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# Definition of Laplace Transforms for Distribution of the First Passage of Zero Level of the Semi-Markov Random Process with Positive Tendency and Negative Jump

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## Abstract

One of the important problems of stochastic process theory is to define the Laplace transforms for the distribution of semi-markov random processes. With this purpose, we will investigate the semi-markov random processes with positive tendency and negative jump in this article. The first passage of the zero level of the process will be included as a random variable. The Laplace transforms for the distribution of this random variable is defined. The parameters of the distribution will be calculated on the basis of the final results.

**Keywords:** Laplace Transforms, Semi-Markov Random Process, Random Variable, Process With Positive Tendency And Negative Jumps

## 1. Introduction

There are number of works devoted to definition of the Laplace transforms for the distribution of the first passage of the zero level. (Borovkov 2004) [1] defined the explicit form of the distribution, while (Klimov 1996) [2] and (Lotov V. I.) [3] indicated implicit form of the distribution of the first passage of zero level. The presented work explicitly defines the Laplace transforms for the unconditional and conditional distribution of the semi-markov random processes with positive tendency and negative jump.

## 2. Problem

Let's assume that $\xi_k$ and $\zeta_k$, $k \ge 1$ random variables of independent $\{\xi_k, \zeta_k\}_{k\ge1}$ random variable sequence evenly distributed in $\{\Omega, F, P(\cdot)\}$ probability face are given. Using these random variables we will derive the following semi-markov random process:

$$X(t) = z + t - \sum_{i=1}^{k-1} \zeta_i ,$$

if

$$\sum_{i=1}^{k-1} \xi_i \le t < \sum_{i=1}^{k} \xi_i k = \overline{1,\infty}$$

$X(t)$ process is the (asymptotic) semi-markov random processes with positive tendency and negative jump.

Let's include the $\tau_1^0$ random variable defined as below:

$$\tau_1^0 = \min\{t : X(t) \le 0\}$$

where $\tau_1^0$, is the time of the first passage of $X(t)$ process.

We need to find Laplace transform for distribution of $\tau_1^0$ random variable.

## 3. Definition of Laplace Transform for the Distribution of $\tau_1^0$ Random Variable

Let us set Laplace transform for the distribution of $\tau_1^0$ random variable as $L(\theta)$:

$$L(\theta) = Ee^{-\theta\tau_1^0},\ \theta > 0,$$

$$L(\theta|z) = E\left(e^{-\theta\tau_1^0}\middle| X(0) = z\right),\ z \ge 0$$

In this case we can express the equation as

$$\tau_1^0 = \begin{cases} \xi_1, & z + \xi_1 - \zeta_1 < 0, \\ \xi_1 + T, & z + \xi_1 - \zeta_1 > 0, \end{cases}$$

Thus, $T$ and $\tau_1^0$ are evenly distributed random variables.

Our goal is to find Laplace transform of relative and non-relative distribution of $\tau_1^0$ random variable.

According to the formula of total probability, we can put it as

 

$$E\left(e^{-\theta\tau_1^0}|X(0)=z\right)=\int_\Omega e^{-\tau_1^0}P(\mathrm{d}\omega)=\int_{\{\omega:z+\xi_1-\zeta_1<0\}}e^{-\theta\xi_1}P(\mathrm{d}\omega)+\int_{\{\omega:z+\xi_1-\zeta_1>0\}}e^{-\theta(\xi_1+T)}P(\mathrm{d}\omega)$$

If to consider the following substitution

$$\xi_1=s;s_1=y;T=\beta$$

we derive

$$\begin{aligned}
E\left(e^{-\theta\tau_1^0}|X(0)=z\right)&=\int_{s=0}^\infty\int_{y=z+s}^\infty e^{-\theta s}P\{\xi_1\in\mathrm{d}s;\zeta_1\in\mathrm{d}y\}\\
&+\int_{s=0}^\infty\int_{y=0}^{z+s}\int_{\beta=0}^\infty e^{-\theta(s+\beta)}P\{\xi_1\in\mathrm{d}s;\zeta_1\in\mathrm{d}y;T\in\mathrm{d}\beta\}\\
&=\int_{s=0}^\infty e^{-\theta s}P\{\xi_1\in\mathrm{d}s\}\int_{y=z+s}^\infty P\{\zeta_1\in\mathrm{d}y\}\\
&+\int_{s=0}^\infty e^{-\theta s}\int_{y=0}^{z+s}\mathrm{d}P\{\zeta_1<y\}\mathrm{d}P\{\xi_1<s\}L(\theta|z+s-y)\\
&=\int_{s=0}^\infty e^{-\theta s}P\{\xi_1\in\mathrm{d}s\}\mathrm{P}\{\zeta_1>z+s\}\\
&+\int_{s=0}^\infty e^{-\theta s}\int_{\beta=z+s}^0 L(\theta|\beta)\mathrm{d}P\{\zeta_1<z+s-\beta\}\mathrm{d}P\{\xi_1<s\}
\end{aligned}$$

or

$$\begin{aligned}
L(\theta|z)&=\int_{s=0}^\infty e^{-\theta s}P\{\zeta_1>z+s\}P\{\xi_1\epsilon\mathrm{d}s\}\\
&+\int_{s=0}^\infty e^{-\theta s}\int_{y=0}^{z+s}L(\theta|z+s-y)\times P\{\zeta_1\epsilon\mathrm{d}y\}P\{\xi_1\epsilon\mathrm{d}s\}
\end{aligned}\tag{1}$$

Let's assume that $z+s-y=\alpha$. In this case we will receive the following integral equation:

$$\begin{aligned}
L(\theta|z)&=\int_{s=0}^\infty e^{-\theta s}P\{\zeta_1>z+s\}P\{\xi_1\epsilon\mathrm{d}s\}\\
&-\int_{s=0}^\infty e^{-\theta s}\int_{\propto=0}^{z+s}L(\theta|\alpha)\mathrm{d}_\propto P\{\zeta_1<z+s-\propto\}\mathrm{d}P\{\xi_1\epsilon\mathrm{d}s\}
\end{aligned}\tag{2}$$

We will solve this integral equation in special case.

Let's assume that $\xi_1$ random variable has the Erlangian distribution of m construction, while $\zeta_1$ random variable has the single construction Erlangian distribution:

$$\begin{aligned}
&P\{\xi_1(\omega)<t\}\\
&=\left\{1-\left[1+\lambda t+\frac{(\lambda t)^2}{2!}+\cdots+\frac{(\lambda t)^{m-1}}{(m-1)!}\right]e^{-\lambda t}\right\}\varepsilon(t),\qquad t>0,\\
&P\{\zeta_1(\omega)<t\}=\left[1-e^{-\mu t}\right]\varepsilon(t),\qquad t>0.
\end{aligned}$$

where

$$\varepsilon(t)=\begin{cases}0,t<0\\1,t>0.\end{cases}$$

In this case Equation (2) will be as follows:

$$\begin{aligned}
L(\theta|z)&=\frac{\lambda^m}{(\lambda+\mu+\theta)^m}e^{-\mu z}\\
&-\frac{\lambda^m\mu}{(m-1)!}e^{-\mu z}\int_{s=0}^\infty e^{-(\lambda+\mu+\theta)s}s^{m-1}\int_{\propto=0}^{z+s}e^{\mu\propto}L(\theta|\propto)\mathrm{d}\propto\mathrm{d}s
\end{aligned}\tag{3}$$

We can derive differential equation from this integral equation. For this purpose, we will multiply both sides of Equation (3) by $e^{\mu z}$:

$$\begin{aligned}
e^{\mu z}L(\theta|z)&=\frac{\lambda^m}{(\lambda+\mu+\theta)^m}\\
&-\frac{\lambda^m\mu}{(m-1)!}\int_{s=0}^\infty e^{-(\lambda+\mu+\theta)s}s^{m-1}\int_{\propto=0}^{z+s}e^{\mu\propto}L(\theta|\propto)\mathrm{d}\propto\mathrm{d}s
\end{aligned}$$

If we increment both sides by z, we will get:

$$\begin{aligned}
&e^{\mu z}L(\theta|z)+e^{\mu z}L'(\theta|z)\\
&=-\frac{\lambda^m\mu e^{\mu z}}{(m-1)!}\int_{s=0}^\infty e^{-(\lambda+\mu+\theta)s}s^{m-1}e^{\mu s}L(\theta|z+s)\mathrm{d}s
\end{aligned}$$

In this case we will receive the differential equation with (m+1) construction:

$$\begin{aligned}
&\sum_{k=0}^m C_m^k\left[\mu L^{(k)}(\theta|z)+L^{(k+1)}(\theta|z)\right]\\
&\times e^{(\lambda+\theta)z}(-1)^{m-k}(\lambda+\theta)^{m-k}\\
&=(-1)^m\lambda^m\mu e^{-(\lambda+\theta)z}L(\theta|z)
\end{aligned}\tag{4}$$

The general solution of this differential equation will be

$$(\theta|z)=C_1(\theta)Le^{k_{1(\theta)}z}C_2(\theta)e^{k_{2(\theta)}z}+\cdots+C_m(\theta)e^{k_{m(\theta)}z}\tag{5}$$

By finding $C_1(\theta),\cdots,C_m(\theta)$ from Equation (3) we will get the following systematic equations:

$$\left.\begin{aligned}
L(\theta|0)&=\frac{\lambda^m}{(\lambda+\mu+\theta)^m}+\frac{\lambda^m\mu}{(m-1)!}\int_{s=0}^\infty e^{-(\lambda+\mu+\theta|)s}\int_{s=0}^\infty\times s^{m-1}\int_{\propto=0}^s e^{\mu\propto}L(\theta|\propto)\mathrm{d}\propto\mathrm{d}s\\
L'(\theta|0)&=-\mu L(\theta|0)+\frac{\lambda^m\mu}{(m-1)!}\int_{s=0}^\infty e^{-(\lambda+\theta)s}\times s^{m-1}L(\theta|s)\mathrm{d}s\\
&\vdots\\
\sum_{k=0}^{m-1}C_m^k\left[\mu L^{(k)}(\theta|0)+L^{(k+1)}(\theta|0)\right]&=(-1)^m\lambda^m\mu\int_{x=0}^\infty e^{-(\lambda+\theta)x}L(\theta|x)\mathrm{d}x
\end{aligned}\right\}\tag{6}$$

 

By exploitation of Equation (5), Equation (6) becomes

$$\left.\begin{array}{c}\sum_{k=0}^{m} C_i(\theta)=\dfrac{\lambda^m}{(\lambda+\mu+\theta)^m}+\dfrac{\lambda^m\mu}{(m-1)!}\int_{s=0}^{\infty} e^{-(\lambda+\mu+\theta)s}\times s^{m-1}\int_{\propto=0}^{s} e^{\mu\propto}\sum_{i=1}^{m} C_i(\theta)e^{k_i(\theta)\alpha} \\ \sum_{i=1}^{m} C_i(\theta)k_i\theta+\mu\sum_{i=1}^{m} C_i(\theta)=\dfrac{\lambda^m\mu}{(m-1)!}\int_{s=0}^{\infty} e^{-(\lambda+\theta)x}\times x^{m-1}\sum_{i=1}^{m} C_i(\theta)e^{k_i(\theta)x}\mathrm{d}x \\ \vdots \\ \sum_{k=0}^{m-1} C_m^k\left[\mu\sum_{i=1}^{m} k^m C_i(\theta)+k^{m+1}\sum_{i=0}^{m} C_i(\theta)\right]=(-1)^m\lambda^m\mu\int_{x=0}^{\infty} e^{-(\lambda+\theta)x}\sum_{i=1}^{m} C_i(\theta)e^{k_i(\theta)x}\mathrm{d}x\end{array}\right\} \quad (7)$$

After simplification of the last system we will get

$$\left.\begin{array}{c}\sum_{i=1}^{m} C_i(\theta)=\dfrac{\lambda^m}{(\lambda+\mu+\theta)^m}+\dfrac{\lambda^m\mu}{(m-1)!}\sum_{i=1}^{m}\dfrac{C_i(\theta)}{\mu+k_i(\theta)}\dfrac{(m-1)!}{(\lambda+\theta-k_i(\theta))^m}-\dfrac{(m-1)!}{(\lambda+\theta+\mu)^m} \\ \sum_{i=1}^{m} C_i\theta k_i\left(\theta+\mu\sum_{i=1}^{m} C_i(\theta)\right)=\dfrac{\lambda^m\mu}{(m-1)!}\sum_{i=1}^{m}\dfrac{(m-1)!}{(\lambda+\theta-k_i(\theta))^m}C_i\theta \\ \vdots \\ \sum_{i=1}^{m} C_m^k\left[\mu\sum_{i=1}^{m} k^m C_i(\theta)+\sum_{i=1}^{m} k^{m+1}C_i(\theta)\right]=(-1)^m\lambda^m\mu\sum_{i=1}^{m}\dfrac{C_i\theta}{(\lambda+\theta-k_i(\theta))^m}\end{array}\right\} \quad (8)$$

or

$$\left.\begin{array}{c}\sum_{i=1}^{m}\left[1-\lambda^m\mu\dfrac{(\lambda+\mu+\theta)^m-(\lambda+\theta-k_i(\theta))^m}{(\mu+k_i(\theta))(\lambda+\mu+\theta)^m(\lambda+\theta-k_i(\theta))^m}\right]C_i(\theta)=\dfrac{\lambda^m}{(\lambda+\mu+\theta)^m} \\ \sum_{i=1}^{m}\left[\mu+k_i(\theta)-\dfrac{\lambda^m\mu}{(\lambda+\theta-k_i(\theta))^m}\right]C_i(\theta)=0 \\ \sum_{k=0}^{m-1} C_m^k k^m\left[\mu+k_i(\theta)-\dfrac{\lambda^m\mu}{(\lambda+\theta-k_i(\theta))^m}\right]C_i(\theta)=0\end{array}\right\} \quad (9)$$

By exploitation of

$$\lambda^m\mu=(\mu+k_i(\theta)(\lambda+\theta-k_i(\theta))^m$$

Equation (9) becomes

$$\left.\begin{array}{c}\sum_{i=1}^{m}\left[1-\dfrac{(\lambda+\mu+\theta)^m-(\lambda+\theta-k_i(\theta))^m}{(\lambda+\mu+\theta)^m}\right]C_i(\theta)=\dfrac{\lambda^m}{(\lambda+\mu+\theta)^m} \\ \sum_{i=1}^{m}\left[\mu+k_i(\theta)-(\mu+k_i(\theta))\right]C_i(\theta)=0 \\ \sum_{k=0}^{m-1} C_m^k k^m\left[\mu+k_i(\theta)-(\mu+k_i(\theta))\right]C_i(\theta)=0\end{array}\right\}$$

or

$$\left.\begin{aligned}\sum_{i=1}^{m}\left[\lambda+\theta-k_i(\theta)\right]^m C_i(\theta)=\lambda^m\\ \sum_{i=1}^{m}0\times C_i(\theta)=0\\ \vdots\\ \sum_{i=1}^{m}0\times C_i(\theta)=0\end{aligned}\right\}\quad(10)$$

Thus, (9) is a linear dependence equation system, as

$$C_2(\theta)=C_3(\theta)=\cdots=C_m(\theta)=0$$

$$C_1(\theta)=\frac{\lambda^m}{\left[\lambda+\theta-k_1(\theta)\right]^m}$$

Then the general solution of integral Equation (3) will be as follows:

$$L(\theta|z)=C_1(\theta)e^{k_1(\theta)z}=\frac{\lambda^m}{\left[\lambda+\theta-k_1(\theta)\right]^m}e^{k_1(\theta)z}$$

This expression is the Laplace transform for relative distribution of $\tau_1^0$ random variable. Then, we will need to find $L(\theta)$. In accordance with formula of total probability,

$$L(\theta)=\int_{z=0}^{\infty}\mathrm{L}(\theta|\mathrm{z})\mathrm{dP}\{\mathrm{X}(0)<z\}$$

and as the distribution of $X(0)$ and $\zeta_1(\omega)$ random variables are same,

$$\begin{aligned}L(\theta)&=\int_{z=0}^{\infty}C_1(\theta)e^{k_1(\theta)z}\mathrm{dP}\{X(0)<z\}\\&=C_1(\theta)\int_{z=0}^{\infty}e^{k_1(\theta)z}\frac{\lambda^m}{(m-1)!}z^{m-1}e^{-\lambda z}\mathrm{d}z\\&=\frac{\lambda^m C_1(\theta)}{(m-1)!}\int_{z=0}^{\infty}e^{-\left[\lambda-k_1(\theta)\right]z}z^{m-1}\mathrm{d}z\\&=\frac{\lambda^m}{\left[\lambda-k_1(\theta)\right]^m}C_1(\theta)\end{aligned}$$

Therefore,

$$L(\theta)=\frac{\lambda^m}{\left[\lambda-k_1(\theta)\right]^m}C_1(\theta)$$

This expression is the Laplace transform for non-relative distribution of $\tau_1^0$ random variable.

Respectively, we will get the following characteristics for $\lambda$m > $\mu$:

$$E\tau_1^0=-L'(0)=\frac{m(\lambda+\mu)}{\lambda(\lambda-m\mu)}$$

$$L'(0)=\frac{m^3\mu^2(3-m)}{\lambda^2(\lambda-m\mu)^2}+2\frac{m^3\mu}{\lambda(\lambda-\mu)(\lambda-m\mu)}+\frac{m(m+1)\lambda}{((\lambda-m\mu)^3}$$

$$D\tau_1^0=L'(0)-\left(L'(0)\right)^2=\frac{m^3\mu^2(3-m)}{\lambda^2(\lambda-m\mu)^2}+2\frac{m^3\mu}{\lambda(\lambda-\mu)(\lambda-m\mu)}+\frac{m(m+1)\lambda}{(\lambda-m\mu)^3}-\frac{m^2(\lambda+m\mu)^2}{\lambda^2(\lambda-m\mu)^2}$$

$$E\left(\tau_1^0|\mathrm{z}\right)=\frac{m(1+z\mu)}{\lambda-m\mu}$$

$$D\left(\tau_1^0|\mathrm{z}\right)=\frac{m}{(\lambda-m\mu)^2}+\frac{m((m+1))(m+z\mu)\mu}{(\lambda-m\mu)^3}$$

## 4. Conclusions

In this article we have defined Laplace transforms for relative and non-relative distribution of the first passage of zero level of semi-markov random process with positive tendency and negative jump.

## 5. References


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