Generalization of Certain Subclasses of Multivalent Functions with Negative Coefficients Defined by Cho-Kwon-Srivastava Operator

Elsayed A. Elrifai, Hanan E. Darwish, Abdusalam R. Ahmed
Department of Mathematics, Faculty of Science, University of Mansoura, Mansoura, Egypt
E-mail: Rifai@mans.edu.eg, Darwish333@yahoo.com, Abdusalam5056@yahoo.com
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Abstract

Making use of the Cho-Kwon-Srivastava operator, we introduce and study a certain SCn(j, p, λ, α, δ) of p-valently analytic functions with negative coefficients. In this paper, we obtain coefficient estimates, distortion theorem, radii of close-to-convexity, starlikeness, convexity and modified Hadamard products of functions belonging to the class SCn(j, p, λ, α, δ). Finally, several applications investigate an integral operator, and certain fractional calculus operators also considered.

Keywords: Multivalent Functions, Cho-Kwon-Srivastava Operator, Modified-Hadamard Product, Fractional Calculus

1. Introduction

Let T(j, p) denote the class of functions of the form:

\[ f(z) = z^p - \sum_{k=j+1}^{p} a_k z^k \quad (a_k \geq 0; \; p, \; j \in \mathbb{N} = \{1, 2, 3, \ldots\}), \]

(1.1)

which are analytic and p-valent in the open unit disc \( U = \{ |z| < 1 \} \). A function \( f(z) \in T(j, p) \) is said to be p-valently starlike of order \( \alpha \) if it satisfies the inequality:

\[ \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in U; 0 \leq \alpha < p; \; p \in \mathbb{N}). \]  

(1.2)

We denote by \( T^*_j(p, \alpha) \) the class of all \( p \)-valently starlike functions of order \( \alpha \). Also a function \( f(z) \in T(j, p) \) is said to be \( p \)-valently convex of order \( \alpha \) if it satisfies the inequality:

\[ \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in U; 0 \leq \alpha < p; \; p \in \mathbb{N}). \]

(1.3)

We denote by \( C_j(p, \alpha) \) the class of all \( p \)-valently convex functions of order \( \alpha \). We note that (see for example Duren [1] and Goodman [2])

\[ f(z) \in C_j(p, \alpha) \iff \frac{zf''(z)}{f'(z)} \in T^*_j(p, \alpha) \]

(1.4)

\( 0 \leq \alpha < p; \; p \in \mathbb{N} \).

The classes \( T^*_j(p, \alpha) \) and \( C_j(p, \alpha) \) are studied by Owa [3].

In [4] Wang et al. defined Cho-kwon-Srivastava operator which

\[ \ell^k_{p,j}(a,c)f(z) = T_j(p,\alpha)f(z) \]

by

\[ \ell^k_{p,j}(a,c)f(z) = z^p - \sum_{k=j+1}^{p} \frac{(\lambda + p)k_p(c)k_p}{(k-p)!a_k}z^k \]

(1.5)

for \( (a,c) \in \mathbb{R}/Z_0^* = \{0,-1,-2,-3,\ldots\}, \; z \in U, \; \lambda > -p \)

and

\[ (\lambda)_k = \begin{cases} 1 & \text{if } k = 0 \\ \lambda(\lambda+1)\cdots(\lambda+k-1) & \text{if } k \in \mathbb{N}. \end{cases} \]

Clearly, \( \ell^k_{p,j}(a,c) \) is the well-known Cho-kwon-Srivastava operator (see [5]) where

\[ \ell^k_{p,j}(p+1,1) = f(z), \; \ell^1_{p,j}(p,1) = \frac{zf''(z)}{f'(z)}. \]
and

$$\ell^+_{p,j}(a,a) = D^{\lambda + p-1} f(z) \left( \lambda > -p \right),$$

where $D^{\lambda + p-1}$ is the well-known Ruscheweyh derivative of $(\lambda + p - 1)$-th order.

With the help of the Cho-Kwon-Srivastava $\ell^+_{p,j}(a,c,f(z))$, we say that a function $f(z)$ belonging to $T(j,p)$ is in the class $SC_n(j,p, \lambda, \alpha, \delta)$ if and only if

$$\text{Re} \left\{ \frac{z\ell^+(a,c,f(z))'}{\ell^+(a,c,f(z))} \right\} > \alpha \quad (p \in N, j \in N_0)$$

(1.6)

We note that:

1) when $\delta = 0$, we have

$$\text{Re} \left\{ \frac{z\ell^+(a,c,f(z))'}{\ell^+(a,c,f(z))} \right\} > \alpha$$

which is the class of starlike of order $\alpha$.

2) when $\delta = 0$, $a = p + 1$, $\lambda = 1$, $c = 1$, we have the class

$$\text{Re} \left\{ \frac{z\ell^+(a,c,f(z))'}{f(z)} \right\} > \alpha; 0 \leq \alpha < p$$

which is the class of starlike functions of order $\alpha$ studied by Owa [3] and Yamakawa [6].

3) when $\delta = 1$, we have

$$\text{Re} \left\{ \frac{z(\ell^+(a,c,f(z))')}{(\ell^+(a,c,f(z))'} \right\} > \alpha; 0 \leq \alpha \leq p$$

which is the class of convex operator of order $\alpha$.

4) when $\delta = 1$, $a = p + 1$, $c = 1$, $\lambda = 1$, we have

$$\text{Re} \left\{ \frac{z\ell^+(a,c,f(z))'}{f(z)} \right\} > \alpha; (0 \leq \alpha < p)$$

which is the class of convex functions of order $\alpha$ studied by Owa [3] and Yamakawa [6].

In our present paper, we shall make use of the familiar $c \in \mathbb{C}$ defined by (c.f. [7,8], see also [9])

$$\int_{-\infty}^{\infty} f(z) \int_{-\infty}^{\infty} f(t) \text{d}t, \quad (1.7)$$

as well as the fractional calculus operator $D_\delta^\mu$ for which it is well known that (see, for details, [10,11]; see also Section 5 below)

$$D_\delta^\mu \{z^\alpha\} = \frac{\Gamma(\rho + 1)}{\Gamma(\rho + 1 - \mu)} z^{\rho - \mu}, \quad (\rho > -1; \mu \in \mathbb{R})$$

(1.8)

in terms of Gamma functions.

2. Coefficient Estimates

**Theorem 1.** Let the function $f(z)$ defined by (1.1). Then $f(z) \in SC_n(j,p, \lambda, \alpha, \delta)$ if and only if

$$\sum_{k=0}^{\infty} \frac{(\lambda + p)_k}{(k-p)!(a)_k} \left[ 1 + \delta(k-1) \right] a_k \leq (p-\alpha) \left[ 1 + \delta(p-1) \right]$$

(2.1)

$$z \in U; 0 \leq \alpha < p, 0 \leq \delta \leq 1, p, j \in N, n \in N_0).$$

**Proof.** Assume that the inequality (2.1) holds true. Then we have

$$(j+p-\alpha) \sum_{k=j+p}^{\infty} \frac{(\lambda + p)_k}{(k-p)!(a)_k} \left[ 1 + \delta(k-1) \right] a_k \leq \sum_{k=j+p}^{\infty} \frac{(\lambda + p)_k}{(k-p)!(a)_k} (k-\alpha) \left[ 1 + \delta(k-1) \right] a_k \leq (p-\alpha) \left[ 1 + \delta(p-1) \right]$$

that is, that

$$\sum_{k=j+p}^{\infty} \frac{(\lambda + p)_k}{(k-p)!(a)_k} \left[ 1 + \delta(k-1) \right] a_k \leq \frac{(p-\alpha) \left[ 1 + \delta(p-1) \right]}{j+p-\alpha}.$$
Then we find that
\[
\frac{\left| z\left(\ell_{p,j}^\alpha(a,c)f(z)\right)' + \delta z^2\left(\ell_{p,j}^\alpha(a,c)f(z)\right)'' \right|}{(1-\delta)\ell_{p,j}^\alpha(a,c)f(z) + \delta z\left(\ell_{p,j}^\alpha(a,c)f(z)\right)'} - p < \sum_{k,j,p} \frac{(\lambda+p)_k(a)_{k-p}}{(k-p)!a_{k-p}} \left[1+\delta(k-1)\right]a_k z^{k-p}
\]
\[
\leq \frac{\left|1+\delta(k-1)\right]a_k z^{k-p}}{1+\delta(k-1)} - \sum_{k,j,p} \frac{(\lambda+p)_k(a)_{k-p}}{(k-p)!a_{k-p}} \left[1+\delta(k-1)\right]a_k z^{k-p}
\]
\[
\leq \frac{\sum_{k,j,p} (\lambda+p)_k(a)_{k-p}(k-p)!a_{k-p} \left[1+\delta(k-1)\right]a_k z^{k-p}}{1+\delta(k-1)} - \sum_{k,j,p} \frac{(\lambda+p)_k(a)_{k-p}}{(k-p)!a_{k-p}} \left[1+\delta(k-1)\right]a_k z^{k-p} \leq p - \alpha.
\]
(2.2)

This shows that the values of the function
\[
\phi(z) = \frac{z\left(\ell_{p,j}^\alpha(a,c)f(z)\right)' + \delta z^2\left(\ell_{p,j}^\alpha(a,c)f(z)\right)''}{(1-\delta)\ell_{p,j}^\alpha(a,c)f(z) + \delta z\left(\ell_{p,j}^\alpha(a,c)f(z)\right)'}
\]
(2.3)
lie in a circle which is centered at \(w = p\) and whose radius is \((p-\alpha)\). Hence \(f(z)\) satisfies the condition (1.6).

Conversely, assume that the function \(f(z)\) is in the class \(SC_n(j,p,\lambda,\alpha,\delta)\). Then we have
\[
\text{Re}\left\{ \frac{z\left(\ell_{p,j}^\alpha(a,c)f(z)\right)' + \delta z^2\left(\ell_{p,j}^\alpha(a,c)f(z)\right)''}{(1-\delta)\ell_{p,j}^\alpha(a,c)f(z) + \delta z\left(\ell_{p,j}^\alpha(a,c)f(z)\right)'} \right\} = \text{Re}\left\{ \frac{p[1+\delta(p-1)] - \sum_{k,j,p} (\lambda+p)_k(a)_{k-p} k\left[1+\delta(k-1)\right]a_k z^{k-p}}{1+\delta(p-1)} - \sum_{k,j,p} (\lambda+p)_k(a)_{k-p} k\left[1+\delta(k-1)\right]a_k z^{k-p} \right\} > \alpha
\]
(2.4)

for some \(\alpha(0 \leq \alpha < p)\), some \(\delta(0 \leq \delta \leq 1)\), \(p, j \in N\), \(n \in N_0\), and \(z \in U\). Choose values of \(z\) on the real axis so that \(\phi(z)\) given by (2.3) is real. Upon clearing the denominator in (2.4) and letting \(z \to 1^-\) through real values, we can see that
\[
p[1+\delta(p-1)] - \sum_{k,j,p} (\lambda+p)_k(a)_{k-p} k\left[1+\delta(k-1)\right]a_k
\]
\[
\geq \alpha\left\{1+\delta(p-1)] - \sum_{k,j,p} (\lambda+p)_k(a)_{k-p} k\left[1+\delta(k-1)\right]a_k \right\}.
\]
(2.5)
Thus we have the inequality (2.1).

**Corollary 1.** Let the function \( f(z) \) defined by (1.1) be in the class \( SC_n(j,p,\lambda,\alpha,\delta) \). Then
\[
a_k \leq \frac{(p-\alpha)[1+\delta(p-1)]}{(\lambda+p)_{k+p}(c)_{k+p}(k-\alpha)[1+\delta(k-1)]}(p-m)!.
\]
(2.6)

The result is sharp for the function \( f(z) \) given by
\[
f(z) = z^p - \frac{(p-\alpha)[1+\delta(p-1)]}{(\lambda+p)_{k+p}(c)_{k+p}(k-\alpha)[1+\delta(k-1)]}z^k
\]
\[(k \geq j+p, p, j \in N, n \in N_0).\] (2.7)

### 3. Distortion Theorem

**Theorem 2.** If a function \( f(z) \) defined by (1.1) is in the class \( SC_n(j,p,\lambda,\alpha,\delta) \) then
\[
f(z) = z^p - \frac{(p-\alpha)[1+\delta(p-1)]}{(\lambda+p)_{k+p}(c)_{k+p}(k-\alpha)[1+\delta(k-1)]}z^{j+p}(p, j \in N, n \in N_0).
\]

Proof. In view of Theorem 1, we have
\[
f(z) = z^p - \frac{(p-\alpha)[1+\delta(p-1)]}{(\lambda+p)_{j+p}(c)_{j+p}(j+\alpha)[1+\delta(j+\alpha)]}z^j.
\]
(3.1)
The result is sharp for the function \( f(z) \) given by
\[
f(z) = z^p - \frac{(p-\alpha)[1+\delta(p-1)]}{j!(a_j)(j+p)}(j+\alpha)[1+\delta(j+\alpha)]
\]
(3.2)

### 4. Radii of Close-to-Convexity, Starlikeness and Convexity

**Theorem 3.** Let the function \( f(z) \) defined by (1.1) be in the class \( SC_n(j,p,\lambda,\alpha,\delta) \) then
1. \( f(z) \) is \( p \)-valently close-to-convex of order

Proof. In view of Theorem 1, we have
\[
f(z) = z^p - \frac{(p-\alpha)[1+\delta(p-1)]}{(\lambda+p)_{j+p}(c)_{j+p}(j+\alpha)[1+\delta(j+\alpha)]}z^{j+p}(p, j \in N, n \in N_0).
\]
(3.3)

Now, by differentiating both sides of (1.1) \( m \) times, we obtain
\[
f^{(m)}(z) = \frac{(p-\alpha)[1+\delta(p-1)]}{j!(a_j)(j+p)}(j+\alpha)[1+\delta(j+\alpha)]z^{j+p}.
\]
(3.4)

Finally, it is easy to see that the bounds in (3.1) are attained for the function \( f(z) \) given by (3.2).
\[
\phi(0 \leq \phi < p) \quad \text{in} \quad |z| < r_i, \quad \text{where}
\]
\[
r_i = \inf_k \left\{ \frac{\left(\lambda + p\right)_{k-p} c_{k-p} (k-\alpha) [1+\delta(k-1)]}{(k-p)! (p-\alpha) [1+\delta(p-1)]} \left( \frac{p-\phi}{k} \right) \right\}^{1/p}, \quad (k \geq j + p, p, j \in N, n \in N_0),
\]
(4.1)

2) \( f(z) \) is \( p \)-valently starlike of order \( \phi(0 \leq \phi < p) \) in \( |z| < r_2 \), where
\[
r_2 = \inf_k \left\{ \frac{\left(\lambda + p\right)_{k-p} c_{k-p} (k-\alpha) [1+\delta(k-1)]}{(k-p)! (p-\alpha) [1+\delta(p-1)]} \left( \frac{p-\phi}{k} \right) \right\}^{1/p}, \quad (k \geq j + p, p, j \in N)
\]
(4.2)

3) \( f(z) \) is \( p \)-valently convex of order \( \phi(0 \leq \phi < p) \) in \( |z| < r_3 \), where
\[
r_3 = \inf_k \left\{ \frac{\left(\lambda + p\right)_{k-p} c_{k-p} (k-\alpha) [1+\delta(k-1)]}{(k-p)! (p-\alpha) [1+\delta(p-1)]} \left( \frac{p(p-\phi)}{k(k-\phi)} \right) \right\}^{1/p}, \quad (k \geq j + p, j, p \in N).
\]
(4.3)

Each of these results is sharp for the function \( f(z) \) given by (2.7).

**Proof.** It is sufficient to show that
\[
\left| \frac{f'(z)}{z^{p-1} - \phi} \right| \leq p - \phi \quad (|z| < r_1; 0 \leq \phi < p, p \in N),
\]
(4.4)

\[
\left| \frac{zf''(z)}{f(z)} - p \right| \leq p - \phi \quad (|z| < r_2; 0 \leq \phi < p, p \in N),
\]
(4.5)

and
\[
\left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \leq p - \phi \quad (|z| < r_3; 0 \leq \phi < p, p \in N),
\]
(4.6)

for a function \( f(z) \in \text{SC}_{n}(j, p, \lambda, \alpha, \delta) \) where \( r_1, r_2 \) and \( r_3 \) are defined by (4.1) - (4.3) respectively. The details involved are fairly straightforward and may omitted.

### 5. Modified Hadamard Products

For the functions \( f_i(z)(i=1,2) \) defined by
\[
f_i(z) = z^p - \sum_{k=j+p}^{\infty} a_{i,k} z^k \quad (a_{i,k} \geq 0; i=1,2),
\]
(5.1)

we denote by \( (f_1 * f_2)(z) \) the modified Hadamard product (or convolution) of the functions \( f_1(z) \) and \( f_2(z) \), defined by
\[
(f_1 * f_2)(z) = z^p - \sum_{k=j+p}^{\infty} a_{1,k} a_{2,k} z^k.
\]
(5.2)

**Theorem 4.** Let the functions \( f_i(z)(i=1,2) \) defined by (5.1) be in the class \( \text{SC}_{n}(j, p, \lambda, \alpha, \delta) \) then \( (f_1 * f_2)(z) \in \text{SC}_{n}(j, p, \lambda, \gamma, \delta) \), where
\[
\gamma = p - \frac{j(p-\alpha)^2 [1+\delta(p-1)]}{j! (a_j)} \frac{(\lambda + p)^{j} [1+\delta(j+p-1)]}{(j+p-\alpha)^{j} [1+\delta(j+p-1)]}.
\]
(5.3)

The result is sharp for the functions \( f_i(z)(i=1,2) \) given by
\[
f_i(z) = z^p - \frac{(p-\alpha)[1+\delta(p-1)]}{j! (a_j) (j+p-\alpha)[1+\delta(j+p-1)]} z^{j+p}, \quad (p, j \in N, i=1,2).
\]
(5.4)
Proof. Employing the technique used earlier by Schild and Silverman [12], we need to find the largest \( \gamma \) such that
\[
\sum_{k=p+1}^{N} \frac{(\lambda + p)_{k-p}(c)_{k-p}(k-\gamma)^{1+\delta(k-1)}}{(k-p)!(a)_{k-p}(p-\gamma)^{1+\delta(p-1)}} a_{k,\gamma} \leq 1.
\]
(5.5)
Since \( f_i(z) \in SC_n(j, p, \lambda, \alpha, \delta)(i=1,2) \), we readily see that
\[
\sum_{k=p+1}^{N} \frac{(\lambda + p)_{k-p}(c)_{k-p}(k-\alpha)^{1+\delta(k-1)}}{(k-p)!(a)_{k-p}(p-\alpha)^{1+\delta(p-1)}} a_{k,j} \leq 1 \quad (i=1,2).
\]
(5.6)
Therefore, by the Cauchy-Schwarz inequality, we obtain
\[
\frac{(p-\alpha)^{1+\delta(p-1)}}{(\lambda + p)_{k-p}(c)_{k-p}(k-\alpha)^{1+\delta(k-1)}} \leq \frac{(k-\gamma)(k-\alpha)}{(p-\alpha)(p-\gamma)} \quad (k \geq j + p, p, j \in \mathbb{N}).
\]
(5.7)

It follows from (5.10) that
\[
\gamma \leq \frac{p(\lambda + p)(p-\alpha)^{1+\delta(p-1)}}{(k-\gamma)(k-\alpha)^{1+\delta(k-1)}-(p-\alpha)^{1+\delta(p-1)}} \quad (k \geq j + p, j, p \in \mathbb{N}).
\]
(5.11)
Now, defining the function \( G(k) \) by
\[
G(k) = \frac{(k-\gamma)(k-\alpha)^{1+\delta(k-1)}-(p-\alpha)^{1+\delta(p-1)}}{(\lambda + p)_{k-p}(c)_{k-p}(k-\alpha)^{1+\delta(k-1)}-(p-\alpha)^{1+\delta(p-1)}} \quad (k \geq j + p, j, p \in \mathbb{N}).
\]
(5.12)
We see that \( G(k) \) is an increasing function of \( k \). Therefore, we conclude that
\[
\gamma \leq G(j + p) = \frac{j(p-\alpha)^{1+\delta(p-1)}}{(j + p - \alpha)^{1+\delta(j + p - \alpha)}-(p-\alpha)^{1+\delta(p-1)}}.
\]
(5.13)
The result is sharp.

Remark: Putting 1) \( a = p + 1, \lambda = 1, c = 1, \delta = 0 \) and 2) \( a = p + 1, \lambda = 1, c = 1, \delta = 1 \) in Theorem 4, we obtain

Corollary 2. Let the functions \( f_i(z)(i=1,2) \) defined by (5.1) be in the class \( T_j(p, \alpha) \). Then \( (f_i * f_i)(z) \in T_j(p, \gamma) \), where
\[
\gamma = p - \frac{j(p-\alpha)^2}{(j + p - \alpha)^2-(p-\alpha)^2}.
\]
(5.14)
The result is sharp.

Corollary 3. Let the functions \( f_i(z)(i=1,2) \) defined by (5.1) be in the class \( C_j(p, \alpha) \). Then \( (f_i * f_i)(z) \in C_j(p, \gamma) \), where
\[
\gamma = p - \frac{j(p-\alpha)^2 p}{(j + p - \alpha)^2(1+j + p - \alpha)-(p-\alpha)^2 p}.
\]
(5.15)
The result is sharp.

Using arguments similar to those in the proof of
Theorem 4, we obtain the following result.

**Theorem 5.** Let the function \( f_1(z) \) defined by (5.1) be in the class \( SC_j(p,\lambda,\alpha,\delta) \). Suppose also that the function \( f_2(z) \) defined by (5.1) be in the class \( SC_j(p,\lambda,\alpha,\delta) \). Then

\[
\zeta = p - \frac{j(p-\alpha)(p-\tau)[1+\delta(p-1)]}{(\lambda+p)_{k-p}(c)_{k-p}(j+p-\alpha)(j+p-\tau)[1+\delta(j+p-1)] - \Omega},
\]

(5.16)

and

\[
\Omega = (p-\alpha)(p-\tau)[1+\delta(p-1)].
\]

(5.17)

The result is the best possible for the functions

\[
f_1(z) = z^p - \frac{(p-\alpha)[1+\delta(p-1)]}{j!(a)_j}(j+p-\alpha)[1+\delta(j+p-1)],
\]

(5.18)

and

\[
f_2(z) = z^p - \frac{(p-\tau)[1+\delta(p-1)]}{j!(a)_j}(j+p-\tau)[1+\delta(j+p-1)].
\]

(5.19)

**Theorem 6.** Let the functions \( f_i(z)(i=1,2,\cdots,m) \) defined by (5.1) be in the class \( SC_j(p,\lambda,\alpha,\delta) \). Then the function

\[
\zeta = p - \frac{j m(p-\alpha)^2[1+\delta(p-1)]}{(\lambda+p)_{k-p}(c)_{k-p}(j+p-\alpha)^2[1+\delta(j+p-1)] - m(p-\alpha)^2[1+\delta(p-1)]}.
\]

(5.21)

The result is sharp for the functions \( f_i(z)(i=1,2,\cdots,m) \) given by (5.4).

**Proof.** Noting that

\[
\sum_{k-j=p}^{\infty} \frac{(\lambda+p)_{k-p}(c)_{k-p}(k-\alpha)[1+\delta(k-1)]}{(k-p)!(a)_{k-p}(p-\alpha)[1+\delta(p-1)]} a_{k,j}^2 \leq \left( \sum_{k-j=p}^{\infty} \frac{(\lambda+p)_{k-j}(c)_{k-j}(k-\alpha)[1+\delta(k-1)]}{(k-p)!(a)_{k-p}(p-\alpha)[1+\delta(p-1)]} a_{k,j} \right)^2 \leq 1,
\]

(5.22)

we have

\[
\sum_{k-j=p}^{\infty} \frac{1}{m} \frac{(\lambda+p)_{k-j}(c)_{k-j}(k-\alpha)[1+\delta(k-1)]}{(k-p)!(a)_{k-p}(p-\alpha)[1+\delta(p-1)]} \left( \sum_{i=1}^{\infty} a_{i,j}^2 \right) \leq 1.
\]

(5.23)

Therefore, we have find largest \( \zeta \) such that
\[
\frac{(k-\zeta)}{(p-\zeta)} \leq \frac{\binom{\lambda+p}{k-p} (c)_{k-p} (k-\alpha)^2 \left[1+\delta(k-1)\right]}{(p-\alpha)^2 \left[1+\delta(p-1)\right]} \quad (k \geq j+p, p, j \in N),
\]

that is,
\[
\zeta \leq p - \frac{m(k-p)(p-\alpha)^2 \left[1+\delta(p-1)\right]}{(\lambda+p)_{k-p} (k-\alpha)^2 \left[1+\delta(k-1)\right] - m(p-\alpha)^2 \left[1+\delta(p-1)\right]}, \quad (k \geq j+p, p, j \in N).
\]

Now, defining the function \( \psi(k) \) by
\[
\psi(k) = p - \frac{m(k-p)(p-\alpha)^2 \left[1+\delta(p-1)\right]}{(\lambda+p)_{k-p} (k-\alpha)^2 \left[1+\delta(k-1)\right] - m(p-\alpha)^2 \left[1+\delta(p-1)\right]}, \quad (k \geq j+p, p, j \in N)
\]
we observe that \( \psi(k) \) is an increasing function of \( k \). We thus conclude that
\[
\zeta \leq \psi(j+p) = p - \frac{mf(p-\alpha)^2 \left[1+\delta(p-1)\right]}{f!(a)_{j-p}}, \quad (j \geq p, p, j \in N)
\]
which completes the proof of Theorem 6.

6. Applications of Fractional Calculus

Various operators of fractional calculus (that is, fractional integral and fractional derivatives) have been studied in (cf., e.g., [9,10,13-15]; see also the various references cited therein).

For our present investigation, we recall the following definitions.

**Definition 1.** The fractional integral of order \( \mu \) is defined, for a function \( f(z) \), by
\[
D^\mu_{z} f(z) = \frac{1}{\Gamma(\mu)} \int_{z-\zeta}^{z} \frac{f(\zeta)}{(z-\zeta)^{\mu}} d\zeta \quad (\mu > 0), \quad (6.1)
\]
where the function \( f(z) \) is analytic in a simply-connected domain of the complex \( z \)-plane containing the origin and the multiplicity of \( (z-\zeta)^{\mu} \) is removed by requiring \( \log(z-\zeta) \) to be real when \( z-\zeta > 0 \).

**Definition 2.** The fractional derivative of order \( \mu \) is defined, for a function \( f(z) \), by
\[
D^\mu_{z} f(z) = \frac{1}{\Gamma(1-\mu)} \int_{0}^{z} \frac{f(\zeta)}{(z-\zeta)^{\mu}} d\zeta \quad (0 \leq \mu < 1), \quad (6.2)
\]
where the function \( f(z) \) is constrained, and the multiplicity of \( (z-\zeta)^{\mu} \) is removed, as in Definition 1.

**Definition 3.** Under the hypotheses of Definition 2, the fractional derivative of order \( n+\mu \) is defined, for a function \( f(z) \), by
\[
D^{n+\mu}_{z} f(z) = \frac{d^n}{dz^n} \left[D^\mu_{z} f(z)\right] \quad (0 \leq \mu < n, n \in N_0). \quad (6.3)
\]

In this section, we shall investigate the growth and distortion properties of functions in the class \( SC_n(j,p,\lambda, \alpha,\delta) \) involving the operators \( J_{c,p} \) and \( D_{c}^{\mu} \). In order to derive our results, we need the following Lemma given by Chen et al. [14].

**Lemma 1 (see [14]).** Let the function \( f(z) \) defined by (1.1). Then
\[
D_{c}^{\mu} \left[J_{c,p} f(z)\right] = \frac{(c+p)\Gamma(p+1)}{(c+p-\mu)\Gamma(p+1-\mu)} \frac{a_z^{k-\mu}}{(c+k)\Gamma(k+1-\mu)} \quad (\mu \in R; c > -p, p, j \in N) \quad (6.4)
\]
and
\[
J_{c,p} \left(D_{c}^{\mu} f(z)\right) = \frac{(c+p)\Gamma(p+1)}{(c+p-\mu)\Gamma(p+1-\mu)} \frac{a_z^{k-\mu}}{(c+k)\Gamma(k+1-\mu)} \quad (\mu \in R; c > -p, p, j \in N) \quad (6.5)
\]
provided that no zeros appear in the denominators in (6.4) and (6.5).
Theorem 7. Let the function $f(z)$ defined by (1.1) be in the class $SC_{cx}(j,p,\lambda,\alpha,\delta)$. Then

$$
|D_z^\mu \{J_{c,p}f\}(z)| \geq \left\{ \begin{array}{ll}
\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} & - \frac{(c+p)\Gamma(j+p+1)(p-\alpha)[1+\delta(p-1)]}{(c+j+p)\Gamma(j+p+1+\mu)\left(\frac{\lambda+p}{j!}c\right)(j+p-\alpha)[1+\delta(j+p-1)]} |z|^\mu,
\end{array} \right.
$$

\[z \in U; 0 \leq \alpha < p; 0 \leq \delta \leq 1; \mu > 0; c > -p, p, j \in N\]  

and

$$
|D_z^\mu \{J_{c,p}f\}(z)| \leq \left\{ \begin{array}{ll}
\frac{\Gamma(p+1)}{\Gamma(p+1+\mu)} & + \frac{(c+p)\Gamma(j+p+1)(p-\alpha)[1+\delta(p-1)]}{(c+j+p)\Gamma(j+p+1+\mu)\left(\frac{\lambda+p}{j!}c\right)(j+p-\alpha)[1+\delta(j+p-1)]} |z|^\mu,
\end{array} \right.
$$

\[z \in U; 0 \leq \alpha < p; 0 \leq \delta \leq 1; \mu > 0; c > -p, p, j \in N\].  

Each of the assertion (6.6) and (6.7) is sharp.

Proof. In view of Theorem 1, we have

$$
\frac{\lambda+p}{j!(a)}(c)(j+p-\alpha)[1+\delta(j+p-1)] \sum_{k=j+p}^{\infty} a_k \leq \frac{(\lambda+p)_{k-p}(c)(k-p)\Gamma(k+p-1+\mu)(k+p-1)}{(k-\alpha)[1+\delta(k-1)]} a_k \leq 1,
$$

which readily yields

$$
\sum_{k=j+p}^{\infty} a_k \leq \frac{(p-\alpha)[1+\delta(p-1)]}{\left(\frac{\lambda+p}{j!}c\right)(j+p-\alpha)[1+\delta(j+p-1)]}.
$$

Consider the function $f(z)$ defined in $U$ by

$$
F(z) = \frac{\Gamma(p+1+\mu)}{\Gamma(p+1)} - z^\mu D_z^\mu \{J_{c,p}f\}(z) = z^\mu - \sum_{k=j+p}^{\infty} a_k \frac{(c+p)\Gamma(k+1+\mu)(p+1+\mu)}{(c+k)\Gamma(k+1+\mu)(p+1)} (k > j+p, p, j \in N, \mu > 0).
$$

where

$$
\phi(k) = \frac{(c+p)\Gamma(k+1+\mu)(p+1+\mu)}{(c+k)\Gamma(k+1+\mu)(p+1)} (k > j+p, p, j \in N, \mu > 0).
$$

Since $\phi(k)$ is a decreasing function of $k$ when $\mu > 0$, we get

$$
0 < \phi(k) \leq \phi(j+p) = \frac{(c+p)\Gamma(j+p+1+\mu)(p+1+\mu)}{(c+j+p)\Gamma(j+p+1+\mu)(p+1)} (c > -p, p, j \in N, \mu > 0).
$$

Thus, by using (6.9) and (6.11), we deduce that

$$
|F(z)| \geq |z|^\mu - |\phi(j+p)| |z|^\mu \sum_{k=j+p}^{\infty} a_k \geq |z|^\mu - \frac{(c+p)\Gamma(j+p+1+\mu)(p-\alpha)[1+\delta(p-1)]}{(c+j+p)\Gamma(j+p+1+\mu)(p+1)\left(\frac{\lambda+p}{j!}c\right)(j+p-\alpha)[1+\delta(j+p-1)]} |z|^\mu,
$$

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which yield the inequalities (6.6) and (6.7) of Theorem 7. The equalities in (6.6) and (6.7) are attained for the function $f(z)$ given by

\[
D^\nu_j \left\{ \left( j, p, f \right) \right\}(z) = \left. \frac{\Gamma(\nu + 1)}{\Gamma(p + 1)} \right\} \left\{ \frac{(c + p)(j + p + 1)(p - \alpha)(1 + \delta(j - 1))}{(c + j + p)(j + p + 1)(p - \alpha)(1 + \delta(j - 1))} \right\} |z|^\nu \phi(p + \mu)
\]

or, equivalently, by

\[
\left( J_{c, p, f} \right)(z) = z^p - \frac{(c + p)(p - \alpha)(1 + \delta(p - 1))}{(c + j + p)(j + p + 1)(p - \alpha)(1 + \delta(j - 1))} z^{j + p}.
\]

Thus we complete the proof of Theorem 7.

Using arguments similar to those in the proof of Theorem 7, we obtain the following result.

\[
\left| D^\nu_j \left\{ \left( j, p, f \right) \right\}(z) \right| \geq \left. \frac{\Gamma(\nu + 1)}{\Gamma(p + 1)} \left\{ \frac{(c + p)(j + p + 1)(p - \alpha)(1 + \delta(j - 1))}{(c + j + p)(j + p + 1)(p - \alpha)(1 + \delta(j - 1))} \right\} |z|^\nu \phi(p + \mu) \right| \left( z \in U; 0 \leq \alpha < p, 0 \leq \delta \leq 1, 0 \leq \mu < 1, c > -p, j, p \in \mathbb{N} \right),
\]

and

\[
\left| D^\nu_j \left\{ \left( j, p, f \right) \right\}(z) \right| \leq \left. \frac{\Gamma(\nu + 1)}{\Gamma(p + 1 - \mu)} \left\{ \frac{(c + p)(j + p + 1)(p - \alpha)(1 + \delta(j - 1))}{(c + j + p)(j + p + 1)(p - \alpha)(1 + \delta(j - 1))} \right\} |z|^\nu \phi(p + \mu) \right| \left( z \in U; 0 \leq \alpha < p, 0 \leq \delta \leq 1, 0 \leq \mu < 1, j, p \in \mathbb{N} \right).
\]

Each of the assertions (6.14) and (6.15) is sharp.

7. References


