Some Actuarial Formula of Life Insurance for Fuzzy Markets*

Qiaoyu Huang, Liang Lin, Tao Sun
College of Science, Guilin University of Technology, Guilin, China
E-mail: Lzcs135@163.com
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Abstract

This paper presents an actuarial model of life insurance for fuzzy markets based on Liu process. At first, some researches about an actuarial model of life insurance for stochastic market and concepts about fuzzy process have been reviewed. Then, an actuarial model of life insurance for fuzzy process is formulated.

Keywords: Fuzzy Actuarial Model, Fuzzy Process, Liu Process, Geometric Liu Process, Actuarial Formula

1. Introduction


The remainder of this paper is structured as follows. Section 2 is intended to introduce some useful concepts of fuzzy process as they are needed. Section 3 reviews Gao and Zhao’s actuarial model for fuzzy market. An actuarial model of life insurance with payouts increased for fuzzy process is formulated in Section 4. Some actuarial formula of pure premium of the n-year continuous life insurance model is discussed in Section 5, as discount factor is a fuzzy process. Finally, some remarks are made in the concluding section.

2. Preliminaries

In this section, we will introduce some useful definitions and properties about fuzzy process.

2.1. Fuzzy Process

Definition 1. (Liu [7]) Given an index set \( T \) and a credibility space \( (\Theta, P, Cr) \), a fuzzy process is a function from \( T \times (\Theta, P, Cr) \) to the set of real numbers. In other words, a fuzzy process \( \{ X_t \}_{t \in T} \) is a function of two variables such that the function \( X^* \) is a fuzzy variable for each \( t \in T \). For simplicity, sometimes we simply use the symbol \( X_t \) instead of longer notation \( X^* \).

Definition 2. (Liu [7]) A fuzzy process \( X_t \) is said to have independent increments if

\[
X_{t_i} - X_{t_{i-1}}, X_{t_{i+1}} - X_{t_i}, \ldots, X_{t_{k-1}} - X_{t_k}
\]

are independent fuzzy variables for any times \( t_0 < t_1 < \cdots < t_k \). A fuzzy process \( X_t \) is said to have stationary increments if, for any given \( s > 0 \), the \( X_{t+s} - X_t \) are identically distributed fuzzy variables for all \( t > 0 \).
2.2. Liu Process

Definition 3. (Liu [7]) A fuzzy process $C_t$ is said to be a Liu process if

1) $C_0 = 0$
2) $C_t$ has stationary and independent increments
3) every increment $C_{t+s} - C_t$ is a normally distributed fuzzy variable with expected value $et$ and variance $\sigma^2 t^2$, whose membership function is

$$\mu(x) = 2 \left( 1 + \exp \left( \frac{\pi|x - et|}{6\sigma^2 t} \right) \right)^{-1}, x \in R$$

(2)

The parameters $e$ and $\sigma$ ($>0$) are called the drift and diffusion coefficients, respectively. The Liu process is said to be standard if $e = 0$ and $\sigma = 1$. The Liu process plays the role of the counterpart of Brownian motion.

Definition 4. (Liu [7]) Let $C_t$ be a standard Liu process. Then the fuzzy process

$$G_t = \exp(\text{et} + \sigma C_t)$$

is called a geometric Liu process, or sometimes exponential Liu process. Li [8] has deduced that

$$G_t$$

is of a lognormal membership function

$$\mu(x) = e^{\sqrt{6\sigma^2} \csc \left( \sqrt{6\sigma^2} \right)}, \sigma < \pi / \sqrt{6}$$

had been proved by Li and Qin [8].

3. Fuzzy Model of Discount Factor

In this section we review a fuzzy model of discount factor which presented by Gao and Zhao, the model is assumed that interests rate is a trapezoidal fuzzy variables.

Accumulation function for force of interest is

$$y(t) = \delta t, 0 \leq t \leq n$$

$V_T$ is present value of payment at the end of $m$ year. As insurer, the present value for Survival annuity paid 1 per year is $y_m$.

$$V_T = \begin{cases} e^{-y(K-m)}, & m \leq K < n \\ 0, & K \geq n \end{cases}$$

where $\delta$ is a trapezoidal fuzzy variable $(a, b, c, d)$, $a \leq b \leq c \leq d$.

4. An Actuarial Model for Fuzzy Process

4.1. The Model

The paper hypothesizes that interest rate has two components: risk-free interest and risk interest. $\delta$ is used to represent force of interest, $y(t) = \delta_0 t + \beta C_t$ is the accumulation function for force of interest, where $\delta_0$ is a part of interests with no risk, $C_t$ is a standard Liu process, used to describe the interests on risk. $\beta$ is $-\text{coefficient}$. We have the discount factor:

$$V_t = \exp(-y(t))$$

The actuarial model for Liu process is:

$$y(t) = \delta_0 t + \beta C_t$$

$$V_t = \exp(-y(t))$$

4.2. Expected Value for $V_t$

$C_t$ is a normally distribution fuzzy variable $C_t \sim N(\text{et}, \sigma^2 t^2)$. Because $C_t$ is a standard Liu process with $e = 0, \sigma = 1$, $C_t \sim N(0, t)$. If $t \beta < \pi / \sqrt{6}$, then it is easy to prove that the expected value is

$$E(V_t) = E\left[ \exp(-\delta_0 t - \beta C_t) \right]$$

$$= e^{-\delta_0 t} E(e^{-\beta C_t})$$

$$= e^{-\delta_0 t} \int_0^\infty C_t \{e^{-\beta x} \geq x\} dx - \int_0^\infty C_t \{e^{-\beta x} \leq x\} dx$$

$$= e^{-\delta_0 t} \int_0^\infty C_t \{ x \leq \frac{1}{\beta} \ln x \} dx$$

$$= e^{-\delta_0 t} \int_0^\infty 1 - \left( 1 + \exp \left( \frac{\pi}{\sqrt{6\beta}} \right) \right) \left( \frac{1}{\beta} \ln x \right) \right) dx$$

$$= e^{-\delta_0 t} \int_0^\infty 1 - \frac{1}{1 + x \left( \frac{\pi}{\sqrt{6\beta}} \right)} dx$$

$$= e^{-\delta_0 t} \int_0^\infty \frac{1}{1 + x \left( \frac{\pi}{\sqrt{6\beta}} \right)} dx$$

$$= e^{-\delta_0 t} \sqrt{6\beta t} \csc \left( \sqrt{6\beta t} \right)$$

Otherwise, we have $E(V_t) = \infty$.

5. Some Actuarial Formula of Pure Premium of the N-Year Continuous Life Insurance

The pure premium of the n-year continuous life insur-
ance with payment \( b(t) \) is
\[
\mathcal{A}_{x\pi} = E[V_t] = \int_0^\infty b(t)E[V_t], p, \mu_{x+i}dt = \int_0^\infty b(t)e^{-\delta t}csc(\sqrt{6}\beta t)p, \mu_{x+i}dt
\]
where \( \beta < \pi/\sqrt{6} \)

5.1. Pure Premium of the N-Year Continuous Life Insurance under the Assumption of de Moivre

Under the assumption of de Moivre, we have
\[
\mu_{x+i} = \frac{1}{\omega-x} , \quad p, \mu_{x+i} = \frac{1}{\omega-x}
\]
Then
\[
\mathcal{A}_{x\pi} = \int_0^\infty b(t)e^{\delta t}csc(\sqrt{6}\beta t)p, \mu_{x+i}dt = \int_0^\infty \frac{\sqrt{6}\beta}{\omega-x} be^{-\delta t}csc(\sqrt{6}\beta t)dt, \beta < \pi/\sqrt{6}
\]
where \( \mathcal{A}_{x\pi} \) respected to pure premium of the n-year continuous life insurance.

If the compensation is a constant, that is to say \( b(t) = b , \) so we have
\[
\mathcal{A}_{x\pi} = \int_0^\infty b e^{\delta t}csc(\sqrt{6}\beta t)p, \mu_{x+i}dt
\]
\[
= \int_0^\infty \frac{\sqrt{6}\beta}{\omega-x} be^{-\delta t}csc(\sqrt{6}\beta t)dt, \beta < \pi/\sqrt{6}
\]
If the payment amount is increasing linearly, \( b(t) = b + t , \) then
\[
\mathcal{A}_{x\pi} = \int_0^\infty (b + t)e^{\delta t}csc(\sqrt{6}\beta t)p, \mu_{x+i}dt = \int_0^\infty \frac{\sqrt{6}\beta}{\omega-x} be^{-\delta t}csc(\sqrt{6}\beta t)dt, \beta < \pi/\sqrt{6}
\]
If the payment amount is increasing geometrically, \( b(t) = t^k , \)
\[
\mathcal{A}_{x\pi} = \int_0^\infty t^ke^{-\delta t}csc(\sqrt{6}\beta t)p, \mu_{x+i}dt = \int_0^\infty \frac{\sqrt{6}\beta}{\omega-x} be^{-\delta t}csc(\sqrt{6}\beta t)dt, \beta < \pi/\sqrt{6}
\]
If the payment amount is increasing exponentially, \( b(t) = e^{\alpha t} \),
\[
\mathcal{A}_{x\pi} = \int_0^\infty e^{\alpha t}e^{-\delta t}csc(\sqrt{6}\beta t)p, \mu_{x+i}dt = \int_0^\infty \frac{\sqrt{6}\beta}{\omega-x} be^{-\delta t}csc(\sqrt{6}\beta t)dt, \beta < \pi/\sqrt{6}
\]

5.2. Pure Premium of the N-Year Continuous Life Insurance under the Assumption of Gompertz

Under the assumption of Gompertz, \( \mu_{x+i} = BC^{x+i}, \) \( B > 0, \quad C \geq 1, \) and
\[
\int_0^\infty p, \mu_{x+i}dt = BC^{x+i}e^{\frac{B}{6}e^{C(1-C)}}
\]
If the compensation is a constant, that is to say \( b(t) = b, \) so we have
\[
\mathcal{A}_{x\pi} = \int_0^\infty b e^{\delta t}csc(\sqrt{6}\beta t)p, \mu_{x+i}dt = \int_0^\infty \frac{\sqrt{6}\beta}{\omega-x} be^{-\delta t}csc(\sqrt{6}\beta t)dt, \beta < \pi/\sqrt{6}
\]
If the payment amount is increasing linearly, \( b(t) = b + t, \) then
\[
\mathcal{A}_{x\pi} = \int_0^\infty (b + t)e^{\delta t}csc(\sqrt{6}\beta t)p, \mu_{x+i}dt = \int_0^\infty \frac{\sqrt{6}\beta}{\omega-x} be^{-\delta t}csc(\sqrt{6}\beta t)dt, \beta < \pi/\sqrt{6}
\]
If the payment amount is increasing geometrically, \( b(t) = t^k, \)
\[
\mathcal{A}_{x\pi} = \int_0^\infty t^ke^{\delta t}csc(\sqrt{6}\beta t)p, \mu_{x+i}dt = \int_0^\infty \frac{\sqrt{6}\beta}{\omega-x} be^{-\delta t}csc(\sqrt{6}\beta t)dt, \beta < \pi/\sqrt{6}
\]
If the payment amount is increasing exponentially, \( b(t) = e^{\alpha t} \),
\[
\mathcal{A}_{x\pi} = \int_0^\infty e^{\alpha t}e^{\delta t}csc(\sqrt{6}\beta t)p, \mu_{x+i}dt = \int_0^\infty \frac{\sqrt{6}\beta}{\omega-x} be^{-\delta t}csc(\sqrt{6}\beta t)dt, \beta < \pi/\sqrt{6}
\]
5.3. Pure Premium of the N-Year Continuous Life Insurance under the Assumption of Makeham

Under the assumption of Makeham,
\[ \mu_{x+t} = A + BC^{x+t} , \quad B > 0, \quad C \geq 1, \quad A \geq -B \]

\[ \alpha = \frac{A}{n} e^{-\theta t} \frac{e^{\xi t}}{C^{(x+t)}} e^{\eta t} \]

If the compensation is a constant, that is to say \( b(t) = b \), so we have
\[ \overline{A}_{x+t} = \int_0^\infty b(t) e^{-\theta t} \sqrt{6} \beta \csc \left( \sqrt{6} \beta t \right) p_{x+t} dt \]
\[ \overline{A}_{x+t} = \int_0^\infty b(t) e^{-\theta t} \sqrt{6} \beta \csc \left( \sqrt{6} \beta t \right) \left( A + BC^{x+t} \right) \]
\[ \beta t < \pi / \sqrt{6} \]

If the payment amount is increasing linearly, \( b(t) = b + t \), then
\[ \overline{A}_{x+t} = \int_0^\infty b(t) e^{-\theta t} \sqrt{6} \beta \csc \left( \sqrt{6} \beta t \right) \left( A + BC^{x+t} \right) \]
\[ \beta t < \pi / \sqrt{6} \]

If the payment amount is increasing geometrically, \( b(t) = t^\theta \)
\[ \overline{A}_{x+t} = \int_0^\infty b(t) e^{-\theta t} \sqrt{6} \beta \csc \left( \sqrt{6} \beta t \right) \left( A + BC^{x+t} \right) \]
\[ \beta t < \pi / \sqrt{6} \]

5.4. Pure Premium of the N-Year Continuous Life Insurance under the Assumption of Weibull

Under the assumption of Weibull,
\[ \mu_{x+t} = k \left( x + t \right)^n , \quad k > 0, n > 0 \]

\[ \alpha = \frac{A}{n} e^{-\theta t} \frac{e^{\xi t}}{C^{(x+t)}} e^{\eta t} \]

If the compensation is a constant, that is to say \( b(t) = b \), so we have
\[ \overline{A}_{x+t} = \int_0^\infty b(t) e^{-\theta t} \sqrt{6} \beta \csc \left( \sqrt{6} \beta t \right) p_{x+t} dt \]
\[ \overline{A}_{x+t} = \int_0^\infty b(t) e^{-\theta t} \sqrt{6} \beta \csc \left( \sqrt{6} \beta t \right) \left( A + BC^{x+t} \right) \]
\[ \beta t < \pi / \sqrt{6} \]

If the payment amount is increasing linearly, \( b(t) = b + t \), then
\[ \overline{A}_{x+t} = \int_0^\infty b(t) e^{-\theta t} \sqrt{6} \beta \csc \left( \sqrt{6} \beta t \right) \left( A + BC^{x+t} \right) \]
\[ \beta t < \pi / \sqrt{6} \]

If the payment amount is increasing geometrically, \( b(t) = t^\theta \)
\[ \overline{A}_{x+t} = \int_0^\infty b(t) e^{-\theta t} \sqrt{6} \beta \csc \left( \sqrt{6} \beta t \right) \left( A + BC^{x+t} \right) \]
\[ \beta t < \pi / \sqrt{6} \]
If the payment amount is increasing exponentially,

\[ b(t) = e^{\alpha t} \]

\[ \overline{X}_n(t) = \int_0^t b(t)e^{-\delta t} \sqrt{6} \beta t \csc \left( \sqrt{6} \beta t \right) dt \]

\[ = \sqrt{6} \beta \int_0^t e^{-\delta t} e^{x(\alpha-\delta)} \csc \left( \sqrt{6} \beta t \right) k(x+1)^n \] \[ \cdot e^{i(x+1)\left[ x^{-1} - (x+1)^{-1} \right]} \] \[ \cdot \int_0^t (x+1)^n e^{(x+1)(\alpha-\delta)} e^{i(x+1)(x+1)^{-1}} \csc \left( \sqrt{6} \beta t \right) dt \]

\[ \beta t < \pi / \sqrt{6} \]

6. Conclusions

The main contribution of this paper is to suggest a new actuarial for fuzzy markets by means of Liu process. Some actuarial formulas of pure premium of the n-year continuous life insurance on the proposed fuzzy model are investigated.

7. References


