An Alternative Method of Stochastic Optimization: The Portfolio Model

Moawia Alghalith
University of the West India, Saint Augustine, Trinidad and Tobago
E-mail: malghalith@gmail.com
Received May 19, 2011; revised May 26, 2011; accepted May 29, 2011

Abstract

We provide a new simple approach to stochastic dynamic optimization. In doing so, we derive the existing (standard) results using a far simpler technique than the duality and the variational methods.

Keywords: Stochastic Optimization, Investment, Portfolio

1. Introduction

Previous studies in stochastic optimization relied on the duality approach and/or variational techniques such as using the Feynman Kac formula and the Hamilton-Jacobi-Bellman partial differential equations. Examples include [1-3], among many others.

In this paper, we offer a new simple approach to stochastic dynamic optimization. That is, we prove the previous results using a simpler method than the duality or the Hamilton-Jacobi-Bellman partial differential equations methods. We apply our method to the standard investment model. Our approach is based on dividing the time horizon into sub-horizons and applying Stein’s lemma.

2. The Portfolio Model

We use the standard investment model (see, for example, [3], among many others). Similar to previous models, we consider a risky asset and a risk-free asset. The risk-free asset price process is given by

\[ 0 = r_s t S_t e^{r_s t} \]

The dynamics of the risky asset price are given by

\[ dS_t = S_t \{ \mu ds + \sigma dW_s \} \]

where \( \mu \) and \( \sigma \) are the deterministic rate of return and the volatility, respectively, and \( W_s \) is a standard Brownian motion.

The wealth process is given by

\[ X^\pi_T = x + \int_T^t \{ rX^\pi_s + (\mu_s - r_s)\pi_s \} ds + \int_T^t \pi_s \sigma_s dW_s, \] (2)

where \( x \) is the initial wealth, \( \{ \pi_s \}_{s \in [t,T]} \) is the risky portfolio process with \( E[\pi_s^2 ds] < \infty \). The trading strategy \( \pi \in A(x) \) is admissible (that is, \( X^\pi_T \geq 0 \)).

The investor’s objective is to maximize the expected utility of the terminal wealth

\[ V(t,x) = \sup_{\pi} E\left[U\left(X^\pi_T\right)|\mathcal{F}_t\right] = E\left[U\left(X^\pi_T\right)|\mathcal{F}_t\right], \] (3)

where \( V(\cdot) \) is the (smooth) value function, \( U(\cdot) \) is continuous, bounded and strictly concave utility function, and \( \mathcal{F}_t \) is the filtration.

We rewrite (2) as

\[ X^\pi_T = x + \int_T^t \{ rX^\pi_s + (\mu_s - r_s)\pi_s \} ds + \int_T^t \pi_s \sigma_s dW_s, \] (4)

Substituting the above equation into (3) and differentiating with respect to \( \pi^*_s \) (and setting the derivative equal to zero) yields

\[ (\mu_s - r_s)E[U'(\cdot)|\mathcal{F}_s] + \sigma_s E[U''(\cdot)W_s|\mathcal{F}_s] = 0. \] (5)

By Stein’s lemma

\[ E[U'(\cdot)W_s|\mathcal{F}_s] = Cov(X_s,W_s)E[U''(\cdot)|\mathcal{F}_s] = \pi^*_s \sigma_s E[U''(\cdot)|\mathcal{F}_s], \] (6)
Substituting this into (5) yields

$$\pi^*_u = -\frac{(\mu_u - r_u) E[U'(\cdot)|\mathcal{F}_u]}{\sigma^2_u E[U''(\cdot)|\mathcal{F}_u]} = -\frac{(\mu_u - r_u) V_\pi(\cdot)}{\sigma^2 V_{xx}(\cdot)}.$$ (7)

This solution can be generalized to any point on time $s$

$$\pi^*_s = -\frac{(\mu_s - r_s) V_\pi(\cdot)}{\sigma^2 V_{xx}(\cdot)}.$$ (8)

This is exactly the solution obtained by the previous literature, but its derivation is far simpler. Furthermore, this approach can be applied to many other stochastic models.

3. References

