Strong Convergence of an Iterative Method for Generalized Mixed Equilibrium Problems and Fixed Point Problems

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Abstract

In this paper, we introduce a hybrid iterative method for finding a common element of the set of common solutions of generalized mixed equilibrium problems and the set of common fixed points of a finite family of nonexpansive mappings. Furthermore, we show a strong convergence theorem under some mild conditions.

Keywords: Generalized Mixed Equilibrium Problem, Hybrid Iterative Scheme, Fixed Point, Nonexpansive Mapping, Strong Convergence

1. Introduction

Equilibrium problems theory provides us with a natural, novel and unified framework for studying a wide class of problems arising in economics, finance, transportation, network and structural analysis, elasticity and optimization. The ideas and techniques of this theory are being used in a variety of diverse areas and proved to be productive and innovative.

Let \( H \) be a Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). Let \( C \) be a nonempty closed convex subset of \( H \) and \( T : C \to 2^H \) a multivalued mapping. Let \( \phi : C \times C \to \mathbb{R} \) be a real-valued function and \( \Theta : H \times C \times C \to \mathbb{R} \) be an equilibrium-like function, i.e., \( \Theta(w,u,v) + \Theta(w,v,u) = 0 \) for each \( (w,u,v) \in H \times C \times C \). The generalized mixed equilibrium problem (for short, GMEP) is to find \( u \in C \) and \( w \in T(u) \) such that

\[
\Theta(T(u),u,v) + \phi(v,u) - \phi(u,u) \geq 0, \quad \forall v \in C.
\]

In particular, if \( T \) is single-valued mapping, this problem is equivalent to finding \( u \in C \) such that

\[
\Theta(T(u),u,v) + \phi(v,u) - \phi(u,u) \geq 0, \quad \forall v \in C.
\]

Denote the set of solutions of GMEP by \( \Omega \).

Now, we recall the following definitions.

A mapping \( f : C \to C \) is said to be contractive if there exists a constant \( \alpha \in (0,1) \) such that

\[
\| f(x) - f(y) \| \leq \alpha \| x - y \| \quad \text{for any} \quad x, y \in C.
\]

A mapping \( g : C \to C \) is said to be firmly nonexpansive if

\[
\| g(x) - g(y) \| \leq \frac{1}{2} \| x - y \| \quad \text{for any} \quad x, y \in C.
\]

A mapping \( T : C \to C \) is said to be nonexpansive if

\[
\| Tx - Ty \| \leq \| x - y \| \quad \text{for any} \quad x, y \in C.
\]

The set of fixed points of \( T \) is denoted by \( F(T) \).

Let \( \{T_i\}_{i=1}^N \) be a finite family of nonexpansive mappings of \( C \) into \( H \) and \( \bigcap_{i=1}^N F(T_i) \neq \emptyset \). Define the mappings

\[
U_{n,1} = \lambda_{n,1} T_1 + (1 - \lambda_{n,1}) I, \\
U_{n,2} = \lambda_{n,2} T_2 U_{n,1} + (1 - \lambda_{n,2}) I, \\
\vdots \\
U_{n,N} = \lambda_{n,N} T_N U_{n,N-1} + (1 - \lambda_{n,N}) I
\]

where \( \{\lambda_{n,i}\}_{i=1}^N \subseteq (0,1) \) for all \( n \geq 1 \). Such a mapping \( W_n \) is called \( W \)-mapping generated by \( T_1, \cdots, T_N \) and \( \{\lambda_{n,i}\}_{i=1}^N \).

2. Preliminaries

Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Then, for any \( x \in H \), there exists a unique nearest point in \( C \), denoted by \( P_C(x) \), such that

\[
\| x - P_C(x) \| \leq \| x - y \| \quad \text{for all} \quad y \in C.
\]

Such a \( P_C \) is called the metric projection.
of $H$ into $C$. We know that $P_C$ is nonexpansive. What's more,
\[ x^* = P_C(x) \Leftrightarrow \{x-x^*, x^*-y\} \geq 0, \forall y \in C. \]

Let $C$ be a convex subset of a real Hilbert space $H$, $\eta : C \times C \rightarrow H$ and $k : C \rightarrow R$ a Frechet differential function. Then $k$ is said to be $\eta$-strongly convex if there exists a constant $\mu > 0$ such that
\[ k(y) - k(x) - (k'(x), \eta(y,x)) \geq \frac{\mu}{2} \|y-x\|^2, \forall x, y \in C. \]

If $\mu = 0$, then $k$ is said to be $\eta$-convex. In particular, if $\eta(y,x) = y - x$ for all $y, x \in C$, then $k$ is said to be strongly convex.

Let $C$ be a nonempty subset of a real Hilbert space $H$. A bifunction $\phi(\cdot, \cdot) : C \times C \rightarrow R$ is said to be skew-symmetric if
\[ \phi(u,v) + \phi(v,u) - \phi(u,u) - \phi(v,v) \leq 0, \forall u, v \in C. \]

It is easy to see that if the skew-symmetric bifunction $\phi(\cdot, \cdot)$ is linear in both arguments, then
\[ \phi(u,v) \geq 0, \forall u \in C. \]

We denote $\rightharpoonup$ for weak convergence and $\rightarrow$ for strong convergence. A bifunction $\phi : C \times C \rightarrow R$ is called weakly sequentially continuous at $(x_n, y_n) \in C \times C$ if $\phi(x_n, y_n) \rightarrow \phi(x_0, y_0)$ as $n \rightarrow \infty$ for each sequence $\{(x_n, y_n)\}$ in $C \times C$ converging weakly to $(x_0, y_0)$.

The function $\phi(\cdot, \cdot)$ is called weakly sequentially continuous on $C \times C$ if it is weakly sequentially continuous at each point of $C \times C$.

Let $CB(X)$ denote the set of nonempty closed bounded subset of $X$. For $A, B \in CB(X)$, define the Hausdorff metric $h$ as follows:
\[ h(A, B) = \max \{\sup_{a \in A} \inf_{b \in B} d(a, b), \inf_{b \in B} \sup_{a \in A} d(a, b)\}. \]

In order to solve the generalized mixed equilibrium problems for an equilibrium-like bifunction $\Theta : H \times C \times C \rightarrow R$, we assume that $\Theta$ satisfies the following conditions with respect to the multivalued mapping $T : C \rightarrow 2^H$:

- For each fixed $v \in C$, $(w, u) \mapsto \Theta(w, u, v)$ is an upper semicontinuous function from $H \times C$ to $R$, that is, $w_n \rightarrow w$ and $u_n \rightarrow u$ imply $\limsup_{n \rightarrow \infty} \Theta(w_n, u_n, v) \leq \Theta(w, u, v)$;
- $(\Theta_2)$ for each fixed $(w, v) \in H \times C$, $u \mapsto \Theta(w, u, v)$ is a concave function;
- $(\Theta_4)$ for each fixed $(w, u) \in H \times C$, $v \mapsto \Theta(w, u, v)$ is a convex function;
- $(\Theta_4)$ $\Theta(w_1, T_1(x), T_2(y)) + \Theta(w_2, T_2(y), T_1(x)) \leq \gamma \|T_1(x) - T_2(y)\|^2$

for all $x, y \in C$ and $r, s \in (0, \infty)$, where $r > 0$, $w_1 \in T(x)$ and $w_2 \in T(y)$.

Let $k : C \rightarrow R$ be a differential function with Frechet derivative $k'(x)$ at $x$ satisfying the following:

- $(k_1)$ $k'$ is continuous from the weak topology to the strong topology;
- $(k_2)$ $k'$ is Lipschitz continuous with constant $\nu > 0$.

Let $\eta : C \times C \rightarrow H$ be a function satisfying the following:

- $(\eta_1)$ $\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in C$;
- $(\eta_2)$ $\eta(\cdot, \cdot)$ is affine in the first coordinate variable;
- $(\eta_3)$ for each fixed $y \in C$, $\eta(x, y)$ is sequentially continuous from the weak topology to the weak topology.

Let $C$ be a nonempty closed convex subset of a real Hilbert space and $T : C \rightarrow 2^H$ a multivalued mapping. For $x \in C$, let $w \in T(x)$. Let $\phi : C \rightarrow R$ be a real-valued function satisfying the following:

- $(\phi_1)$ $\phi(\cdot, \cdot)$ is skew symmetric;
- $(\phi_2)$ for each fixed $y \in C$, $\phi(\cdot, y)$ is convex and upper semicontinuous;
- $(\phi_3)$ $\phi(\cdot, \cdot)$ is weakly continuous on $C \times C$.

Recently Wei-You Zeng, Nan-Jing Huang and Chang-Wen Zhao [1] introduce and consider a new class of equilibrium problems, which is known as the generalized mixed equilibrium problems. Furthermore, they introduce an iterative scheme (1.4) by the viscosity approximation method for finding a common element of the set of common solutions for generalized mixed equilibrium problems and the set of common fixed points of a sequence of nonexpansive mappings in Hilbert space.
Motivated and inspired by the research going on in this important field, we introduce the following hybrid iterative scheme (1.5) for finding a common element of the set of common solutions for generalized mixed equilibrium problems and the set of common fixed points of a sequence of nonexpansive mappings. We show that the approximation solution converges strongly to a unique solution of a class of variational inequalities under some mild conditions. Results obtained in this paper can be viewed as an improvement and refinement of the recent results in this direction.

**Algorithm 1.1.** Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$, $T : C \to CB(H)$ be a multivalued mapping, $f$ be a contraction of $C$ into itself with coefficient $\alpha \in (0,1)$. Let $W_n : C \to C$ be defined by (1.3), and $r > 0$. For given $x_1 \in C$ and $w_i \in T(x_i)$, there exists sequences $\{x_n\}, \{u_n\}$ in $C$ and $\{w_n : w_n \in T(x_n)\}$ in $H$ such that for all $n = 1, 2, \cdots$

\[
\begin{align*}
\left\|w_n - w_{n+1}\right\| & \leq \left(1 + \frac{1}{n}\right)h(T(x_n), T(x_{n+1})), \\
\Theta(w_n, u_n, v) + \phi(v, u_n) - \phi(u_n, u_n) + \frac{1}{r}(k'(u) - k'(x_n), \eta(v, u_n)) & \geq 0, \forall v \in C
\end{align*}
\]  

(2.2)  

where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are three sequences in $(0, 1)$ such that $a_n + b_n + c_n = 1$.

It is easy to see that the iterative scheme (1.5) may be well defined.

Let $r$ be a positive number. For a given point $x \in C$ and $w_n \in T(x)$, consider the following auxiliary problem for GMEP: find $u \in C$ such that

\[
\begin{align*}
\Theta(w_n, u, v) + \phi(v, u) - \phi(u, u) + \frac{1}{r}(k'(u) - k'(x), \eta(v, u)) & \geq 0, \forall v \in C,
\end{align*}
\]  

(2.3)  

It is easy to see that if $u = x$, then $u$ is a solution of GMEP.

Then there hold the following:
1) the auxiliary problem (1.6) has a unique solution;
2) $T_r$ is single-valued;
3) if $\lambda \nu / \mu \leq 1$, it follows that $T_r$ is firmly nonexpansive;
4) $F(T_r) = \Omega$;
5) $\Omega$ is closed and convex.

**Lemma 1.2.** [3] Let $H$ be a real Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $\{T_{n,r}\}$ be a finite family of nonexpansive mappings of $C$ into $H$ and $\bigcap_{n=1}^{N} F(T_n) \neq \emptyset$, and let $\{\lambda_{n,i}\}_{i=1}^{N}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Then,

\[
F(W_n) = \bigcap_{n=1}^{N} F(T_n).
\]  

**Lemma 1.3.** [4] If the sequences $\{u_n\}$ and $\{x_n\}$ are bounded and $W_n$ is defined by (1.3), then the following estimates hold:

\[
\left\|W_n x_{n+1} - W_n u_n\right\| \leq \left\|x_{n+1} - x_n\right\| + 2M \sum_{i=1}^{N} \left|\lambda_{n+1,i} - \lambda_{n,i}\right|, \quad \forall n \geq 0
\]

and

\[
\left\|W_n u_{n+1} - W_n x_n\right\| \leq \left\|u_{n+1} - x_n\right\| + 2M \sum_{i=1}^{N} \left|\lambda_{n+1,i} - \lambda_{n,i}\right|, \quad \forall n \geq 0
\]
for some constant \( M > 0 \).

**Lemma 14.** [4] In a real Hilbert space \( H \), \( \forall x, y, z \in H \) and \( t_1, t_2, t_3 \in [0,1] \) with \( t_1 + t_2 + t_3 = 1 \), there holds the following equality:

\[ \|x + t_2y + t_3z\| \leq t_1\|x\| + t_2\|y\| + t_3\|z\| \]

**Lemma 15.** [6] Let \( \{x_n\} \) and \( \{u_n\} \) be bounded sequences in a Banach space \( X \) and let \( \{b_n\} \) be a sequence in \([0,1]\) with \( 0 < \liminf_{n \to \infty} b_n \leq \limsup_{n \to \infty} b_n < 1 \). Suppose

\[ x_{n+1} = (1-b_n)x_n + b_nu_n \]

for all integers \( n \geq 0 \) and

\[ \limsup_{n \to \infty} \left( \|x_{n+1} - x_n\| - \|x_{n+1} - x\| \right) \leq 0. \]

Then,

\[ \lim_{n \to \infty} \|x_n - x\| = 0. \]

**Lemma 16.** [5] Let \( \{a_n\} \) is a sequence of nonnegative real numbers such that

\[ a_{n+1} \leq (1-\delta_n)a_n + b_n, \quad \forall n = 1,2,\ldots \]

where \( \{\delta_n\} \) is a sequence in \((0,1), \sum_{n=1}^{\infty} \delta_n = \infty \) and \( \limsup_{n \to \infty} b_n/\delta_n \leq 0 \), then \( \lim_{n \to \infty} a_n = 0 \).

**Lemma 17.** [2] Let \( \{x_n\} \) be a sequence in a normed space \((X,\|\cdot\|)\) such that

\[ \|x_{n+1} - x_{n}\| \leq \theta \|x_n - x_{n-1}\| + r_n, \forall n = 1,2,\ldots \]

where \( \theta \in (0,1) \), \( \{s_n\} \) and \( \{r_n\} \) are sequences satisfying the following conditions:

1) \( s_n \geq 1 \) and \( \sum_{n=1}^{\infty} (s_n - 1) < \infty \);
2) \( r_n \geq 0 \) and \( \sum_{n=1}^{\infty} r_n < \infty \).

Then \( \{x_n\} \) is a Cauchy sequence.

**Lemma 18.** [7] Let \( A, B \in CB(X) \) and \( a \in A \). Then for \( \rho > 1 \), there must exist a point \( b \in B \) such that \( d(a,b) \leq \rho d(A,B) \).

**Lemma 19.** [5] In a real Hilbert space \( H \), there holds the following equality:

\[ \|x+y\| \geq \|x\| + 2\{y, x+y\}, \forall x, y \in H. \]

### 3. Main Results

**Theorem 21.** Let \( C \) be a nonempty closed convex bounded subset of a real Hilbert space \( H \) and \( r > 0 \), \( T : C \to CB(H) \) be a multivalued \( h \)-Lipschitz continuous mapping with constant \( L > 0 \), and let \( \phi : C \times C \to R \) be a real-valued function satisfying \( (\phi_1) \cdot (\phi_2) \) and \( \Theta : H \times C \times C \to R \) be an equilibrium-like function satisfying the conditions \( (\Theta_1) \cdot (\Theta_2) \). Assume that \( \eta : C \times C \to H \) is a Lipschitz function with lipschitz constant \( \lambda > 0 \) which satisfies the conditions \( (\eta_1) \cdot (\eta_2) \). Let \( k : C \to R \) be an \( \eta \)-strongly convex function with constant \( \mu > 0 \) which satisfies the conditions \( (k_1) \cdot (k_2) \) with \( \lambda + \mu \leq 1 \). Let \( \{T_n\}_{n=1}^{\infty} \) be a finite family of nonexpansive mappings on \( H \) such that \( \bigcap_{n=1}^{\infty} F(T_n) \cap \Omega \neq \emptyset \). Let \( f \) be a contraction of \( C \) into itself with coefficient \( \alpha \in (0,1) \). Let \( \{x_n\}, \{u_n\}, \{w_n\} \) be sequences generated by (1.5), where \( \{a_n\}, \{b_n\} \) and \( \{c_n\} \) are three sequences in \((0,1)\) with \( a_n + b_n + c_n = 1 \) satisfying the following conditions:

1) \( \lim_{n \to \infty} a_n = 0, \sum_{n=1}^{\infty} a_n = \infty \) and
2) \( 0 < \liminf_{n \to \infty} b_n \leq \limsup_{n \to \infty} b_n < 1; \)
3) \( \lim_{n \to \infty} \|\lambda_{n+1} - \lambda_n\| = 0; \)
4) \( \sum_{n=1}^{\infty} c_n = \infty \).

Then

1) \( \lim_{n \to \infty} \|x_n - u_n\| = 0, \lim_{n \to \infty} \|x_n - x_{n-1}\| = 0; \)
2) \( \lim_{n \to \infty} \|x_n - W_n^\alpha\| = 0, \lim_{n \to \infty} \|x_n - u_n\| = 0. \)
and set \( z_n = \frac{x_n - b_n x_n}{1 - b_n} \), we obtain

\[
\begin{align*}
  z_{n+1} - z_n &= \frac{x_{n+1} - b_n x_{n+1}}{1 - b_n} - \frac{x_n - b_n x_n}{1 - b_n} \\
  &= \frac{a_{n+1}}{1 - b_{n+1}} f(W_{n+1}x_{n+1}) + \frac{a_n}{1 - b_n} f(W_n x_n) - \frac{a_n}{1 - b_n} f(W_n x_n) \\
  &= \frac{c_{n+1}}{1 - b_{n+1}} (W_{n+1}u_{n+1} - W_n u_n) + \frac{c_n}{1 - b_n} W_n u_n \\
  &= \frac{c_{n+1}}{1 - b_{n+1}} (W_{n+1}u_{n+1} - W_n u_n) + \frac{c_n}{1 - b_n} W_n u_n
\end{align*}
\]

By Lemma 1.3, we arrive at

\[
\begin{align*}
  \|z_{n+1} - z_n\| &\leq \frac{\alpha_{n+1}}{1 - b_{n+1}} \|f(W_{n+1}x_{n+1}) - f(W_n x_n)\| + \frac{\alpha_n}{1 - b_n} \left(\|f(W_n x_n)\| + \|W_n u_n\|\right) \\
  &\quad + \frac{\alpha_n}{1 - b_n} \left(\|f(W_n x_n)\| + \|W_n u_n\|\right) \\
  &\quad + \frac{\alpha_{n+1}}{1 - b_{n+1}} \left(\|W_{n+1}u_{n+1} - W_n u_n\| + \sum_{i=1}^{N} \lambda_{n+1,i} - \lambda_{n,i}\right) \\
  &\quad + \frac{\alpha_n}{1 - b_n} \left(\|W_n u_n\| + \sum_{i=1}^{N} \lambda_{n+1,i} - \lambda_{n,i}\right) \\
  &\leq \frac{\alpha_{n+1}}{1 - b_{n+1}} \|x_{n+1} - x_n\| + 2M \sum_{i=1}^{N} \lambda_{n+1,i} - \lambda_{n,i} \\
  &\quad + \frac{\alpha_n}{1 - b_n} \left(\|f(W_n x_n)\| + \|W_n u_n\|\right) + 2M \sum_{i=1}^{N} \lambda_{n+1,i} - \lambda_{n,i} \\
  &\quad \leq \|x_{n+1} - x_n\| + \frac{\alpha_n}{1 - b_n} \left(\|f(W_n x_n)\| + \|W_n u_n\|\right) + 2M \sum_{i=1}^{N} \lambda_{n+1,i} - \lambda_{n,i}
\end{align*}
\]

It follows from conditions (a) and (c), we have

\[
\limsup_{n \to \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.
\]

Hence by Lemma 1.5, we can see that

\[
\lim_{n \to \infty} \|z_n - x_n\| = 0
\]

Consequently

\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} (1 - b_n) \|z_n - x_n\| = 0
\]

From (2.1), we get

\[
\lim_{n \to \infty} \|u_{n+1} - u_n\| = 0
\]

2) In view of (1.5), we conclude that

\[
\begin{align*}
  \|x_n - W_n u_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - W_n u_n\| \\
  &\leq \|x_n - x_{n+1}\| + \alpha_n \|f(W_n x_n) - W_n u_n\| \\
  &\quad + \beta_n \|x_n - W_n u_n\|
\end{align*}
\]

that is

\[
\|x_n - W_n u_n\| \leq \|x_n - x_{n+1}\| + \alpha_n \|f(W_n x_n) - W_n u_n\| \\
  + \beta_n \|x_n - W_n u_n\|
\]

For \( p \in \Gamma = \cap_{i=1}^{N} F(T_i) \cap \Omega \), note that \( T_i \) is firmly nonexpansive, we can see that

\[
\|u_n - p\| \leq \|T_i x_n - p\| \leq \langle T_i x_n - T_i p, x_n - p \rangle \\
  = \langle u_n - p, x_n - p \rangle \\
  = \frac{1}{2} \left( \|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - x_n\|^2 \right)
\]

and so

\[
\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - x_n\|^2 \leq \|x_n - p\|^2
\]

In view of Lemma 1.4, (2.6) and (2.7), we compute
which follows that
\[ c_n \|x_n - u_n\| \leq (\|x_n - p\| + \|x_{n+1} - p\|)(\|x_n - x_n\|) + a_n \|f(W_{x_n}) - p\| \]
and hence
\[ \lim_{n \to \infty} \|x_n - u_n\| = 0 \]

This completes the proof.

Proof of Theorem 2.1. We divide our proof into 3 steps.

**Step 1.** We prove that there exists \( x^* \in C \), such that \( x_n \to x^* \), \( u_n \to x^* \) and \( w_n \to w \) as \( n \to \infty \), where \( w \in T(x^*) \). From (1.5), (2.1) and Lemma 1.3, we compute

By Lemma 1.7 and conditions (a)-(d), we conclude that \( \{x_n\} \) is a Cauchy sequence in \( C \) such that \( \lim_{n \to \infty} x_n = x^* \). On the other hand, \( \lim_{n \to \infty} \|x_n - u_n\| = 0 \) implies that \( \lim_{n \to \infty} u_n = x^* \). From (1.5), we have

\[ \|w_n - w_m\| \leq \left( 1 + \frac{1}{n} \right) h(T(x_n), T(x_{n+1})) \]

Hence
\[ \sum_{i=n}^{m-1} \|x_i - x_{i+1}\| \leq \frac{\theta}{1 - \theta} \|x_n - x_{n+1}\| + \sum_{i=n}^{m-1} r_i \]

In view of (2.4) and (2.8), we obtain
\[ \lim_{m \to \infty} \|w_m - w_n\| = 0 \]

which implies that \( \{w_n\} \) is a Cauchy sequence in \( H \) and therefore there exists an element \( w \) in \( H \) such that \( \lim_{n \to \infty} w_n = w \). Next we can see that

\[ d \left( w, T(x^*) \right) = \inf_{b \in T(x^*)} d \left( w, b \right) \leq \|w - w_n\| + d \left( w, T(x^*) \right) \]

Hence, we derive that \( d \left( w, T(x^*) \right) = 0 \), that is
w ∈ T (x') as T (x') ∈ CB (H).

**Step 2.** Let Q = P_T (T (x')) f. Then Q is a contraction of C into itself. In fact, for all x, y ∈ C

\[ \| Q (x) - Q (y) \| ≤ \| f (x) - f (y) \| ≤ \alpha \| x - y \| \]

Therefore there exists a unique element q ∈ C such that q = Q (q). Noting that q ∈ C and Q(q) ∈ ∩_i=1^n F (T_i) ∩ Ω, we get that q ∈ ∩_i=1^n F (T_i) ∩ Ω.

Then

\[ \langle f (q) - q, p - q \rangle ≤ 0, \quad \forall p ∈ ∩_i=1^n F (T_i) ∩ Ω \]  

(3.13)

Next, we show that x' ∈ ∩_i=1^n F (T_i) ∩ Ω. Since x_n → x' and u_n → x', we know that k'(u_n) − k'(x_n) → 0. From (1.5) and (2.7), we have

\[ Θ (w, x', v) + φ (v, x') - φ (x', x') ≥ 0 \]

that is x' ∈ Ω. We shall show x' ∈ F (W_n). Assume

\[ x' ∈ F (W_n), \text{ that is } x' ≠ W_n x'. \]  

Since \{u_n\} is bounded, there exists a subsequence \{u_{n_j}\} of \{u_n\} which converges weakly to x'. By Lemma 2.1, we conclude that \[ \| W_n u_n - u \| → 0. \]  

From Opial’s condition, we have

\[ \liminf_{j → ∞} \| u_{n_j} - x \| < \liminf_{j → ∞} \| u_{n_j} - W_n x' \| \]

\[ ≤ \liminf_{j → ∞} \| u_{n_j} - W_n u_{n_j} + \| W_n u_{n_j} - W_n x' \| \]

\[ ≤ \liminf_{j → ∞} \| u_{n_j} - x' \| \]

This is a contradiction. So, we get

\[ \lim_{n → ∞} (f (q) - q, x_n - q) = (f (q) - q, x_n - q) ≤ 0 \]  

(3.14)

By Lemma 1.9, (1.5) and (2.7), we compute

\[ \| x_{n+1} - q \| ≤ \frac{1 - a_n}{1 - a_n} \| x_n - q \| + \frac{a_n}{1 - a_n} \| q - x_n \| \]

Hence

\[ \| x_{n+1} - q \| ≤ \frac{1 - a_n^2}{1 - a_n} \| x_n - q \| + \frac{a_n}{1 - a_n} \| q - x_n \| + \frac{2a_n}{1 - a_n} \| (f (q) - q, x_{n+1} - q) \| \]

\[ ≤ \left( 1 - \frac{1 - a_n^2}{1 - a_n} \right) \| x_n - q \| + \frac{a_n}{1 - a_n} \| q - x_n \| + \frac{2a_n}{1 - a_n} \| (f (q) - q, x_{n+1} - q) \| \]

\[ ≤ \left( 1 - \frac{1 - a_n^2}{1 - a_n} \right) \| x_n - q \| + \frac{a_n}{1 - a_n} \| q - x_n \| + \frac{2a_n}{1 - a_n} \| (f (q) - q, x_{n+1} - q) \| \]

where \( M_i = \sup \{ \| x_n - q \| : n ≥ 1 \} \), \( δ_n = \frac{2a_n}{1 - a_n} \), and \( σ_n = \frac{a_n M_i}{2 (1 - α)} + \frac{1}{1 - a_n} \]  

It is easy to see that \( δ_n → 0, \sum_1^∞ δ_n = ∞, \) and \( \limsup_{n → ∞} σ_n ≤ 0. \) Hence, by Lemma 1.6, the sequence \{x_n\} converges strongly to q. Consequently, we can obtain that \{u_n\} also converges strongly to q, and so x' = q. This completes the proof.

Putting T,x = x for all i ≥ 1 in Theorem 2.1, we obtain

**Corollary 2.1.** Let C be a nonempty closed convex bounded subset of a real Hilbert space H,

\[ T : C → CB (H) \]  

be a multivalued h-Lipschitz continuous mapping with constant L > 0, and let \( φ : C × C → R \) be a real-valued function satisfying \( (φ) - (φ) \) and \( Θ : H × C × C → R \) be an equilibrium-like function satisfying the conditions \( (Θ) - (Θ) \) and \( Ω ≠ \emptyset. \) Assume that \( η : C × C → H \) is a Lipschitz function with lipschitz constant λ > 0 which satisfies the conditions \( (η) - (η) \). Let k : C → R be an η-strongly convex function with constant μ > 0 which satisfies the conditions \( (k) - (k) \) and \( λ + μ ≤ 1. \) Let F be a contraction of C into itself with coefficient \( α < 0, 1). \) Then the sequences \{x_n\}, \{u_n\}, and \{w_n\} generated iteratively by
converge strongly to $x^* \in \Omega$, and $\{w_n\}$ converges strongly
to $w^* \in T(x^*)$, where $x^* = P_\Omega f(x^*)$, and $\{a_n\}$, $\{b_n\}$
and $\{c_n\}$ are sequences in $(0, 1)$ with $a_n + b_n + c_n = 1$, and
$r > 0$ satisfying the following conditions:
1) $\lim_{n \to \infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$ and
$\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$;
2) $0 < \liminf_{n \to \infty} b_n \leq \limsup_{n \to \infty} b_n < 1$ and
$\sum_{n=1}^{\infty} |b_{n+1} - b_n| < \infty$;
3) $\sum_{n=1}^{\infty} |c_{n+1} - c_n| < \infty$.

4. References


