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Persistence in Non-Autonomous Lotka-Volterra System with Predator-Prey Ratio-Dependence and Density Dependence

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Abstract

The main purpose of this article is considering the persistence non-autonomous Lotka-Volterra system with predator-prey ratio-dependence and density dependence. We get the sufficient conditions of persistence of system, further have the necessary conditions, also the uniform persistence condition, which can be easily checked for the model is obtained.

Keywords: Uniform Persistence, Density Dependent Predator, Ratio-Dependent, Non-Autonomous Lotka-Volterra System

1. Introduction

Predator-prey behavior is a form of very common biological interaction in nature. There are many mathematical models to model predator-prey behavior such as Lotka-Volterra system [1-6], Rosenzweig-MacArthur system, Kolmogorov system, etc. Recently, models with such a prey-dependent-only response function have been facing challenges from the biology and physiology communities. Some biologists [7-9] have argued that in many situations, especially when predators have to search for food (and therefore, have to share or compete for food), the functional response in a predator-prey model should be predator-dependent.

The certain environment confines for the predator to be density dependent. The theories on the model of the predator-prey in which the predator has density dependence are not perfect [10-12]. Kartina [13] shows that predator dependence is important at not only very high predator densities on per captia predation rate but also at low predator densities. In ecology, we should consider both prey and predator density dependence, and need to take into account realistic levels of predator dependence. The qualitative analysis for the model will be difficult compared to the model with only density dependent prey [10-12].

In this paper, we will consider the permanence of non-autonomous density dependent and ratio-dependent predator-prey system

$$\begin{aligned} x' &= x\left[a(t)-b(t)x-\frac{c(t)y}{m(t)y+x}\right], \\ y' &= y\left[-d(t)-e(t)y+\frac{f(t)x}{m(t)y+x}\right], \end{aligned} \tag{1}$$

where $x(t)$ and $y(t)$ stand for the density of the prey and the predator at time t, respectively, $a(t), c(t), d(t)$, $e(t)$ and $f(t)$ are functions about time t and stand for the prey intrinsic growth rate, capture rate, death rate of the predator, predator density dependence rate, and the conversion rate, respectively, $a(t)/b(t)$ gives the carrying capacity of the prey, and $f(t)$ is the half saturation function.

2. Preparation

In this paper, we will always assume that the parameters in system (1) are periodic continuous on R and with common period $\omega > 0$.

$$\text{Denote } \overline{f} := \frac{1}{\omega}\int_0^{\omega} f(t)\mathrm{d}t\ ,$$

where $f(t)$ is a continuous and periodic function with period ω.

Motivated by the biological background of system (1), this paper only considers positive solutions of system (1). We can directly integrate the two equations of system (1) to obtain

$$x(t)=x(0)e^{\int_0^t\left[a(s)-b(s)x(s)-\frac{c(s)y(s)}{m(s)y(s)+x(s)}\right]\mathrm{d}s},$$

$$y(t)=y(0)e^{\int_0^t\left[-d(s)-e(s)y(s)+\frac{f(s)x(s)}{m(s)y(s)+x(s)}\right]\mathrm{d}s}.$$

Hence, it is obvious that the solutions $x(t), y(t)$ is positive if and only if the initial value

$$x(0)>0,\ y(0)>0.$$

In order to describe in the following results, we need first to discuss system (1) in the absence of the predator, namely, the Riccati equation

$$x'=x\left[a(t)-b(t)x\right] \tag{2}$$

with initial value $x(t_0)>x_0(x_0\neq 0)$, the solution is given by

$$x(t)=\left(\frac{1}{x_0}e^{-\int_{t_0}^t a(s)\mathrm{d}s}+\int_{t_0}^t b(s)e^{-\int_s^t a(\tau)\mathrm{d}\tau}\mathrm{d}s\right)^{-1}. \tag{3}$$

Clearly, the null solution $x(t)=0$ exists in equation (2). By the uniqueness of solutions, we can see that solutions with positive initial values remain positive.

One can easily show that $x^*(t+\omega)=x^*(t)$ is a periodic ω solution of (2), and

$$x^*(t)=\left(e^{\int_0^\omega a(s)ds}-1\right)\left(\int_t^{t+\omega}b(s)e^{-\int_s^t a(\tau)d\tau}ds\right)^{-1}.$$

The coming lemma will play a key role in proof of the following important results.

Definition 2.1. System (1) is said to be permanent if there exist positive constants δ,Δ with $0<\delta\leq\Delta$ such that

$$\min\left\{\liminf_{t\to+\infty}x(t),\ \liminf_{t\to+\infty}y(t)\right\}\geq\delta,$$

$$\max\left\{\limsup_{t\to+\infty}x(t),\ \limsup_{t\to+\infty}y(t)\right\}\leq\Delta$$

For all solutions of (1) with positive initial values. System (1) is said to be nonpermanent if there is a positive solution $\left(x(t),y(t)\right)$ of (1) satisfying

$$\min\left\{\liminf_{t\to+\infty}x(t),\ \liminf_{t\to+\infty}y(t)\right\}=0.$$

Lemma 2.1 If $b(t)\geq 0$ for all $t\in R$ and $\bar{b}>0$, then the Equation (2) has a unique nonnegative ω-periodic solution $x^*(t)$ is globally asymptotically stable for $x(t)$ with positive initial value $x(t_0)=x_0>0$. Moreover, if $\bar{a}>0$, then $x^*(t)>0$.

In order to prove the important theorem, firstly, we need to prove the following theorems.

Theorem 2.1 If $\bar{a}>0,\bar{b}>0,\bar{e}>0,$ then exist positive constants M_x^0, M_y^0 and

$$\limsup_{t\to\infty}x(t)\leq M_x^0,\quad \limsup_{t\to\infty}y(t)\leq M_y^0,$$

For all solution $\left(x(t),y(t)\right)$ of (1) with positive initial values.

Proof. If $u(t)$ is solution of the following equation

$$u'=u\left[a(t)-b(t)u\right],$$

and assume $M_x^0=\max\limits_{0\leq t<\omega}\left\{u(t)\right\}$ is fixed. From the system (1), we can obtain that

$$x'\leq x\left[a(t)-b(t)x\right],$$

by comparison theorem and Lemma 2.1, we have there is a constant $T_1>0$ such that

$$\limsup_{t\to\infty}x(t)\leq M_x^0,\quad t\geq T_1.$$

And if $v(t)$ is solution of the following equation

$$v'=v\left[\left|d(t)\right|+f(t)-e(t)v\right],$$

assume $M_y^0=\max\limits_{0\leq t<\omega}\left\{v(t)\right\}$. Further we have

$$y'\leq y\left[\left|d(t)\right|+f(t)-e(t)y\right],$$

similarly, we can obtain that there is a constant $T_2>T_1$ such that

$$\limsup_{t\to\infty}y(t)\leq M_y^0,\quad t\geq T_2.$$

We complete the proof of Theorem 2.1.

Lemma 2.2. If the condition

$$(H_0)\quad \bar{a}>0,\bar{b}>0,\bar{e}>0,\bar{a}>\overline{c/m}$$

holds, then exist positive constants α and

$$\limsup_{t\to\infty}x(t)\geq\alpha, \tag{5}$$

for all solution $\left(x(t),y(t)\right)$ of (1) with positive initial values.

Theorem 2.2. If the condition (H_0) holds, then exist positive constant β and

$$\liminf_{t\to\infty}x(t)\geq\beta, \tag{6}$$

for all solution $\left(x(t),y(t)\right)$ of (1) with positive initial values.

Proof. If the conclusion (6) is not true, then there is a sequence $\{z_n\}\subset R_+^2$ such that

$$\liminf_{t\to\infty}x(t,z_n)<\frac{\alpha}{2n^2},\quad n=1,2,\cdots$$

by Lemma 2.2, we have

$$\limsup_{t\to\infty}x(t,z_n)>\alpha,\quad n=1,2,\cdots$$

there are two time sequences $\{s_q^{(n)}\}$ and $\{t_q^{(n)}\}$ satisfying the following conditions

$$0<s_1^{(n)}<t_1^{(n)}<s_2^{(n)}<t_2^{(n)}<\cdots<s_q^{(n)}<$$

$$t_q^{(n)}<\cdots,s_q^{(n)}\to\infty,\ t_q^{(n)}\to\infty,\ q\to\infty$$

$$x\left(s_q^{(n)}, z_n\right) = \frac{\alpha}{n},\ x\left(t_q^{(n)}, z_n\right) = \frac{\alpha}{n^2},$$

$$\frac{\alpha}{n^2} < x(t, z_n) < \frac{\alpha}{n}, \quad t \in \left(s_q^{(n)},\ t_q^{(n)}\right).$$

By Theorem 2.1, for a given positive integer n, there is a $T^{(n)} > 0$ such that

$$x(t, z_n) \le M_x^0.$$

Thus, for any $t \ge T^{(n)}$,

$$x'(t, z_n) \ge x(t, z_n)\left(a(t) - \frac{c(t)}{m(t)} - b(t)M_x^0\right). \tag{7}$$

For $s_q^{(n)} \to \infty,\ q \to \infty$, there is a positive integer $K^{(n)}$ such that $s_q^{(n)} > T^{(n)}$ for all $q \ge K^{(n)}$. By integrating (7) from $s_q^{(n)}$ to $t_q^{(n)}$ for any $q \ge K^{(n)}$, we obtain

$$x\left(t_q^{(n)}, z_n\right) \ge$$

$$x\left(s_q^{(n)}, z_n\right)\exp\int_{s_q^{(n)}}^{t_q^{(n)}}\left(a(t) - \frac{c(t)}{m(t)} - b(t)M_x^0\right)\mathrm{d}t,$$

hence,

$$\int_{s_q^{(n)}}^{t_q^{(n)}}\left(-a(t) + \frac{c(t)}{m(t)} + b(t)M_x^0\right)\mathrm{d}t \ge \ln n,$$

for $q \ge K^{(n)}$.

If

$$\int_0^{\omega}\left(a(t) - \frac{c(t)}{m(t)} - b(t)M_x^0\right)\mathrm{d}t \ge 0,$$

this leads to a contradiction. Otherwise

$$\int_0^{\omega}\left(a(t) - \frac{c(t)}{m(t)} - b(t)M_x^0\right)\mathrm{d}t < 0,$$

we have

$$t_q^{(n)} - s_q^{(n)} \to \infty \text{, as } n \to \infty, q \ge K^{(n)}.$$

There are constants $p > 0,\ N_0 > 0$, such that for $n \ge N_0, q \ge K^{(n)}$,

$$\frac{\alpha}{n} < \varepsilon_x, t_q^{(n)} - s_q^{(n)} > 2p.$$

Thus, we can get as $n \ge N_0, q \ge K^{(n)}$

$$x(t, z_n) < \varepsilon_x, \quad t \in \left[s_q^{(n)},\ t_q^{(n)}\right].$$

Therefore,

$$x'(t, z_n) \ge x(t, z_n)\left(a(t) - \frac{c(t)}{m(t)} - b(t)\varepsilon_x\right),$$

and choosing sufficiently small positive numbers $\varepsilon_x < 1$ such that

$$\int_0^{\omega}\left(a(t) - \frac{c(t)}{m(t)} - b(t)\varepsilon_x\right)\mathrm{d}t > 0, \tag{8}$$

and by the following Equation (8), we can obtain

$$\frac{\beta}{n^2} = x\left(t_q^{(n)}, z_n\right) \ge x\left(s_q^{(n)} + p, z_n\right)$$

$$\exp\int_{s_q^{(n)}+p}^{t_q^{(n)}}\left(a(t) - \frac{c(t)}{m(t)} - b(t)\varepsilon_x\right)dt > \frac{\beta}{n^2},$$

which is a contradiction. This completes the proof of Theorem 2.2.

By Lemma 2.1 and (H_0), the following equation,

$$x' = x\left[a(t) - \frac{c(t)}{m(t)} - b(t)x\right], \tag{9}$$

has a unique positive $\omega-$ periodic solution

$$x^{**}(t) = \left(e^{\int_0^{\omega}\left(a(s) - \frac{c(s)}{m(s)}\right)\mathrm{d}s} - 1\right)$$

$$\left(\int_t^{t+\omega} b(s) e^{-\int_s^t\left(a(\tau) - \frac{c(\tau)}{m(\tau)}\right)\mathrm{d}\tau}\mathrm{d}s\right)^{-1}.$$

Lemma 2.3. If system (1) satisfies (H_0) and

$$(H_1) \quad \int_0^{\omega}\left(-\mathrm{d}(t) + \frac{f(t)x^{**}(t)}{m(t) + x^{**}(t)}\right)\mathrm{d}t > 0,$$

where $x^{**}(t)$ is the unique periodic solution of equation (9), then exist positive constants γ and

$$\limsup_{t\to\infty} y(t) \ge \gamma, \tag{10}$$

for all solution $\left(x(t), y(t)\right)$ of (1) with positive initial values.

Theorem 2.3. If system (1) satisfies (H_0) and (H_1), then exist positive constants η and

$$\liminf_{t\to\infty} y(t) \ge \eta, \tag{11}$$

for all solution $\left(x(t), y(t)\right)$ of (1) with positive initial values.

Proof. If the conclusion (11) is not true, then there is a sequence $\{z_m\} \subset R_+^2$ such that

$$\liminf_{t\to\infty} y(t, z_m) < \frac{\gamma}{(m+1)^2}, \quad m = 1, 2, \cdots$$

but, by Lemma 2.3,

$$\limsup_{t\to\infty} y(t, z_m) \ge \gamma, \quad m = 1, 2, \cdots$$

hence, there are two time sequences $\left\{s_q^{(m)}\right\}$ and $\left\{t_q^{(m)}\right\}$ satisfying the following conditions

$$0 < s_1^{(m)} < t_1^{(m)} < s_2^{(m)} < t_2^{(m)} < \cdots < s_q^{(m)} <$$

$$t_q^{(m)} < \cdots, s_q^{(m)} \to \infty,\ t_q^{(m)} \to \infty,\ q \to \infty$$

and

$$y\left(s_q^{(m)}, z_m\right) = \frac{\gamma}{m+1},\ y\left(t_q^{(m)}, z_m\right) = \frac{\gamma}{\left(m+1\right)^2},$$

$$\frac{\gamma}{(m+1)^2} < y(t, z_m) < \frac{\gamma}{m+1}, \quad t \in \left(s_q^{(m)}, \ t_q^{(m)}\right).$$

By Theorem 2.1, for a given integer $m > 0$, there is a $T_{(1)}^m > 0$, such that

$$y(t, z_m) \le M_y^0, \quad t \ge T_1^{(m)}.$$

Because of $s_q^{(m)} \to \infty, \ q \to \infty$, there is a positive integer $K^{(m)}$, such that $s_q^{(m)} > T_{(1)}^{(m)}$ as $q \ge K^{(m)}$. Hence,

$$y'(t, z_m) \ge y(t, z_m)\left(-|d(t)| - e(t)M_y^0\right), \quad t \in \left[s_q^{(m)}, t_q^{(m)}\right].$$

Integrating the above inequality from $s_q^{(m)}$ to $t_q^{(m)}$, we have

$$y\left(t_q^{(m)}, z_m\right) \ge y\left(s_q^{(m)}, z_m\right) \exp \int_{s_q^{(m)}}^{t_q^{(m)}} \left(-|d(t)| - e(t)M_y^0\right) \mathrm{d}t,$$

that is to say,

$$\int_{s_q^{(m)}}^{t_q^{(m)}} \left(|d(t)| + e(t)M_y^0\right) \mathrm{d}t \ge \ln(m+1), \quad q \ge K^{(m)}.$$

Thus, we obtain

$$t_q^{(m)} - s_q^{(m)} \to \infty, \text{ as } \ m \to \infty, q \ge K^{(m)}.$$

By (H_1), we can choose constant $\varepsilon_0 > 0$, such that

$$\int_0^{\omega} \left(-d(t) + \frac{f(t)\left(x^{**}(t) - \varepsilon_0\right)}{m(t)\varepsilon_0 + x^{**}(t) - \varepsilon_0} - e(t)\varepsilon_0 \right) \mathrm{d}t > 0. \quad (12)$$

By (12), we obtain that there are constants $p > 0, N_0 > 0$, such that for $m \ge N_0, q \ge K^{(m)}$, and $r \ge p$,

$$\frac{\gamma}{m+1} < 2\varsigma < \varepsilon_x, t_q^{(m)} - s_q^{(m)} > 2p, \quad (13)$$

and

$$\int_0^{r} \left(-d(t) + \frac{f(t)\left(x^{**}(t) - \varepsilon_0\right)}{m(t)\varepsilon_0 + x^{**}(t) - \varepsilon_0} - e(t)\varepsilon_0 \right) \mathrm{d}t > 0. \quad (14)$$

In addition,

$$y(t, z_m) < 2\varsigma < \varepsilon_0, \quad t \in \left[s_q^{(m)}, \ t_q^{(m)}\right],$$
$$m \ge N_0, q \ge K^{(m)}$$

It is easy to obtain that,

$$x'(t, z_m) \ge x(t, z_m)\left(a(t) - \frac{c(t)}{m(t)} - b(t)x(t, z_m) \right),$$

From $x^{**}(t)$ of (12) and the comparison theorem, we can get

$$x(t, z_m) \ge x^{**}(t), \quad t \in \left[s_q^{(m)}, \ t_q^{(m)}\right].$$

Hence, there exists $T^0 > p$, which is dependent of m and q, such that

$$x(t, z_m) \ge x^{**}(t) - \varepsilon_0, \quad t \in \left[s_q^{(m)} + T^0, \ t_q^{(m)}\right].$$

Hence,

$$y'(t, z_m) \ge y(t, z_m) \left(-d(t) + \frac{f(t)\left(x^{**}(t) - \varepsilon_0\right)}{m(t)\varepsilon_0 + x^{**}(t) - \varepsilon_0} - e(t)\varepsilon_0 \right) := y(t, z_m)g(t)$$

Integrating the above inequality from $s_q^{(m)} + T^0$ to $t_q^{(m)}$ yields

$$\frac{\gamma}{(m+1)^2} = g\left(t_q^{(m)}, z_m\right) \ge y\left(s_q^{(m)} + T^0, z_m\right) \exp \int_{s_q^{(m)}+T^0}^{t_q^{(m)}} g(t) \mathrm{d}t \ge \frac{\gamma}{(m+1)^2},$$

From (12). This is a contradiction. We complete proof of Theorem 2.3.

3. The Main Conclusions

From Definition 2.1 and Theorems 2.1-2.3, we can get the following theorem.

Theorem 3.1. If the condition (H_0) and (H_1) hold, the system (1) will be uniform persistent.

Lemma 3.1. If system (1) satisfies (H_0) and

$$(H_2) \quad \int_0^{\omega} \left(-d(t) + \frac{f(t)x^{**}(t)}{m(t) + x^{**}(t)} \right) \mathrm{d}t \le 0,$$

where $x^{**}(t)$ is the unique periodic solution of equation (9), then $\lim\limits_{t \to +\infty} y(t) = 0,$ for all solution $\left(x(t), y(t)\right)$ of (1) with positive initial values.

Proof By (H_2), for any given $0 < \varepsilon < 1$, there exists $\varepsilon_1 (0 < \varepsilon_1 < \varepsilon)$ and $\varepsilon_0 > 0$, such that

$$\int_0^{\omega} \left(-d(t) + \frac{f(t)(x^{**}(t) + \varepsilon_1)}{m(t)\varepsilon + (x^{**}(t) + \varepsilon_1)} - e(t)\varepsilon \right) \mathrm{d}t \le -\bar{e}\frac{\varepsilon}{2} < -\varepsilon_0. \quad (15)$$

From (1), easily get

$$x' \le x(a(t) - b(t)x),$$

hence, for given ε_1, there exists $T_1 > 0$, have

$$x(t) \le x^{**}(t) + \varepsilon_1, \quad t \ge T_1. \quad (16)$$

Together with (15), we have

$$\int_0^{\omega} \left(-d(t) + \frac{f(t)x(t)}{m(t)\varepsilon + x(t)} - e(t)\varepsilon \right) \mathrm{d}t \le \varepsilon_0, \quad t > T_1. \quad (17)$$

There will exists $T_2 > T_1$, and $y(T_2) < \varepsilon$. Otherwise,

$$\varepsilon \le y(t)$$
$$\le y(T_1)\exp\int_{T_1}^{t}\left(-d(s)+\frac{f(s)x(s)}{m(s)\varepsilon + x(s)} - e(s)\varepsilon\right)\mathrm{d}s \to 0,$$
$$t \to +\infty,$$

it implies $\varepsilon \le 0$, which is a contradiction. In the following we will prove

$$y(t) \le \varepsilon\, e^{D(\varepsilon)\omega}, \quad t \ge T_2,$$

where

$$D(\varepsilon) = \max_{0\le t\le\omega}\left\{d(t)+\frac{f(t)(x^{**}(t)+\varepsilon)}{m(t)\varepsilon + x^{**}(t)+\varepsilon}+e(t)\varepsilon\right\}.$$

Otherwise, there exists $T_3 > T_2$, such that

$$y(T_3) > \varepsilon\, e^{D(\varepsilon)\omega}.$$

By the continuity of $y(t)$, there must exist $T_4 \in (T_2, T_3)$ such that $y(T_4) = \varepsilon$ and $y(t) > \varepsilon$, for $t \in (T_4, T_3)$. Let N_1 be the nonnegative integer, such that $T_3 \in (T_4 + N_1\omega, T_4 + (N_1+1)\omega)$, by (16) and (17), we have

$$\varepsilon\, e^{D(\varepsilon)\omega} < y(T_3)$$
$$< y(T_4)\exp\int_{T_4}^{T_3}\left(-d(t)+\frac{f(t)x(t)}{m(t)\varepsilon + x(t)} - e(t)\varepsilon\right)\mathrm{d}t$$
$$= \varepsilon\exp\left(\int_{T_4}^{T_4+N_1\omega}+\int_{T_4+N_1\omega}^{T_3}\right)$$
$$\cdot\left(-d(t)+\frac{f(t)x(t)}{m(t)\varepsilon + x(t)} - e(t)\varepsilon\right)\mathrm{d}t$$
$$< \varepsilon\exp\int_{T_4+N_1\omega}^{T_3}\left(d(t)+\frac{f(t)x(t)}{m(t)\varepsilon + x(t)} + e(t)\varepsilon\right)\mathrm{d}t$$
$$< \varepsilon\exp\max\left\{d(t)+\frac{f(t)(x^{**}(t)+\varepsilon)}{m(t)\varepsilon + x^{**}(t)+\varepsilon}+e(t)\varepsilon\right\}\omega = \varepsilon\, e^{D(\varepsilon)\omega},$$

which is a contradiction. That is to say, $y(t) \le \varepsilon e^{D(\varepsilon)\omega}$, by the arbitrariness of ε, we derive $\lim_{t\to+\infty} y(t) = 0$, this completes the proof of Lemma 3.1.

From Theorem 3.1 and Lemma 3.1, we can easily get the following theorem.

Theorem 3.2. Supposed that the condition (H_0) holds, then the system (1) is uniform persistent if and only if (H_1) is true.

Example 3.1. In system (1), let

$$a(t) = 3,\ b(t) = 2+\cos t,\ c(t) = 2,$$
$$d(t) = \frac{1}{10}+\frac{1}{20}\sin t,\ e(t) = 1+\frac{1}{2}\sin t,$$
$$f(t) = 1,\ m(t) = 2 \text{ and } \omega = 2\pi$$

then Equation (1) becomes

$$x' = x\left[3-(2+\cos t)x - \frac{2y}{2y+x}\right],\ y' = y\left[-\frac{1}{10}-\frac{1}{20}\sin t-\left(1+\frac{1}{2}\sin t\right)y+\frac{x}{2y+x}\right] \tag{18}$$

By simple computation, we have

$$\bar{a} > 0, \bar{b} > 0, \bar{e} > 0, \bar{a} - \overline{c/m} > 0.$$

From (4), we can obtain solution of equation

$$x' = x\left[3-(2+\cos t)x\right]$$

is

$$x^{**}(t) = \frac{1}{1+\frac{2}{5}\cos t+\frac{1}{5}\sin t}.$$

Hence,

$$\int_0^{2\pi}\left(-d(t)+\frac{f(t)x^{**}(t)}{m(t)+x^{**}(t)}\right)\mathrm{d}t \approx 1.566 > 0.$$

That is to say, system (18) is uniform persistent by Theorem 3.1.

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