore, given functions $g_i \in L^2(\mathbb{R}^n)$ one can find constants $\lambda_i, \mu_i (i = 1, 2)$ large enough for the hypotheses of the theorem to hold.

We will prove in this manuscript that under the hypotheses of theorem 2.1, for $t \in \mathbb{R}$ the ground state of $H(t)$ is not degenerate.

Let $-E(t)$ be the ground state eigenvalue of the time dependent operator given by Equation (1). We define

$$\Theta(p) := \frac{p^2}{2} + E(t)$$

and

$$a_\mu = \left( \frac{\dot{g}}{\Theta(p)} \right)^2; \quad d_\mu = \left( \frac{\dot{g}_2}{\Theta(p)} \right)^2$$

for $i = 1, 2$. (6)

Moreover,

$$a_{\mu} = -\left( \dot{g}_i, \Theta^{-1} \dot{g}_{1,\mu} \right),$$

$$b_{\mu} = -\left( \dot{g}_i, \Theta^{-1} \dot{g}_{2,\mu} \right),$$

$$d_{\mu} = -\left( \dot{g}_{2,\rho}, \Theta^{-1} \dot{g}_{1,\rho} \right)$$

(7)

**Lemma 2.1.** Let $-E(t)$ be the ground state eigenvalue of the time dependent operator $H(t)$ given by Equation (1). Then, the matrix equation

$$M = \begin{pmatrix} x \\ y \end{pmatrix} = 0 \in \mathbb{R}^n,$$

has a nontrivial solution. Furthermore

$$\det(D) = \det \begin{pmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{pmatrix} > 0 \quad (\forall t \in \mathbb{R}).$$

**Proof:** Let $\psi(t)$ the eigenvector for $H(t)$ with respective eigenvalue $-E(t)$, then the Fourier transform of $\psi(t)$ is given by

$$\hat{\psi}(t) = \lambda_1 (g_1, \psi) \hat{g}_1(\Theta(p)) + \lambda_2 (g_{1,\rho}, \psi) \hat{g}_{1,\rho}(\Theta(p))$$

$$+ \mu_1 (g_2, \psi) \hat{g}_2(\Theta(p)) + \mu_2 (g_{2,\rho}, \psi) \hat{g}_{2,\rho}(\Theta(p)),$$

(9)

where $\Theta(p) := \frac{p^2}{2} + E(t)$. The Plancherel theorem implies that $(u, v) = (\hat{u}, \hat{v}) \forall u, v \in L^2(\mathbb{R}^n)$. Taking inner products in (9) with $\hat{g}_i$ and $\hat{g}_{1,\rho}$ for $i = 1, 2$, we get
\[
\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 (\hat{g}_1, \hat{\psi}) \\ \lambda_2 (\hat{g}_1, \hat{\psi}) \end{pmatrix} \\
\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \mu_1 (\hat{g}_2, \hat{\psi}) \\ \mu_2 (\hat{g}_2, \hat{\psi}) \end{pmatrix}
\]

(11)

From Theorem 2.1 we deduce the existence of a non-trivial solution to Equation (8).

Now we fix \( E > 0 \). For every \( t \in \mathbb{R} \) let us consider the function,

\[
R_E(t) = \left( \frac{1}{\mu_1} - \frac{\hat{g}_2}{(p^2 + E)^{1/2}} \right) \left( \frac{1}{\mu_2} - \frac{\hat{g}^*_2}{(p^2 + E)^{1/2}} \right)
\]

and observe that for \( E_0(2) < E \),

\[
H(0) = \frac{1}{2} \Delta - (\lambda_1 + \lambda_2) V_1 - (\mu_1 + \mu_2) V_2
\]

is not degenerate.

Proof: Lemma (2.1) assures that \( D^{-1} \) exists. Equation (8) implies,

\[
y = -D^{-1}Bx, \quad (A - B^T D^{-1} B) x = 0.
\]

We take \( C := A - B^T D^{-1} B \), so that,

\[
C = \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix}
\]

where
\[ c_{11} = a_{11} - \frac{\left(b_{12}d_{22} - 2b_1b_2d_{12} + b_{21}d_{11}\right)}{\det D}, \]
\[ c_{12} = a_{12} - \frac{\left(b_1b_2d_{22} + b_1b_2d_{11} - b_2^2d_{12} - b_2b_{12}d_{12}\right)}{\det D}, \]
\[ c_{22} = a_{22} - \frac{\left(b_1^2d_{11} - 2b_1b_2d_{12} + b_{21}^2d_{22}\right)}{\det D}. \]

From Theorem 2.1, we know that there exists a non-trivial solution to system (8). Thus \( \det C = 0 \). Accordingly,
\[ C = \begin{pmatrix} c_{11} & c_{12} \\ k_{c_{11}} & k_{c_{12}} \end{pmatrix}, \] (15)
for some constant \( k = k(t) \). Moreover, for \( t = 0 \) the matrix \( C = C(t) \) is not null. In fact, for this value of \( t \), the following terms simplify
\[ a_{12} = -\left(\frac{p^2}{2} + E_0\right)^{-1/2} \hat{g}_1, \]
\[ d_{12} = -\left(\frac{p^2}{2} + E_0\right)^{-1/2} \hat{g}_2, \]
\[ b_{11} = b_{12} = b_{11} = -\left(\frac{p^2}{2} + E_0\right)^{-1} \hat{g}_2. \]

It follows that,
\[ c_{12} = a_{12} - \frac{b_{11}^2 (d_{11} + d_{22}) - 2d_{12} b_{11}}{\det D} = -\left(\frac{p^2}{2} + E_0\right)^{-1/2} \hat{g}_1 \left(\frac{p^2}{2} + E_0\right)^{-1/2} \hat{g}_1, \]
\[ \left(\hat{g}_1, \hat{g}_2 \right) \left(\frac{p^2}{2} + E_0\right)^{-1} \hat{g}_2 \left(\frac{p^2}{2} + E_0\right)^{-1} \hat{g}_2 \right) \]
\[ \frac{\mu_1 + \mu_2 - 2}{\mu_1 \mu_2} \left(\frac{p^2}{2} + E_0\right)^{-1/2} \hat{g}_2 \left(\frac{p^2}{2} + E_0\right)^{-1/2} \hat{g}_2 \right) \]
\[ \left(\frac{p^2}{2} + E_0\right)^{-1/2} \hat{g}_2 \right)^2 \frac{\mu_1 + \mu_2 - 2}{\mu_1 \mu_2} \left(\frac{p^2}{2} + E_0\right)^{-1/2} \hat{g}_2 \left(\frac{p^2}{2} + E_0\right)^{-1/2} \hat{g}_2 \right) \]
\[ \left(\frac{p^2}{2} + E_0\right)^{-1/2} \hat{g}_2 \right)^2 \frac{\mu_1 + \mu_2 - 2}{\mu_1 \mu_2} \left(\frac{p^2}{2} + E_0\right)^{-1/2} \hat{g}_2 \left(\frac{p^2}{2} + E_0\right)^{-1/2} \hat{g}_2 \right) \]
\[ \left(\frac{p^2}{2} + E_0\right)^{-1/2} \hat{g}_2 \right)^2 \frac{\mu_1 + \mu_2 - 2}{\mu_1 \mu_2} \left(\frac{p^2}{2} + E_0\right)^{-1/2} \hat{g}_2 \left(\frac{p^2}{2} + E_0\right)^{-1/2} \hat{g}_2 \right) \]
\[ < 0, \] (16)
where we use equation (5), the hypothesis \( E_0(2) < E_1 \) and statement (3) of theorem 2.1. Therefore,

Equations (11)-(16) imply,
\[ \mu_i \left(\hat{g}_2, \hat{\psi}\right) = \left(\frac{b_1b_2d_{22} - b_1d_{12} + c_{11}b_1d_{22} - b_2b_{12}d_{12}}{\det D} \right) x_i \]
\[ \mu_j \left(\hat{g}_2, \hat{\psi}\right) = \left(\frac{b_1d_{12} - b_2d_{11} + c_{11}b_{12}d_{11} - b_2b_{12}d_{12}}{\det D} \right) x_i \] (17)

Substitution of these equalities in Equation (9) gives,
\[ \hat{\psi}(0) = x_i \left(\frac{\hat{g}_2 + k_1 \hat{g}_{1,\rho}}{\frac{p^2}{2} + E_0} \right) \]
\[ + k_2 \left(\frac{\hat{g}_2 + k_1 \hat{g}_{1,\rho}}{\frac{p^2}{2} + E_0} \right). \] (18)

Here,
\[ k_1 = -\frac{c_{11}}{c_{12}} \]
\[ k_2 = \frac{\left(c_{12}b_2d_{12} + c_1b_1d_{12}\right) - \left(c_{12}b_2d_{12} + c_1b_1d_{22}\right)}{c_{12} \det D} \]
\[ k_3 = \frac{\left(c_{12}b_1d_{11} + c_1b_2d_{11}\right) - \left(c_{12}b_1d_{11} + c_1b_2d_{22}\right)}{c_{12} \det D}. \] (19)

This determines the vector \( \hat{\psi}(0) \) up to a multiplicatively constant, and from the Plancherel theorem, also the eigenspace associated to the ground state for \( H(0) \), proving the statement of the lemma. \( \square \)

**Theorem 2.2.** Let \( H(t) \) be defined by Equation (1) and suppose the hypotheses of theorem 2.1 hold true. Moreover, we take the curve \( \rho : \mathbb{R} \to \mathbb{R}^n \) so that \( \rho(t) = a + vt, \ \forall |t| \leq M, \) for some positive constant \( M \) and fixed vectors \( a, v \in \mathbb{R}^n \). Then the dimension of the spectral projection onto the interval \( [-E_0, -E_\alpha] \), associated with the selfadjoint operator \( H(t) \), is equal to one for each \( t \in \mathbb{R} \).

**Proof:** The resolvent \( R(A) \) of a self-adjoint operator \( A \) at \( t \in \mathbb{C} \) is defined by \( (iI - A)^{-1} \) with \( I \) denoting the identity operator on \( L^2(\mathbb{R}^n) \). We take \( H_t = H(t_1) \), \( H_1 = H(t_1) \), for two distinct values \( t_1 \) and \( t_2 \) and calculate the difference \( R(H_2) - R(H_1) \).

\[ R(H_2) - R(H_1) = \left[ (iI - H_2^{-1}) - (iI - H_1^{-1}) \right] R(H_1) \]
\[ = \lambda_2 R(H_2) \left[ V_{1,\rho} - V_{2,\rho} \right] R(H_1) \]
\[ + \mu_2 R(H_2) \left[ V_{2,\rho} - V_{1,\rho} \right] R(H_1) \] (20)
Here $V_{1,\rho}$ is given as in Equation (3) with $g_{i,\rho}(x) = g_i(x - \rho(t))$ replaced with $g_{1,\rho}(x) = g_1(x - \rho(t_1))$. Also $V_{1,\rho}, V_{2,\rho^1}, \text{ and } V_{2,\rho^2}$ being defined similarly. It follows from Equation (1) and standard arguments that

$$\| R(\lambda_1) - R(\lambda_2) \| \leq \| Y(t_2) - t_1 \|,$$

where $Y$ is a constant uniform in $t_1, t_2 \in \mathbb{R}/\mathbb{Z}$. Depending on $\| \lambda_1 \|_\ell$ and $\| \lambda_2 \|_\ell$, $\ell = 1, 2$. This implies that $R(\lambda(t))$ is uniformly continuous on $\mathbb{R}$ with respect to the norm topology. Let $P_\lambda(B)$ denote the spectral projection of a self-adjoint operator $B$ corresponding to the Borel set $S \subseteq \mathbb{R}$. By functional calculus, we get

$$P_{\lambda_2} - P_{\lambda_1} \in \mathcal{L}(H(t_2)) \rightarrow P_{\lambda_2} - P_{\lambda_1} \in \mathcal{L}(H(t_1))$$

as $t_2 \rightarrow t_1$ in the operator norm. Therefore, by standard arguments

$$\dim P_{\lambda_2} - \dim P_{\lambda_1} \leq 1$$

For $t_2$ close enough to $t_1$. It follows from Lemma 2.2 that

$$\dim P_{\lambda_2} - \dim P_{\lambda_1} = 1 \quad (\forall t \in \mathbb{R}).$$

Remark: We mention that the hypothesis for the curve $\rho(t)$ can be relaxed to the condition that $\rho(t)$ is asymptotic to a straight line.

3. References


