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Meromorphic Functions Sharing Three Values*

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Abstract

In this paper, we prove a result on the uniqueness of meromorphic functions sharing three values counting multiplicity and improve a result obtained by Xiaomin Li and Hongxun Yi.

Keywords: Uniqueness, Meromorphic Functions, Sharing Three Values

1. Introduction and Main Results

Let f and g be two non-constant meromorphic functions in the complex plane. It is assumed that the reader is familiar with the standard notations of Nevanlinna's theory such as $T(r,f)$, $m(r,f)$, $N(r,f)$ $\bar{N}(r,f)$ and so on, which can be found in [1]. We use E to denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. The notation $S(r,f)$ denotes any quantity satisfying $S(r,f)=\circ(T(r,f))(r\to\infty, r\notin E)$. A meromorphic function $b(\not\equiv\infty)$ is called a small function with respect to f provided that $T(r,b)=S(r,f)$. A meromorphic function $b(\not\equiv\infty)$ is called a exceptional function of f provided that $N\left(r,\frac{1}{f-b}\right)=S(r,f)$.

Let a be a complex number, we say that f and g share the value a CM provided $f-a$ and $g-a$ have the same zeros counting multiplicities (see [2]). We say that f and g share ∞ CM provided that $1/f$ and $1/g$ share 0 CM.

Xiaomin Li and Hongxun Yi prove the following theorem:

Theorem A ([3]). Let f and g be two distinct nonconstant meromorphic functions sharing three values $0,1$ and ∞ CM, if there exists a finite complex number $a\neq 0,1$ such that a is not a Picard value of f, and

$$N_{1)}\left(r,\frac{1}{f-a}\right)\neq T(r,f)+S(r,f),$$

then

$$N_{1)}\left(r,\frac{1}{f-a}\right)=\frac{k-2}{k}T(r,f)+S(r,f)$$

and one of the following cases will hold:

1) $f=\dfrac{e^{(k+1)\gamma}-1}{e^{s\gamma}-1}$, $g=\dfrac{e^{-(k+1)\gamma}-1}{e^{-s\gamma}-1}$, with $\dfrac{(a-1)^{k+1-s}}{a^{k+1}}=\dfrac{s^s(k+1-s)^{k+1-s}}{(k+1)^{k+1}}$ and $a\neq\dfrac{k+1}{s}$;

2) $f=\dfrac{e^{s\gamma}-1}{e^{(k+1)\gamma}-1}$, $g=\dfrac{e^{-s\gamma}-1}{e^{-(k+1)\gamma}-1}$, with $a^s(1-a)^{k+1-s}=\dfrac{s^s(k+1-s)^{k+1-s}}{(k+1)^{k+1}}$ and $a\neq\dfrac{s}{k+1}$;

3) $f=\dfrac{e^{s\gamma}-1}{e^{-(k+1-s)\gamma}-1}$, $g=\dfrac{e^{-s\gamma}-1}{e^{(k+1-s)\gamma}-1}$, with $\dfrac{(-a)^s}{(1-a)^{k+1}}=\dfrac{s^s(k+1-s)^{k+1-s}}{(k+1)^{k+1}}$ and $a\neq-\dfrac{s}{k+1-s}$;

4) $f=\dfrac{e^{k\gamma}-1}{\lambda e^{s\gamma}-1}$, $g=\dfrac{e^{-k\gamma}-1}{(1/\lambda)e^{-s\gamma}-1}$, with $\lambda^k\neq 0,1$ and $\dfrac{(a-1)^{k-s}}{\lambda^k a^k}=\dfrac{s^s(k-s)^{k-s}}{k^k}$;

5) $f=\dfrac{e^{s\gamma}-1}{\lambda e^{k\gamma}-1}$, $g=\dfrac{e^{-s\gamma}-1}{(1/\lambda)e^{-k\gamma}-1}$, with $\lambda^s\neq 0,1$ and $\lambda^s a^s(1-a)^{k-s}=\dfrac{s^s(k-s)^{k-s}}{k^k}$;

6) $f=\dfrac{e^{s\gamma}-1}{\lambda e^{-(k-s)\gamma}-1}$, $g=\dfrac{e^{-s\gamma}-1}{(1/\lambda)e^{(k-s)\gamma}-1}$, with

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$\lambda^s \neq 0,1$ and $\dfrac{(-\lambda a)^s}{(1-a)^k} = \dfrac{s^s(k-s)^{k-s}}{k^k}$.

where γ is a nonconstant entire function, s and $k(\geq 2)$ are positive integers such that s and $k+1$ are mutually prime and $1 \leq s \leq k$ in 1), 2), 3), s and k are mutually prime and $1 \leq s \leq k-1$ in 4), 5), 6).

Xinhou Hua and Mingliang Fang proved the following theorem:

Theorem B ([4]). Let f and g be two nonconstant meromorphic functions sharing three values $0,1$ and ∞ CM, if

$$T(r,f) \neq N(r,b(z),f) + S(r,f)$$

$b(z)(\not\equiv 0,1,\infty)$ is a small function of f, then one of the following holds:

1) $f \equiv g$;

2) $f \equiv bg$, and b, 1 are exceptional functions of f;

3) $(f-1) \equiv (1-b)(g-1)$, and b, 0 are exceptional functions of f;

4) $(f-b)(g-1+b) \equiv b(1-b)$, and b, ∞ are exceptional functions of f.

As we all know, many results on constants are also valid for small functions, although some times they are more difficult. In this paper, we improve the above theorems and obtain the following result.

Theorem 1.1. Let f and g be two distinct nonconstant meromorphic functions sharing three values $0,1$ and ∞ CM, if there exists a small function $b(z)(\not\equiv 0,1,\infty)$ of f such that $b(z)$ is a exceptional function of f, and

$$N_{1)}(r,b(z),f) \neq T(r,f) + S(r,f) \tag{1.1}$$

then

$$N_{1)}(r,b(z),f) = \frac{k-2}{k}T(r,f) + S(r,f) \tag{1.2}$$

and one of the following cases will hold:

1) $f = \dfrac{e^{(k+1)\gamma}-1}{e^{s\gamma}-1}$, $g = \dfrac{e^{-(k+1)\gamma}-1}{e^{-s\gamma}-1}$, with

$\dfrac{(b-1)^{k+1-s}}{b^{k+1}} \equiv \dfrac{s^s(k+1-s)^{k+1-s}}{(k+1)^{k+1}}$ and $b \not\equiv \dfrac{k+1}{s}$;

2) $f = \dfrac{e^{s\gamma}-1}{e^{(k+1)\gamma}-1}$, $g = \dfrac{e^{-s\gamma}-1}{e^{-(k+1)\gamma}-1}$, with

$b^s(1-b)^{k+1-s} \equiv \dfrac{s^s(k+1-s)^{k-s}}{(k+1)^{k+1}}$ and $b \not\equiv \dfrac{s}{k+1}$;

3) $f = \dfrac{e^{s\gamma}-1}{e^{-(k+1-s)\gamma}-1}$, $g = \dfrac{e^{-s\gamma}-1}{e^{(k+1-s)\gamma}-1}$, with

$\dfrac{(-b)^s}{(1-b)^{k+1}} \equiv \dfrac{s^s(k+1-s)^{k+1-s}}{(k+1)^{k+1}}$ and $b \not\equiv -\dfrac{s}{k+1-s}$;

4) $f = \dfrac{e^{k\gamma}-1}{\lambda(1+cb)e^{s\gamma}-1}$, $g = \dfrac{e^{-k\gamma}-1}{(1/\lambda(1+cb))e^{-s\gamma}-1}$,

with $\lambda^k \neq 0,1$ and

$$\frac{\left(k(b-1)\gamma' + b'\right)^k}{\left(\lambda sb(1+cb)(b-1)\gamma' + \lambda\left(2cb-cb^2+1\right)b'\right)^s}$$
$$\equiv \left(\lambda(1+2cb)b' + \lambda(k-s)b(1+cb)\gamma'\right)^{k-s};$$

5) $f = \dfrac{e^{s\gamma}-1}{\lambda(1+cb)e^{k\gamma}-1}$, $g = \dfrac{e^{-s\gamma}-1}{\left(1/\lambda(1+cb)\right)e^{-k\gamma}-1}$,

with $\lambda^s \neq 0,1$ and

$$\frac{\left(s(b-1)\gamma' + b'\right)^s}{\left(\lambda kb(1+cb)(b-1)\gamma' + \lambda\left(2cb-cb^2+1\right)b'\right)^k}$$
$$\equiv \left(\lambda(1+2cb)b' + \lambda(s-k)b(1+cb)\gamma'\right)^{s-k};$$

6) $f = \dfrac{e^{s\gamma}-1}{\lambda e^{-(k-s)\gamma}-1}$, $g = \dfrac{e^{-s\gamma}-1}{\left(1/\lambda(1+cb)\right)e^{(k-s)\gamma}-1}$, with

$\lambda^s \neq 0,1$ and

$$\frac{\left(s(b-1)\gamma' + b'\right)^s}{\left(\lambda(s-k)b(1+cb)(b-1)\gamma' + \lambda\left(2cb-cb^2+1\right)b'\right)^{(s-k)}}$$
$$\equiv \left(\lambda(1+2cb)\right)b' + \left(\lambda kb(1+cb)\gamma'\right)^k.$$

where γ is a nonconstant entire function, s and $k(\geq 2)$ are positive integers such that s and $k+1$ are mutually prime and $1 \leq s \leq k$ in 1), 2), 3), s and k are mutually prime and $1 \leq s \leq k-1$ in 4), 5), 6), c and λ are constants.

2. Some Lemmas

Lemma 2.1 ([4]). Let f and g be two nonconstant meromorphic functions sharing three values $0,1$ and ∞ CM. If $f \not\equiv g$, then for any small function $b(z)(\not\equiv 0,1,\infty)$ we have

$$N_{(3}(r,b(z),f) + N_{(3}(r,b(z),g) = S(r,f).$$

Lemma 2.2 ([3]). Let f be a nonconstant merorphic function, a_1 a_2 and a_3 be three distinct small functions of f, if

$$\overline{N}\left(r,\frac{1}{f-a_1}\right) + \overline{N}\left(r,\frac{1}{f-a_1}\right) = S(r,f),$$

then

$$N_{1)}\left(r,\frac{1}{f-a_3}\right)=T(r,f)+S(r,f).$$

Using the same method of [3] in Lemma 2.2, we get the following result:

Lemma 2.3. Let f and g be two nonconstant meromorphic functions sharing three values 0,1 and ∞ CM. If f is a fractional linear transformation of g, for any small function $b(z)(\not\equiv 0,1,\infty)$, then either $b(z)$ is a exceptional function of f, or

$$N_{1)}\left(r,\frac{1}{f-b}\right)=T(r,f)+S(r,f).$$

Lemma 2.4. Let s and t are two integers, and ω be a nonconstant meromorphic function and $b(z)$ is a small function of ω, if $b^s\not\equiv 1$, then

$$N_0\left(r,\omega^s-1,\omega^t-b\right)=S(r,\omega),$$

where $N_0\left(r,\omega^s-1,\omega^t-b\right)$ denotes the reduced counting function of the common zero of ω^s-1 and ω^t-b.

Proof. If z_0 is a zero of ω^s-1 and ω^t-b, then we have

$$\omega^s(z_0)=1, \tag{2.1}$$

and

$$\omega^t(z_0)=b(z_0). \tag{2.2}$$

From (2.1) and (2.2) we get $b^s(z_0)=1$, thus $N_0\left(r,\omega^s-1,\omega^t-b\right)=S(r,\omega)$, since $b^s\not\equiv 1$.

Lemma 2.5. Let

$$P(\omega)=\omega^n+a\omega^m+b, \tag{2.3}$$

where $\omega=\mathrm{e}^\gamma$, γ is a nonconstant entire function, $a(\not\equiv\infty)$ and $b(\not\equiv\infty)$ are two small functions of ω, n and m are positive integers such that $n>m$.

1)

$$N_{(3}\left(r,\frac{1}{P(\omega)}\right)=S(r,\omega). \tag{2.4}$$

2) If

$$\left(\frac{-b'+ab\omega'/\omega}{a'+(m-n)a\omega'/\omega}\right)^n\not\equiv\left(\frac{-mab\omega'/\omega}{a'+(m-n)a\omega'/\omega}\right)^m, \tag{2.5}$$

then

$$N_{(2}\left(r,\frac{1}{P(\omega)}\right)=S(r,\omega). \tag{2.6}$$

3) If n and m are mutually prime, and

$$\left(\frac{-b'+ab\omega'/\omega}{a'+(m-n)a\omega'/\omega}\right)^n\equiv\left(\frac{-mab\omega'/\omega}{a'+(m-n)a\omega'/\omega}\right)^m, \tag{2.7}$$

then

$$N_{(2}\left(r,\frac{1}{P(\omega)}\right)=2T(r,\omega)+S(r,\omega). \tag{2.8}$$

Proof. 1) Differentiating $P(\omega)$ two times and eliminating ω^n and ω^m from the three equations we obtain

$$P(\omega)+h_1P'(\omega)+h_2P''(\omega)=1, \tag{2.9}$$

with $T(r,h_i)=S(r,\omega)$ $(i=1,2)$. Thus (2.4) holds.

2) Suppose $N_{(2}\left(r,\frac{1}{P(\omega)}\right)\neq S(r,\omega)$, and let z_0 be a zero of $P(\omega)$ with multiplicity ≥ 2, then from (2.3) we have

$$\omega^n(z_0)+a(z_0)\omega^m(z_0)+b(z_0)=0, \tag{2.10}$$

and

$$\begin{aligned}&\frac{n\omega'(z_0)}{\omega(z_0)}\omega^n(z_0)+(a'(z_0)\\&+\left(\frac{a(z_0)\omega'(z_0)}{\omega(z_0)}\right)\omega^m(z_0)+b'(z_0)=0.\end{aligned} \tag{2.11}$$

From (2.10) and (2.11) we get

$$\omega^m(z_0)=\frac{-b'(z_0)+nb(z_0)\omega'(z_0)/\omega(z_0)}{a'(z_0)+(m-n)a(z_0)\omega'(z_0)/\omega(z_0)}, \tag{2.12}$$

and

$$\omega^n(z_0)=\frac{-ma(z_0)b(z_0)\omega'(z_0)/\omega(z_0)}{a'(z_0)+(m-n)a(z_0)\omega'(z_0)/\omega(z_0)}. \tag{2.13}$$

Since $\omega=\mathrm{e}^\gamma$, γ is a nonconstant entire function, we have

$$T(r,\omega'/\omega)=S(r,\omega). \tag{2.14}$$

From (2.7) (2.12) (2.13) and (2.14), we get (2.6) holds.

3) Let z_0 be a zero of $P(\omega)$ with multiplicity ≥ 2, using proceeding as in 2) we can get (2.12) and (2.13). On the other hands, since n and m are mutually prime, there exist one and only one pair of integers s and t such that

$$ns-mt=1(0<s<m,0<t<n) \tag{2.15}$$

From (2.12) (2.13) and (2.15) we can get z_0 is a root of

$$\begin{aligned}\omega&=\omega^{ns-mt}\\&=\left(\frac{-b'+nb\omega'/\omega}{a'+(m-n)a\omega'/\omega}\right)^s\left(\frac{-ab\omega'/\omega}{a'+(m-n)a\omega'/\omega}\right)^t\end{aligned}$$

which implies (2.5) holds since ω has two distinct exceptional functions.

Lemma 2.6 ([5]) Let f_1 and f_2 be two non-constant meromorphic functions satisfying

$$\overline{N}(r,f_i)+\overline{N}\left(r,\frac{1}{f_i}\right)=S(r),\ i=1,2.$$

Then either

$$N_0(r,1;f_1,f_2)=S(r)$$

or there exist two integers s, t $(|s|+|t|>0)$ such that

$$f_1^s f_2^t\equiv 1$$

where $N_0(r,1;f_1,f_2)$ denotes the reduced counting function of f_1 and f_2 related to the common 1-point and

$$T(r)=T(r,f_1)+T(r,f_2),$$
$$S(r)=o(T(r))\ (r\to\infty, r\notin E)$$

only depending on f_1 and f_2.

Lemma 2.7 ([6]) Let f be a nonconstant meromorphic function and $R(f)=\frac{P(f)}{Q(f)}$, where

$$P(f)=\sum_{k=1}^{p}a_k f^k \text{ and } Q(f)=\sum_{j=1}^{q}b_j f^j$$

are two mutually prime polynomials in f. If the coefficients $a_k(z)$, $b_j(z)$ are small functions of f and $a_p\not\equiv 0$, $b_q\not\equiv 0$, then

$$T(r,R(f))=\max(p,q)T(r,f)+S(r,f).$$

3. Proof of Theorem 1.1

If f is a fractional transformation of g, by Lemma 2.3 we have that either $b(z)$ is a exceptional function of f, or $N_{1)}\left(r,\frac{1}{f-b}\right)=T(r,f)+S(r,f)$, which contradicts with the assumption of Theorem 1.1. Thus f is not a fractional transformation of g. By Theorem B we have

$$N\left(r,\frac{1}{f-b}\right)=T(r,f)+S(r,f). \tag{3.1}$$

From (1.1) and (3.1) we obtain

$$N_{(2}\left(r,\frac{1}{f-b}\right)\neq S(r,f). \tag{3.2}$$

By Lemma 2.1 we have

$$N_{(3}\left(r,\frac{1}{f-b}\right)=S(r,f). \tag{3.3}$$

Combining (3.2) and (3.3) we get

$$\overline{N}_{(2}\left(r,\frac{1}{f-b}\right)\neq S(r,f). \tag{3.4}$$

Noting that f and g share 0,1 and ∞ CM, we have

$$\frac{f}{g}=\mathrm{e}^{\alpha},\ \frac{f-1}{g-1}=\mathrm{e}^{\beta}. \tag{3.5}$$

where α and β are two entire functions. From (3.5) we get

$$f=\frac{\mathrm{e}^{\alpha}-1}{\mathrm{e}^{\beta}-1},\ g=\frac{\mathrm{e}^{-\alpha}-1}{\mathrm{e}^{-\beta}-1}, \tag{3.6}$$

and

$$f-b=\frac{\mathrm{e}^{\alpha}-b\mathrm{e}^{\beta}+b-1}{\mathrm{e}^{\beta}-1}. \tag{3.7}$$

Assume that $T(r,e^{\beta})=S(r,f)$, Noting 0 and ∞ are Picard values of e^{α}, from (3.6) we have $\frac{-1}{\mathrm{e}^{\beta}-1}$ and ∞ are exceptional functions of f, by Lemma 2.2 we get

$$N_{1)}\left(r,\frac{1}{f-b}\right)=T(r,f)+S(r,f),$$

which contradicts with the assumption of Theorem 1.1.

Thus $T(r,e^{\beta})\neq S(r,f)$.

Similarly, we have $T(r,e^{\alpha})\neq S(r,f)$ and $T(r,e^{\alpha-\beta})\neq S(r,f)$.

Let z_0 be a multiple zero of $f-b$, but not a zero of α',β', and $\beta'-\alpha'$. From (3.7) we obtain

$$e^{\alpha(z_0)}-b(z_0)e^{\beta(z_0)}+b(z_0)-1=0. \tag{3.8}$$

and

$$\alpha'(z_0)-b'(z_0)\mathrm{e}^{\beta(z_0)}-b(z_0)\beta'(z_0)\mathrm{e}^{\beta(z_0)}+b'(z_0)=0. \tag{3.9}$$

From (3.8) and (3.9) we have

$$\begin{aligned}\mathrm{e}^{\alpha(z_0)}&=\frac{b'(z_0)+b(z_0)\beta'(z_0)-b^2(z_0)\beta'(z_0)}{b'(z_0)+b(z_0)\beta'(z_0)-b(z_0)\alpha'(z_0)},\\ \mathrm{e}^{\beta(z_0)}&=\frac{b'(z_0)+\alpha'(z_0)-b(z_0)\alpha'(z_0)}{b'(z_0)+b(z_0)\beta'(z_0)-b(z_0)\alpha'(z_0)}.\end{aligned} \tag{3.10}$$

Set

$$f_1=\frac{b'+b\beta'-b\alpha'}{b'+b\beta'-b^2\beta'}\mathrm{e}^{\alpha},\ f_2=\frac{b'+b\beta'-b\alpha'}{b'+\alpha'-b\alpha'}\mathrm{e}^{\beta}, \tag{3.11}$$

and

$$\begin{aligned}&T(r)=T(r,f_1)+T(r,f_2),\\ &S(r)=o(T(r))\ (r\to\infty, r\notin E).\end{aligned} \tag{3.12}$$

From (3.5) (3.11) and (3.12) we get

$$S(r,f)=S(r). \tag{3.13}$$

From (3.11) (3.12) and (3.13) we get

$$\bar{N}(r,f_i)+\bar{N}\left(r,\frac{1}{f_i}\right)=S(r),\ i=1,2. \tag{3.14}$$

From (3.10) and (3.11) we have $f_1(z_0)=1$, $f_2(z_0)=1$. Thus

$$\bar{N}_{(2}\left(r,\frac{1}{f-b}\right)\le N_0(r,1;f_1,f_2)+S(r,f). \tag{3.15}$$

$N_0(r,1;f_1,f_2)$ denotes the reduced counting function of the common 1-points of f_1 and f_2. From (3.4) (3.13) and (3.15), we obtain

$$N_0(r,1;f_1,f_2)\ne S(r). \tag{3.16}$$

From (3.16) and Lemma 2.5, we know there exist two integers p and $q(|p|+|q|>0)$ such that

$$f_1^p f_2^q\equiv 1. \tag{3.17}$$

Noting $T(r,\mathrm{e}^{\alpha})\ne S(r,f)$, $T(r,\mathrm{e}^{\beta})\ne S(r,f)$ and $T(r,\mathrm{e}^{\alpha-\beta})\ne S(r,f)$, from (3.11) and (3.17), we have $p\ne 0$, $q\ne 0$ and $p\ne -q$, and

$$\mathrm{e}^{p\alpha+q\beta}=\left(\frac{b'+b\beta'-b^2\beta'}{b'+b\beta'-b\alpha'}\right)^p\left(\frac{b'+\alpha'-b\alpha'}{b'+b\beta'-b\alpha'}\right)^q. \tag{3.18}$$

Let $Q(z)=\dfrac{\alpha'-b\beta'}{b'+b\beta'-b\alpha'}$, then from (3.18) we get

$$\mathrm{e}^{p\alpha+q\beta}=(1+Q)^p(1+bQ)^q. \tag{3.19}$$

Noting that $b(z)$ is a small function of f, we obtain that

$$Q(z)\equiv c, \tag{3.20}$$

where c is a constant. From (3.19) and (3.20) we obtain

$$\mathrm{e}^{p\alpha+q\beta}=(1+c)^p(1+bc)^q. \tag{3.21}$$

Without loss of generality, From (3.21) we may assume that p and q are mutually prime and $q>0$. Let $\lambda=(1+c)^{\frac{p}{q}}$ and $\alpha=q\gamma$, where γ is an entire function. Then from (3.6) and (3.21) we obtain

$$f=\frac{\mathrm{e}^{q\gamma}-1}{\lambda(1+cb)\mathrm{e}^{-p\gamma}-1},\quad g=\frac{\mathrm{e}^{-q\gamma}-1}{1/(\lambda(1+cb))\mathrm{e}^{p\gamma}-1}. \tag{3.22}$$

Noting that $p\ne 0$, $q\ne 0$ and $p\ne -q$, We discuss the following three cases.

Case 1. Suppose that $q>-p>0$, we discuss the following two subcases.

Subcase 1.1. If $(\lambda(1+cb))^q\equiv 1$. Setting $k+1=q$ and $s=-p$, let $\omega=\lambda(1+cb)$ and $\mathrm{e}^{s\delta}=\omega\mathrm{e}^{s\gamma}$. From (3.22) and (3.7) we get

$$f=\frac{\mathrm{e}^{(k+1)\delta}-1}{\mathrm{e}^{s\delta}-1},\ g=\frac{\mathrm{e}^{-(k+1)\delta}-1}{\mathrm{e}^{-s\delta}-1}. \tag{3.23}$$

And

$$f-b=\frac{\mathrm{e}^{(k+1)\delta}-b\mathrm{e}^{s\delta}+b-1}{\mathrm{e}^{s\delta}-1}. \tag{3.24}$$

Since in this subcase b is a constant, let $\delta=\gamma$, (3.23) assume the form (1) in Theorem 1.1. From the proof of Theorem A we know (1.2) holds with

$$\frac{(b-1)^{k+1-s}}{b^{k+1}}\equiv\frac{s^s(k+1-s)^{k+1-s}}{(k+1)^{k+1}}$$

and

$$b\not\equiv\frac{k+1}{s}$$

Subcase 1.2. If $(\lambda(1+cb))^q\not\equiv 1$. Setting $k=q$ and $s=-p$. From (3.22) we get

$$f=\frac{\mathrm{e}^{k\gamma}-1}{\lambda(1+cb)\mathrm{e}^{s\gamma}-1},\quad g=\frac{\mathrm{e}^{-k\gamma}-1}{1/(\lambda(1+cb))\mathrm{e}^{-s\gamma}-1}. \tag{3.25}$$

which assume the form (iv) in **Theorem 1.1**.

We have from (3.25) and (3.7)

$$f-b=\frac{\mathrm{e}^{k\gamma}-\lambda b(1+cb)\mathrm{e}^{s\gamma}+b-1}{\lambda(1+cb)\mathrm{e}^{s\gamma}-1} \tag{3.26}$$

Since $(\lambda(1+cb))^k\not\equiv 1$, from (3.25) (3.26) Lemma 2.4 and Lemma 2.7 we get

$$T(r,f)=kT(r,e^{\gamma})+S(r,f). \tag{3.27}$$

$$N_0\left(r,0;\mathrm{e}^{k\gamma}-\lambda b(1+cb)\mathrm{e}^{s\gamma}+b-1,\lambda(1+cb)\mathrm{e}^{s\gamma}-1\right)=S(r,f) \tag{3.28}$$

where

$$N_0\left(r,0;\mathrm{e}^{k\gamma}-\lambda b(1+cb)\mathrm{e}^{s\gamma}+b-1,\lambda(1+cb)\mathrm{e}^{s\gamma}-1\right)$$

denotes the reduced counting function of common zeros of $\mathrm{e}^{k\gamma}-\lambda b(1+cb)\mathrm{e}^{s\gamma}+b-1$ and $\lambda(1+cb)\mathrm{e}^{s\gamma}-1$.

If

$$\frac{\left(k(b-1)\gamma'+b'\right)^k}{\left(\lambda sb(1+cb)(b-1)\gamma'+\lambda\left(2cb-cb^2+1\right)b'\right)^s}$$
$$\not\equiv\left(\lambda(1+2cb)b'+(k-s)b(1+cb)\gamma'\right)^{k-s}$$

by Lemma 2.5 (2), we get a contradiction with (3.4).

Thus From (3.27) (3.28) and Lemma 2.5 (3) we obtain (1.2) holds with

$$\frac{\left(k(b-1)\gamma'+b'\right)^k}{\left(\lambda sb(1+cb)(b-1)\gamma'+\lambda\left(2cb-cb^2+1\right)b'\right)^s}$$
$$\equiv\left(\lambda(1+2cb)b'+\lambda(k-s)b(1+cb)\gamma'\right)^{k-s}.$$

Case 2. Suppose that $-p>q>0$, we discuss the following two subcases.

Subcase 2.1. If $\left(\lambda(1+cb)\right)^q\equiv 1$. Setting $k+1=-p$ and $s=q$, let $\omega=\lambda(1+cb)$ and $e^{k\delta}=\omega e^{k\gamma}$. From (3.22) and (3.7) we get

$$f=\frac{e^{s\delta}-1}{e^{(k+1)\delta}-1},\ g=\frac{e^{-s\delta}-1}{e^{-(k+1)\delta}-1} \tag{3.29}$$

and

$$f-b=\frac{e^{s\delta}-be^{k\delta}+b-1}{e^{k\delta}-1} \tag{3.30}$$

Since in this subcase b is a constant, let $\gamma=\delta$, (3.29) assume the form of 2) in Theorem 1.1. By the proof of Theorem A we know (1.2) holds with

$$b^s(1-b)^{k+1-s}\equiv\frac{s^s(k+1-s)^{k+1-s}}{(k+1)^{k+1}}$$

and

$$b\not\equiv\frac{s}{k+1}.$$

Subcase 2.2 If $\left(\lambda(1+cb)\right)^q\not\equiv 1$. Setting $k=-p$ and $s=q$, from(3.22) we get

$$f=\frac{e^{s\gamma}-1}{\lambda(1+cb)e^{k\gamma}-1},$$
$$g=\frac{e^{-s\gamma}-1}{\left(1/\lambda(1+cb)\right)e^{-k\gamma}-1}. \tag{3.31}$$

Which assume the form 5) in Theorem 1.1. We have from (3.31) and (3.7)

$$f-b=\frac{e^{s\gamma}-\lambda b(1+cb)e^{k\gamma}+b-1}{\lambda(1+cb)e^{k\gamma}-1}. \tag{3.32}$$

In the same manner as Subcase 1.2 we know (1.2) holds with

$$\frac{\left(s(b-1)\gamma'+b'\right)^s}{\left(\lambda sb(1+cb)(b-1)\gamma'+\lambda\left(2cb-cb^2+1\right)b'\right)^k}$$
$$\equiv\left(\lambda(1+2cb)b'+\lambda(s-k)b(1+cb)\gamma'\right)^{s-k}.$$

Case 3. Suppose that $p>0$, we discuss the following two subcases.

Subcase 3.1. If $\left(\lambda(1+cb)\right)^q\equiv 1$. Setting $k+1=p+q$ and $s=q$, let $\omega=\lambda(1+cb)$ and $e^{-(k-s)\delta}=\omega e^{-(k-s)\gamma}$. From (3.22) and (3.7) we get

$$f=\frac{e^{s\delta}-1}{e^{-(k+1-s)\delta}-1},\ g=\frac{e^{-s\delta}-1}{e^{(k+1-s)\delta}-1}. \tag{3.33}$$

and

$$f-b=\frac{e^{s\delta}-be^{-(k+1-s)\delta}+b-1}{e^{-(k+1-s)\delta}-1}. \tag{3.34}$$

Since in this subcase b is a constant, let $\gamma=\delta$, (3.33) assume the form (3) in Theorem 1.1. By the proof of Theorem A we know (1.2) holds with

$$\frac{(-b)^s}{(1-b)^{k+1}}\equiv\frac{s^s(k+1-s)^{k+1-s}}{(k+1)^{k+1}}$$

and

$$b\not\equiv-\frac{s}{k+1-s}$$

Subcase 3.2. If $\left(\lambda(1+cb)\right)^q\not\equiv 1$. Setting $k=p+q$ and $s=q$, we have from (3.22)

$$f=\frac{e^{s\gamma}-1}{\lambda(1+cb)e^{-(k-s)\gamma}-1},$$
$$g=\frac{e^{-s\gamma}-1}{1/\left(\lambda(1+cb)\right)e^{(k-s)\gamma}-1}. \tag{3.35}$$

which assume the form (6) in Theorem 1.1. From (3.35) and (3.7) we get

$$f-b=\frac{e^{s\gamma}-\lambda b(1+cb)e^{-(k-s)\gamma}+b-1}{\lambda(1+cb)e^{-(k-s)\gamma}-1}. \tag{3.36}$$

In the same manner as Subcase 1.2 we get (1.2) holds with

$$\frac{\left(s(b-1)\gamma'+b'\right)^s}{\left(\lambda sb(1+cb)(b-1)\gamma'+\lambda\left(2cb-cb^2+1\right)b'\right)^k}$$
$$\equiv\left(\lambda(1+2cb)b'+\lambda kb(1+cb)\gamma'\right)^k.$$

Theorem 1.1 is thus completely proved.

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5. References

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