Existence and Uniqueness of Solution for a Fractional Order Integro-Differential Equation with Non-Local and Global Boundary Conditions

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Abstract

In this paper, we prove an important existence and uniqueness theorem for a fractional order Fredholm – Volterra integro-differential equation with non-local and global boundary conditions by converting it to the corresponding well known Fredholm integral equation of second kind. The considered problem in this paper has been solved already numerically in [1].

Keywords: Fractional Order Integro-Differential Equation, Non-Local Boundary Conditions, Fundamental Solution

1. Introduction

Let’s consider a problem under boundary condition containing non-local and global terms for a fractional order integro-differential equation

\[ D^q y(x) = f(x) + \int_a^x K_1(x,t)y(t)dt + \int_a^b K_2(x,t)y(t)dt, \]

where \( m-q \in [0,1] \), \( f(x) \), \( K_1(x,t) \), \( K_2(x,t) \) and \( H_i(t) \), \( i=1,m \) are continuous, real-valued functions, \( \gamma_i \), \( \eta_i \), \( \lambda_i \) and \( d_i \), \( i=1,m \), \( j=1,m \) are real constants, and boundary conditions (2) are linearly independent.

2. Existence and Uniqueness of Solution

Theorem. Let the functions \( f(x) \), \( K_j(x,t), j=1,2 \) and \( H_i(t), \ i=1,m \) are continuous, \( \gamma_i \), \( \eta_i \), \( \lambda_i \) and \( d_i \), \( i,j=1,m \) are real constant, the boundary conditions (2) are linearly independent and condition (15) is satisfied. Then the boundary value problem (1)-(2) has unique solution.

Proof: Acting in Equation (1) by fractional order derivative operator \( D^{m-q} \) [2], we get

\[ D^{m-q}D^q y(x) = D^{m-q} f(x)D^{m-q} \left( \int_a^x K_1(x,t)y(t)dt \right) + D^{m-q} \int_a^b K_2(x,t)y(t)dt, \]

since \( D^{m-q}D^q y(x) = D^m y(x) \), then we get the equation

\[ D^m y(x) = F(x) + \int_a^x M_1(x,t)y(t)dt + \int_a^b M_2(x,t)y(t)dt, \]

where \( m-q \in [0,1] \), \( f(x) \), \( K_1(x,t) \), \( K_2(x,t) \) and \( H_i(t) \), \( i=1,m \) are continuous, real-valued functions, \( \gamma_i \), \( \eta_i \), \( \lambda_i \) and \( d_i \), \( i=1,m \), \( j=1,m \) are real constants, and boundary conditions (2) are linearly independent.
where

\[
\begin{align*}
F(x) &= D^{m-q} f(x) = \frac{d}{dx} \int_x^{x-q} f(\zeta) d\zeta, \\
M_1(x,t) &= \frac{d}{dx} \int_x^{x-q} K_1(\zeta,t) d\zeta, \\
M_2(x,t) &= \frac{d}{dx} \int_x^{x-q} K_2(\zeta,t) d\zeta.
\end{align*}
\] (4)

Now, we write Equation (3) in the general form

\[
D^m y(x) = G(x,y),
\] (3.1)

and accept that \( G(x,y) \) is known, then the fundamental solution (see [3]) is in the form

\[
Y(x-\eta) = (x-\eta)^{-m+1}\theta(x-\eta). \tag{5}
\]

where

\[
-\sum_{k=0}^{m-1} (-1)^k \left[ D^{m-k} y(x) \right]^{(k)}(x-\eta) \bigg|_{k=a}^{b} + \int_a^b G(x,y) Y(x-\eta) dx = (-1)^m \begin{cases} y(\eta), & \eta \in (a,b), \\
\frac{1}{2} y(\eta), & \eta = a, \eta = b. \end{cases} \tag{8}
\]

Hence, the first expressions for the necessary conditions are obtained in the form

\[
\begin{align*}
\frac{(-1)^m}{2} y(a) &= -\sum_{k=0}^{m-1} (-1)^k \left[ D^{m-k} y(x) \right]^{(k)}(x-a) \bigg|_{k=a}^{b} + \int_a^b G(x,y) Y(x-a) dx, \\
\frac{(-1)^m}{2} y(a) &= -\sum_{k=0}^{m-1} (-1)^k \left[ D^{m-k} y(x) \right]^{(k)}(x-b) \bigg|_{k=a}^{b} + \int_a^b G(x,y) Y(x-b) dx.
\end{align*} \tag{9}
\]

It is easy to see that the second expression in (9) turns into an identity. Indeed, as it is seen from (5)-(6), the integral at the right side of the second condition contains the value of the function \( \theta(x-\eta) \), which is zero for \( \eta = b \). For \( x = a \) the summation in the second expression contains the Heaviside function which is zero for \( s = 0, m-1 \).

Finally, the first summand contains positive degrees of \( (x-\eta) \) for \( s = 0, m-2 \) these terms become zero at \( x = \eta = b \). Here, for \( s = m-1 \), the expression of fundamental solution for \( s = m-1 \) yields the Heaviside function. For \( x = b, \eta = b \) this becomes \( \frac{1}{2} \), therefore, the second one of necessary conditions (9) turns into identity.

Now, we construct the second basic expression to get the second group of necessary conditions. For that, we multiply both sides of (3) by the derivative of (5) and integrate on \( (a,b) \) [6,7]:

\[
\int_a^b D^m y(x) Y'(x-\eta) dx = \int_a^b G(x,y) Y'(x-\eta) dx.
\]

Integrating by parts on the left side of the obtained expression and taking into account (5) and (6), we get the second basic relation as follows:

\[
-\sum_{k=0}^{m-1} (-1)^k \left[ D^{m-k} y(x) \right]^{(k+1)}(x-\eta) \bigg|_{k=a}^{b} + \int_a^b G(x,y) Y'(x-\eta) dx = (-1)^{m-1} \begin{cases} y'(\eta), & \eta \in (a,b), \\
\frac{1}{2} y'(\eta), & \eta = a, \eta = b, \end{cases} \tag{10}
\]

and so the second group of the necessary conditions are obtained as
Similar to the second expression of (9), we can show that the second expression of (11) turns into identity. If we continue this process, in order to get the \(m\)-th basic relation, we multiply (3) the \((m-1)\)-th order derivative of (5) and integrate on \((a,b)\) to get:

\[
\int_a^b D^m y(x) Y^{(m-1)}_x (x-\eta) \, dx = \int_a^b G(x,y) Y^{(m-1)}_x (x-\eta) \, dx.
\]

Here, once integrating by parts on the left side of the obtained expression gives

\[
D^m y(x) Y^{(m-1)}_x (x-\eta) \bigg|_{x=a}^{x=b} - \int_a^b D^m y(x) Y^{(m-1)}_x (x-\eta) \, dx = \int_a^b G(x,y) Y^{(m-1)}_x (x-\eta) \, dx.
\]

Thus, if we take into account that (5) is the fundamental solution, the last relation \((m\)-th\) will be as follows:

\[
-\frac{1}{2} Y^{(m-1)}(a) = -D^m y(x) Y^{(m-1)}_x (x-a) \bigg|_{x=a}^{x=b} + \int_a^b G(x,y) Y^{(m-1)}_x (x-a) \, dx,
\]

\[
-\frac{1}{2} Y^{(m-1)}(b) = -D^m y(x) Y^{(m-1)}_x (x-b) \bigg|_{x=a}^{x=b} + \int_a^b G(x,y) Y^{(m-1)}_x (x-b) \, dx,
\]

here, as above, the second necessary condition turns into identity.

Now, we join to the given \(m\) linearly independent boundary condition (7), the necessary conditions in (9), (11) and etc. (13) that are not identities, and write the system of \(2m\) linear algebraic equations obtained with respect to the boundary values of the unknown function in the following way.

\[
\begin{align*}
\gamma_{11} y(a) + \gamma_{12} y'(a) + \cdots + \gamma_{1m} y^{(m-1)}(a) + \eta_{11} y(b) + \eta_{12} y'(b) + \cdots + \eta_{1m} y^{(m-1)}(b) &= d_1 - \Lambda_1 \int_a^b H_1(t) y(t) \, dt, \\
& \quad \vdots \\
\gamma_{m1} y(a) + \gamma_{m2} y'(a) + \cdots + \gamma_{mm} y^{(m-1)}(a) + \eta_{m1} y(b) + \eta_{m2} y'(b) + \cdots + \eta_{mm} y^{(m-1)}(b) &= d_m - \Lambda_m \int_a^b H_m(t) y(t) \, dt, \\
(-1)^m y(a) + y'(a) + \cdots + y^{(m-1)}(a) - (-1)^m y(b) - (-1)^m Y^{(m-2)}(b-a) y^{(m-1)}(b) &= \int_a^b G(x,y) Y(x-a) \, dx, \\
& \quad \vdots \\
y(a) + (-1)^m y'(a) + y^{(m-1)}(a) - y(b) - (-1)^m y'(b) - (-1)^m Y^{(m-2)}(b-a) y^{(m-1)}(b) - (-1)^m y'(b-a) y^{(m-1)}(b) &= \int_a^b G(x,y) Y'(x-a) \, dx, \\
& \quad \vdots \\
(-1) y^{(m-1)}(a) - y(b) - \cdots - y^{(m-2)}(b) - (-1)^m Y^{(m-1)}(b-a) y^{(m-1)}(b) &= \int_a^b G(x,y) Y^{(m-1)}(x-a) \, dx,
\end{align*}
\]
For solving the system (14) by the Cramer’s rule, it is necessary that its basic determinant differ from zero.

Accept that the following condition is satisfied

\[
\begin{vmatrix}
\gamma_{11} & \gamma_{12} & \cdots & \gamma_{1m} & \eta_{11} & \eta_{12} & \cdots & \eta_{1m} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\gamma_{n1} & \gamma_{n2} & \cdots & \gamma_{nm} & \eta_{n1} & \eta_{n2} & \cdots & \eta_{nm} \\
(-1)^m & 0 & \cdots & 0 & (-1)^m & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & -1 & 0 & \cdots & \cdots & (-1)^{m-1}(b-a) \\
\end{vmatrix} \neq 0,
\]

(15)

Then, from system (14), we get

\[
y^{(l-1)}(a) = \frac{1}{\Delta} \sum_{x=1}^{n} d_x - \lambda_x \int_a^b H_x(t) y(t) dt \Delta^{(x, k)} + \sum_{x=1}^{m} \int_a^b G(x, y) y^{(l-1)}(x-a) dx \Delta^{(m, x, k)},
\]

(16)

where \(\Delta^{(p, q)}\) denotes the cofactor of the elements at the intersection of \(p\)-th row and \(q\)-th column of the determinant \(\Delta\). Calculate the following expression:

\[
\int_a^b G(x, y) Y^{(l)}(x-a) dx = \int_a^b F(x) + \int_a^b M_1(x, t) y(t) dt + \int_a^b M_2(x, t) y(t) dt + \int_a^b [G(x, y) Y^{(l-1)}(x-a) dx] \Delta^{(m, x, k)},
\]

Then, we get:

\[
\int_a^b G(x, y) Y^{(l)}(x-a) dx = \tilde{F}_y + \int_a^b M_1(t) y(t) dt,
\]

(17)

and so,

\[
\begin{align*}
\tilde{F}_y &= \int_a^b F(x) Y^{(l)}(x-a) dx, \\
\tilde{M}_1(t) &= \int_a^b y^{(l)}(x-a) M_2(x, t) dx + \int_a^b Y^{(l)}(x-a) M_1(x, t) dx.
\end{align*}
\]

(18)

Finally, coming back to (8), we take into account (16) and (17) and write the second kind Fredholm type integral equation [8] for which the boundary value problem (1)-(2) is reduced to:

\[
y(\eta) = A(\eta) + \int_a^b B(\eta, t) y(t) dt,
\]

(19)

where

\[
A(\eta) = \sum_{s=0}^{m-1} (-1)^{s+m+1} Y^{(s)}(b-\eta) \Delta \sum_{k=1}^{m} \frac{d_k}{\Delta} \Delta^{(k, 2m-s)} + \sum_{s=0}^{m-1} (-1)^{s+m+1} Y^{(s)}(b-\eta) \Delta \sum_{k=1}^{m} \frac{\Delta^{(m, k, 2m-s)}}{\Delta} \tilde{F}_{k-1} + \int_a^b F(x) Y(x-\eta) dx,
\]

(20)
\[ B(\eta, t) = \sum_{s=0}^{m-1} (-1)^{s+m} y^{(s)}(b - \eta) \sum_{k=1}^{m} \frac{d_k}{\Delta^{(k,m-s)}} H_k(t) + \sum_{s=0}^{m-1} (-1)^{s+m+1} y^{(s)}(b - \eta) \sum_{k=1}^{m} \frac{\Delta^{(m+k,2m-s)}}{\Delta} \tilde{M}_{k,s}(t) \]
\[ + \sum_{s=0}^{m-1} (-1)^{s+m+1} y^{(s)}(a - \eta) \sum_{k=1}^{m} \frac{d_k}{\Delta^{(k,m-s)}} H_k(t) + \sum_{s=0}^{m-1} (-1)^{s+m} y^{(s)}(a - \eta) \sum_{k=1}^{m} \frac{\Delta^{(m+k,2m-s)}}{\Delta} \tilde{M}_{k,s}(t) \]
\[ + \int_a^b Y(x - \eta) M_1(x,t) \, dx + \int_a^b Y(x - \eta) M_2(x,t) \, dx. \]

By the hypothesis of theorem on the functions \( f(x), K_j(x,t), j = 1, 2 \) and \( H_i(t), i = 1, m \) the integral Equation (19) has unique solution and so in all conducted operations we can come back and we conclude that the solution of (19) is the unique solution of boundary value problem (1)-(2).

3. References


