A Precise Asymptotic Behaviour of the Large Deviation Probabilities for Weighted Sums

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Abstract
Let \( \{X_n, n \geq 1\} \) be a sequence of independent and identically distributed positive valued random variables with a common distribution function \( F \). When \( F \) belongs to the domain of partial attraction of a semi stable law with index \( \alpha, 0 < \alpha < 1 \), an asymptotic behavior of the large deviation probabilities with respect to properly normalized weighted sums have been studied and in support of this we obtained Chover’s form of law of iterated logarithm.

Keywords: Large Deviations, Law of Iterated Logarithm, Semi-Stable Law, Domain of Partial Attraction, Weighted Sums

1. Introduction
Let \( \{X_n, n \geq 1\} \) be a sequence of independent and identically distributed (i.i.d) positive valued random variables (r.v.s) with a common distribution function \( F \). Let \( BV [0,1] \) be the set of all continuous bounded variation functions over \([0,1]\). Set
\[
S_n = \sum_{k=1}^{n} X_k, n \geq 1, \quad T_n = \sum_{k=1}^{n} f(X_k),
\]
where \( f \) is a member of \( BV[0,1] \). Let \( \{n_k, k \geq 1\} \) be a strictly increasing subsequence of positive integers such that
\[
\lim_{k \to \infty} \frac{n_{k+1}}{n_k} = r(1) \quad \text{as} \quad k \to \infty.
\]
Kruglov [1] established that, if there exists sequences \( (a_k) \) and \( (b_k) \) of real constants, \( b_k \to \infty \) as \( k \to \infty \), such that
\[
\lim_{k \to \infty} \mathbb{P}(\frac{S_{n_k}}{b_k} - a_k \leq x) = G_{\alpha}(x)
\]
at all continuity points \( x \) of \( G_{\alpha} \), then \( G_{\alpha} \) is necessarily a semi stable d.f with characteristic exponent \( \alpha, 0 < \alpha \leq 2 \). When \( \alpha = 2 \), semi-stable becomes normal.

It is known that probabilities of the type \( \mathbb{P}(S_n > x_n) \), or either of the one sided components, are called large deviation probabilities, where \( \{x_n, n \geq 1\} \) is a monotone sequence of positive numbers with \( x_n \to \infty \) as \( n \to \infty \) such that \( \frac{S_n}{x_n} \to 0 \). In fact, under different conditions on sequence of r.v.s, Heyde [2-4] studied the large deviation problems for partial sums. In brief, for the r.v.s which are in the domain of attraction of a stable law and r.v.s which are not belong to the domain of partial attraction of the normal law, Heyde [2] and [3] established the order of magnitude of the larger deviation probabilities, whereas in Heyde [3], he obtained the precise asymptotic behavior of large deviation probabilities for r.v.s in the domain of attraction of stable law.

When r.v.s. has i.i.d symmetric stable r.v.s, Chover [5] obtained the law of iterated logarithm (LIL) for partial sums by normalizing in the power and for r.v.s which are in the domain of attraction of a stable law, Peng and Qi [6] obtained Chover’s type LIL for weighted sums, where the weights are belongs to \( BV[0,1] \). Many authors studied the non-trivial limit behavior for different weighted sums. See Peng and Qi [6] and references therein.

Probability of large values plays an important role in studying non-trivial limit behavior for stable like r.v.s. As far as properly normalized partial sums of stable like r.v.s, we can use the asymptotic results of Heyde [2-4]. (See Divanji [7]). However the observations made by Heyde [2-4] on the large deviation probabilities implicitly motivated us to study the large deviation probabilities for weighted sums. In fact, when the underlying i.i.d positive valued r.v.s are in the domain of partial attraction of a semi stable law of Kruglov’s [1] setup, denoted as \( F \in DP(\alpha), 0 < \alpha < 1 \), a precise asymptotic behavior...
of the large deviation probabilities of Heyde [2-4] can be obtained for weighted sums. In support of this can be considered for Chover’s type of non-trivial limit behavior for weighted sums.

In the next section we present some lemmas and main result in Section 3. In the last section, we discuss the existence of Chover’s form of LIL for weighted sums. In the process i.o, a.s and s.v. mean ‘infinitely often’, ‘almost surely’ and ‘slowly varying’ respectively. C, ε, k and n with or without a super script or subscript denote positive constants with k and n confined to be integers. In the sequel, observe that when α < 1, a_k can always be chosen to be zero.

2. Lemmas

Lemma 2.1
Let F ∈ DP(α), 0 < α < 1. Then there exists s. v. function L, such that

\[
\lim_{x \to \infty} \frac{x^\alpha (1 - F(x))}{L(x)} = 1.
\]

Lemma 2.2
Let F ∈ DP(α), 0 < α < 1 and let

\[ B_n = \inf \{ x > 0 : 1 - F(x) \geq 1/n \}. \]

Then \( B_n = n^{1/\alpha} l(n) \),

where l is a function s. v. at \( \infty \).

The above lemmas can be referred to Divanji and Vasudeva [8].

Lemma 2.3
Let L be any s. v. function and let \( (x_n) \) and \( (y_n) \) be sequence of real constants tending to \( \infty \) as \( n \to \infty \). Then for any \( \delta > 0 \), \n
\[
\lim_{n \to \infty} \frac{L(x_n y_n)}{L(x_n)} = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{L(x_n y_n)}{L(y_n)} = 0.
\]

This lemma can be referred to Drasin and Seneta [9].

Lemma 2.4
Let F ∈ DP(α), 0 < α < 1. Let \( (x_n) \) be a monotone sequence of real numbers tending to \( \infty \) as \( n \to \infty \) and \( B_n \) defined in Lemma 2.2. Then \( B_n \) \( \sum_{k=1}^{\infty} f \left( \frac{k}{n} \right) S_k + f(1)S_n \leq 2C \max_{1 \leq k \leq n} S_k \)

Dividing on both sides by \( x_n B_n \), we have

\[
T_n = \sum_{k=1}^{\infty} f \left( \frac{k}{n} \right) X_k \quad \text{and} \quad \lim_{n \to \infty} \frac{T_n}{x_n B_n} = 0 \quad \text{as} \quad n \to \infty.
\]

Lemma 2.5
Let F ∈ DP(α), 0 < α < 1. Let \( (x_n) \) be a monotone sequence of real numbers tending to \( \infty \) as \( n \to \infty \) and \( B_n \) defined in Lemma 2.2.

Proof
Since \( f \in BV[0,1] \). Hence there exists a constant C such that \( |f(x)| \leq C \) and

\[
\sum_{k=1}^{n-1} \left| f \left( \frac{k}{n} \right) - f \left( \frac{k+1}{n} \right) \right| \leq C, \quad \text{for all} \quad n \geq 1.
\]

Therefore

\[
T_n = \sum_{k=1}^{\infty} f \left( \frac{k}{n} \right) X_k = \sum_{k=1}^{\infty} \left( f \left( \frac{k}{n} \right) - f \left( \frac{k+1}{n} \right) \right) S_k + f(1)S_n \leq 2C \max_{1 \leq k \leq n} S_k
\]

3. Main Results

Theorem 3.1
Let F ∈ DP(α), 0 < α < 1. Let \( (x_n) \) be a monotone sequence of real numbers tending to \( \infty \) as \( n \to \infty \) and \( B_n \) defined in Lemma 2.2. Then \n
\[
\sup_{n \to \infty} \frac{P(T_n \geq x_n B_n)}{n P(X \geq x_n B_n)} = 1.
\]

Proof
To prove the assertion, it is enough to show that

\[
\liminf_{n \to \infty} \frac{P(T_n \geq x_n B_n)}{n P(X \geq x_n B_n)} \leq \limsup_{n \to \infty} \frac{P(T_n \geq x_n B_n)}{n P(X \geq x_n B_n)} < \infty.
\]

Let \( \epsilon > 0 \) and define

\[
A_i = \left\{ f \left( \frac{i}{n} \right) X_i \geq (1+\epsilon) x_n B_n \right\}, \quad i = 1, 2, \ldots, n.
\]

and

\[
B_i = \left\{ \sum_{j=1}^{i} f \left( \frac{j}{n} \right) X_j \leq \epsilon x_n B_n \right\}, \quad i = 1, 2, \ldots, n.
\]

Proceeding on the lines of Heyde [4] and Lemma 3.1 of Vasudeva [12], we get,

\[
P(T_n \geq x_n B_n) \geq \sum_{i=1}^{n} P(A_i) \left( P(B_i) - n P(A_i) \right) \geq n P(A_i) \left( P(B_i) - n P(A_i) \right)
\]

From Lemma 2.5, we have \( T_n \to \infty \) as \( n \to \infty \) and given \( \delta > 0 \) with \( 1 - 2\delta > 0 \), we can choose \( N_1 \) so
large such that \( P(B_i) > 1 - 2\delta \) for all \( n \geq N_1 \) and for all 
\( i = 1, 2, 3, \ldots, n \). Further from Lemma 2.5, we see that 
\( nP(A_i) \to 0 \) as \( n \to \infty \), so that we can choose \( N_2 \) so large 
that \( P(A_i) < \delta_0 \) for \( n \geq N_2 \). Thus for \( n \geq N = \max(N_1, \ 
N_2) \), we obtain from (1), 
\[ P(T_n \geq x_n B_n) \geq n(1 - 2\delta)P(X \geq (1 + \varepsilon)x_n B_n), \] this implies 
\[ \frac{P(T_n \geq x_n B_n)}{nP(X \geq x_n B_n)} \geq \frac{n(1 - 2\delta)P(X \geq (1 + \varepsilon)x_n B_n)}{nP(X \geq x_n B_n)} \]
\[ \geq (1 - 2\delta) \frac{P(X \geq (1 + \varepsilon)x_n B_n)}{P(X \geq x_n B_n)}. \]

Using Lemma 2.1, we have 
\[ \frac{P(T_n \geq x_n B_n)}{nP(X \geq x_n B_n)} \geq \frac{(1 - 2\delta)}{(1 + \varepsilon)^n} \frac{x_n B_n}{L(x_n B_n)} \]
\[ \geq \frac{(1 - 2\delta)}{(1 + \varepsilon)^n} \frac{L(1 + \varepsilon)x_n B_n}{L(x_n B_n)}. \]

Choose \( \varepsilon > 0 \) sufficiently very small such that 
\[ \lim_{n \to \infty} \frac{L((1 + \varepsilon)x_n B_n)}{L(x_n B_n)} = 1, \] one can find a constant \( C_1 > 0 \) 
such that 
\[ \liminf_{n \to \infty} \frac{P(T_n \geq x_n B_n)}{nP(X \geq x_n B_n)} > C_1 > 0. \]

In order to complete the proof, we use truncation method.
Define 
\[ Y_k = \begin{cases} 
X_k, & \text{if } f \left( \frac{k}{n} \right) X_k \leq x_n B_n \\
0, & \text{otherwise} 
\end{cases} \]
Let 
\[ R_k = f \left( \frac{k}{n} \right) X_k - f \left( \frac{k}{n} \right) Y_k, \]
\[ T_{1,n} = \sum_{k=1}^{n} f \left( \frac{k}{n} \right) Y_k \quad \text{and} \quad T_{2,n} = \sum_{k=1}^{n} R_k. \]
Notice that 
\[ P(T_n \geq x_n B_n) \leq P(T_{1,n} \geq x_n B_n) + P(T_{2,n} \neq 0). \] This implies 
\[ \frac{P(T_n \geq x_n B_n)}{nP(X \geq x_n B_n)} \leq \frac{P(T_{1,n} \geq x_n B_n)}{nP(X \geq x_n B_n)} + \frac{P(T_{2,n} \neq 0)}{nP(X \geq x_n B_n)} \] 
(2)

Observe that 
\[ P(T_{2,n} \neq 0) \leq nP(R_i \neq 0) = nP \left( f \left( \frac{1}{n} \right) X \geq x_n B_n \right), \] for 
fixed \( n \) and \( f \) is continuous \( BV \) \([0,1]\) and it attains bounds. Hence using Lemma 1, we have,
\[ \frac{P(T_{2,n} \neq 0)}{nP(X \geq x_n B_n)} = \frac{n P \left( f \left( \frac{1}{n} \right) X \geq x_n B_n \right)}{nP(X \geq x_n B_n)} \]
\[ \leq \frac{\left( \frac{1}{n} \right)^{\alpha}}{L(x_n B_n)} \frac{\int \left( \frac{1}{n} \right)^{\alpha} \phi(y) dy}{\int \phi(y) dy} \] 
(3)

Using Karamata’s representation of s.v. function, one gets that 
\[ \frac{L(x_n B_n)}{L(x_n B_n)} = \frac{a(a(x_n B_n) - \int \phi(y) dy \right) \exp \left( \int \frac{\phi(y)}{y} dy - \int \frac{\phi(y)}{y} dy \right) \]
\[ = \frac{a(a(x_n B_n) - \int \phi(y) dy \right) \exp \left( \int \frac{\phi(y)}{y} dy \right) \]
\[ \leq C_0, \] for \( y \geq x_n B_n \).
This yield 
\[ \frac{L(x_n B_n)}{L(x_n B_n)} \leq C_0 \exp \left( \delta_0 \log \left( \frac{1}{n} \right)^{\alpha} \right) \leq \frac{C_0}{n^{\delta_0}}. \] (4)

Substituting (4) in (3), one can find some constant \( C_1 \) such that the second term in (2) becomes
\[
\frac{P(T_{2n} \neq 0)}{nP(X \geq x_n B_n)} \leq C_1 \left( f\left( \frac{1}{n} \right) \right)^{\alpha - \delta}.
\]
Since \( f \in \text{BV}[0,1] \) and \( f\left( \frac{1}{n} \right) \to f(0) \) (\( \epsilon \in \text{BV}[0,1] \)), as \( n \to \infty \). Therefore we can find some constant \( C_2 (>C_1) \) such that
\[
\lim_{n \to \infty} \frac{P(T_{2n} \neq 0)}{nP(X \geq x_n B_n)} \leq C_2 < \infty. \quad (5)
\]

Now consider the first term in the right of (2). By Tchebychev’s inequality, we get
\[
\frac{P(T_{n} \geq x_n B_n)}{nP(X \geq x_n B_n)} \leq \frac{E(T_{n}^2)}{n x_n^\alpha B_n^\alpha P(X \geq x_n B_n)}.
\]
For \( k \neq m \), hence
\[
P(T_{n} \geq x_n B_n) \leq \frac{E(T_{n}^2)}{n x_n^\alpha B_n^\alpha P(X \geq x_n B_n)} \leq \frac{\sum_{k=1}^{n} f^2\left( \frac{k}{n} \right) E(Y_k^2)}{n x_n^\alpha B_n^\alpha P(X \geq x_n B_n)}.
\]
By Theorem 1, on page 544, of Feller [12] and Lemma 2.1, one gets that
\[
\sum_{k=1}^{n} f^2\left( \frac{k}{n} \right) E(Y_k^2) \leq \frac{x_n^\alpha B_n^\alpha \sum_{k=1}^{n} f^2\left( \frac{k}{n} \right) x_n^{2-\alpha} B_n^{2-\alpha} L(x_n B_n)}{n x_n^\alpha B_n^\alpha P(X \geq x_n B_n)}. \leq \frac{\sum_{k=1}^{n} f^2\left( \frac{k}{n} \right) x_n^{2-\alpha} B_n^{2-\alpha} L(x_n B_n)}{n x_n^\alpha B_n^\alpha P(X \geq x_n B_n)}
\]
Using similar steps of (4), one can find some constant \( C_3 \) such that
\[
\sum_{k=1}^{n} f^2\left( \frac{k}{n} \right) E(Y_k^2) \leq \frac{x_n^\alpha B_n^\alpha \sum_{k=1}^{n} f^2\left( \frac{k}{n} \right) x_n^{2-\alpha} B_n^{2-\alpha} L(x_n B_n)}{n x_n^\alpha B_n^\alpha P(X \geq x_n B_n)}. \leq \frac{\sum_{k=1}^{n} f^2\left( \frac{k}{n} \right) x_n^{2-\alpha} B_n^{2-\alpha} L(x_n B_n)}{n x_n^\alpha B_n^\alpha P(X \geq x_n B_n)}
\]
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Following similar steps of (4), we can find some constant $C_5$ and $\delta_0 > 0$ such that

$$\frac{L(x)}{L(x_n B_n^a)} \leq C_5 (1+\delta_0) \left( \frac{x_n B_n^a}{x} \right)^{\delta_0}.$$ 

Hence

$$D \leq \left( C_5 (1+\delta_0) \sum_{k=1}^{n} \left( \frac{k}{n} \right) \right)^2 \frac{n x_n B_n^a}{\left( x_n B_n^a \right)^{2-a}}.$$ 

Also

$$\int_0^x x^{-\alpha-\delta_0} dx = \frac{1}{1-\alpha-\delta_0} x^{1-\alpha-\delta_0} B_n^{1-\alpha-\delta_0} f^{\alpha+\delta_0-1} \left( \frac{k}{n} \right)$$

and there exists $C_6 (>C_5)$ such that

$$C_6 \left( \sum_{k=1}^{n} \left( \frac{k}{n} \right) \right)^2 \frac{n x_n B_n^a}{L(x_n B_n^a)}.$$ 

Let $M_n = x_n B_n$, where $x_n \to \infty$ and $B_n \to \infty$ as $n \to \infty$. Since $F \in DP (\alpha), 0 < \alpha < 1$, then

$$\frac{nL(M_n)}{M_a^a} \to C_7.$$ 

Using (8) one can find some constant $C_8$ such that

$$D \leq C_8 \left( \sum_{k=1}^{n} \left( \frac{k}{n} \right) \right)^2.$$ 

Since $f$ is Continuous BV $[0,1]$, therefore there exists $C_9$ such that

$$\sum_{k=1}^{n} \left( \frac{k}{n} \right) \leq n C_9$$

and hence

$$D \leq C_9 \Rightarrow B \leq C_9 \Rightarrow A \leq C_9.$$ 

From (7) and (9), we claim that

$$\frac{P \left( T_{1,n} \geq x_n B_n \right)}{nP \left( X \geq x_n B_n \right)} \to 0 \quad \text{as } n \to \infty, \text{i.e., holds.}$$

Substituting (5) and (6) in (2), we get

$$\lim \sup_{n \to \infty} \frac{P \left( T_n \geq x_n B_n \right)}{nP \left( X \geq x_n B_n \right)} < \infty.$$ 

The proof of the theorem is completed.

4. Chover’s Form of LIL

Theorem 4.1

Let $F \in DP (\alpha), 0 < \alpha < 1$. Then

$$\lim \sup_{n \to \infty} \frac{T_n B_n}{B_n^{1+\epsilon}} = e^\sigma \quad \text{a.s.}$$

Proof

To prove the assertion, it suffices to show for any $\varepsilon \in (0, 1)$, that

$$P \left( T_n \geq B_n \left( \log n \right)^\frac{1+\varepsilon}{\varepsilon} \text{ i.o.} \right) = 0 \quad (10)$$

and

$$P \left( T_n \geq B_n \left( \log n \right)^\frac{1+\varepsilon}{\varepsilon} \text{ i.o.} \right) = 1 \quad (11)$$

To prove (10), let

$$A_n = \left\{ T_n \geq B_n \left( \log n \right)^\frac{1+\varepsilon}{\varepsilon} \right\} \quad \text{and} \quad x_n = B_n \left( \log n \right)^\frac{1+\varepsilon}{\varepsilon}.$$ 

By the above Theorem 3.1, one can find a $C_{10}$ such that,

$$P(A_n) \leq C_{10} n P(X \geq x_n).$$ 

Using Lemma 2.1,

$$P(A_n) \leq C_{10} n x_n L(x_n) \leq C_{10} n B_n \log n \frac{B_n}{L(B_n) \log n} \log (1+\varepsilon).$$

Applying Lemma 2.3 with $\delta = \frac{\varepsilon}{2}$ and using the boundedness of $\theta$, $P(A_n) \leq C_{11} \left( \log n \right)^{(1+\varepsilon)/2}$ for some $C_{11} > 0$. Consequently $\sum_{n=1}^{\infty} P(A_n) < \infty$ and (3) follows from the Borel-Cantelli Lemma.

Define, for large $k$,

$$m_k = \min \left\{ j : n_j \geq \beta^{(k-1)\alpha} \right\},$$

where $\beta > 1$ and $\delta > 0$ and from the relation $T_n = T_{n_k} - T_{n_{k-1}} + T_{n_{k-1}}$, $k \geq 1$, and in order to establish (11), it is enough if we show that $\varepsilon \in (0, 1)$, that

$$P \left( T_{n_k} - T_{n_{k-1}} \geq 2B_{n_k} \left( \log n_{m_k} \right)^\frac{1+\varepsilon}{\varepsilon} \text{ i.o.} \right) = 1 \quad (13)$$

and

$$P \left( T_{n_k} - T_{n_{k-1}} \geq B_{n_k} \left( \log n_{m_k} \right)^\frac{1+\varepsilon}{\varepsilon} \text{ i.o.} \right) = 0 \quad (14)$$

Define $z_n = B_n \left( \log n \right)^{(1+\varepsilon)/2}$ and

$$D_k = \left( T_{n_k} - T_{n_{k-1}} \right) \geq z_{n_{k-1}}, \quad k \geq 1.$$ 

Note that

$$T_{n_k} - T_{n_{k-1}} \geq z_{n_{k-1}}, \quad k \geq 1.$$ 

By the above Theorem 3.1, one can find a constant $C_{12} > 0$ and $k_1$ such that for all $k \geq k_1$,

$$P(D_k) \geq C_{12} \left( n_{m_k} - n_{m_{k-1}} \right) P(X \geq 2z_{n_k})$$

$$= C_{12} n_{m_k} \left( 1 - \frac{n_{m_{k-1}}}{n_{m_k}} \right) P(X \geq 2z_{n_k}).$$
Since \( F \in \text{DP} (\omega), 0 < \alpha < 1 \) and under Kruglov’s [9] setup i.e.,\( \lim_{k \to \infty} \frac{n_{k+1}}{n_k} = r (>1) \) implies that there exists \( \lambda = r^{-1} (>1) \) such that
\[
\frac{n_{m_{k+1}}}{n_{m_k}} < \lambda < 1 \quad \text{for all } k \geq k_1. \tag{15}
\]
\[P(D_k) \geq C_{13} n_{m_k} P(X \geq 2z_{m_k}) \text{, for some } C_{13} > 0.
\]
Now following the steps similar to those used to get an upper bound of \( P(A_n) \), one can find a \( k_3 \) such that for all \( k \geq k_3 \), \( P(D_k) \geq C_{14} (\log n_k) \left(\frac{1}{2}\right) \), for some \( C_{14} > 0 \).

Hence \( \sum_{k=k_3}^{\infty} P(D_k) = \infty \). In view of the fact that \( D_k \)'s are mutually independent, by applying the Borel-Cantelli Lemma, (13) is established. Observe that
\[
P\left(T_{m_{k-1}} \geq B_{m_k} \left(\log n_k\right)^{\frac{1}{\alpha}}\right)
= P\left(T_{m_{k-1}} \geq B_{m_k} \left(\log n_k\right)^{\frac{1}{\alpha}}\right).
\]
Again by Theorem 3.1, one can find a constant \( C_{15} \) and \( k_4 \) such that for all \( k \geq k_4 \),
\[
P\left(T_{m_{k-1}} \geq B_{m_k} \left(\log n_k\right)^{\frac{1}{\alpha}}\right) \leq C_{15} n_{m_{k-1}} \frac{1}{n_{m_k}} \left(\frac{1}{2}\right)\frac{1}{\alpha}.
\]
By (12) we have \( n_{m_k} \geq \beta^{(k-1)j} \) implies
\[
n_{m_{k+1}} \geq \beta^{(k-1)j} \geq n_{m_k} \quad \text{and} \quad (15), \text{ we have, } n_{k+1} \geq \lambda n_k. \text{ Therefore, } n_{m_{k-1}} \geq \beta^{(k-1)j} \geq \lambda n_{m_{k-1}} \Rightarrow \lambda n_{m_{k-1}} \leq \beta^{(k-1)j} \Rightarrow n_{m_{k-1}} \leq \frac{1}{\lambda \beta^{(k-1)j}} = \lambda_i \beta^{k^2}, \text{ where } \lambda_i = \frac{1}{\lambda}. \text{ Hence}
\[
\frac{n_{m_{k-1}}}{n_{m_k}} \leq \frac{\lambda_i \beta^{k^2}}{\beta^{(k-1)j}} = \frac{\lambda_i}{\beta^k} \text{ and}
\]
\[
\sum_{k=k_4}^{\infty} \frac{n_{m_{k-1}}}{n_{m_k}} \left(\frac{1}{\alpha}\right) \leq \lambda_i \sum_{k=k_4}^{\infty} \frac{1}{\beta^k} \left(\frac{1}{2}\right) < \infty.
\]
Therefore \( P\left(T_{m_{k-1}} \geq B_{m_k} \left(\log n_k\right)^{\frac{1}{\alpha}} i.o\right) = 0 \),
which implies the proof of (11) follows from (13) and (14) and the proof of the theorem is completed.

Another direct application of Theorem 3.1 is for the Cesàro sums of index \( r \). Here we may write
\[
f\left(\frac{k}{n}\right) = \frac{A_{n-k}}{A_n} \text{, where } A_n = \frac{\Gamma(n+1)}{\Gamma(n+1)\Gamma(r+1)}.
\]
Using Sterling approximation, we get \( A_n = \frac{n!}{(r+1)n} \) so that
\[
f\left(\frac{k}{n}\right) = \left(1 - \frac{k}{n}\right)^{\frac{1}{r}}.
\]
The following result of Vasudeva [11] can be extended to domain of partial attraction of semi stable law and proof follows on similar lines of Theorem 2, we omit the details.

**Theorem 4.2**
Let \( F \in \text{DP} (\omega), 0 < \alpha < 1. \) Then
\[
\lim_{n \to \infty} \text{Sup} \left\{\frac{T_n}{B_n}\right\} = e^{\frac{1}{\alpha}} \text{ a.s.}, \text{ where}
\]
\[
T_n = \sum_{k=1}^{n} \left(1 - \frac{k}{n}\right)^{\frac{1}{r}} X_k \text{ and } r > 0.
\]

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**6. References**


