Multi-Item EOQ Model with Both Demand-Dependent Unit Cost and Varying Leading Time via Geometric Programming

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Received March 10, 2011; revised March 17, 2011; accepted March 21, 2011

Abstract

The objective of this paper is to derive the analytical solution of the EOQ model of multiple items with both demand-dependent unit cost and leading time using geometric programming approach. The varying purchase and leading time crashing costs are considered to be continuous functions of demand rate and leading time, respectively. The researchers deduce the optimal order quantity, the demand rate and the leading time as decision variables then the optimal total cost is obtained.

Keywords: Inventory, Geometric Programming, Leading Time, Demand-Dependent, Economic Order Quantity

1. Introduction

The problem of the EOQ model with demand-dependent unit cost had been treated by some researchers. Cheng [1] studied an EOQ model with demand-dependent unit cost of single-item. The problem of inventory models involving lead time as a decision variable have been succinctly described by Ben-Daya and Abdul Raouf [2]. Abou-El-Ata and Kotb [3] developed a crisp inventory model under two restrictions. Also, Teng and Yang [4] examined deterministic inventory lot-size models with time-varying demand and cost under generalized holding costs. Other related studies were written by Jung and Klein [5], Das et al. [6] and Mandal et al. [7]. Recently, Kotb and Fergany [8] discussed multi-item EOQ model with varying holding cost: a geometric programming approach.

The aim of this paper is to derive the optimal solution of EOQ inventory model and minimize the total cost function based on the values of demand rate, order quantity and leading time using geometric programming approach.

2. Notations and Assumptions

To construct the model of this problem, we define the following variables:

- \( D_r \) = Annual demand rate (decision variable).
- \( C_{pr} \) = Unit purchase (production) cost.
- \( C_{hr} \) = Unit holding (inventory carrying) cost per item per unit time.
- \( C_{or} \) = Ordering cost.
- \( SS \) = \( K \sigma \sqrt{L_r} \) = Safety stock.
- \( n \) = Number of different items carried in inventory.
- \( L_r \) = Leading rate time (decision variable).
- \( Q_r \) = Production (order) quantity batch (decision variable).
- \( TC(D_r, Q_r, L_r) \) = Average annual total cost. For the \( r \)th item.

The following basic assumptions about the model are made:

1) Demand rate \( D_r \) is uniform over time.
2) Time horizon is finite.
3) No shortages are allowed.
4) Unit production cost \( C_{pr}(D_r) = C_{pr} D_r^{-b} \), \( r=1,2,3,\cdots,n \), \( b>1 \) is inversely related to the demand rate. Where \( b \) is called the price elasticity.
5) Lead time crashing cost is related to the lead time by a function of the form \( R(L_r) = \alpha L_r^{-\beta}, r=1,2,3,\cdots,n, \alpha>0, 0<\beta\leq0.5 \). where \( \alpha, \beta \) are real constants selected to provide the best fit of the estimated cost func-
tion.
6) Our objective is to minimize the annual relevant total cost.

3. Mathematical Formulation

The annual relevant total cost (sum of production, order, inventory carrying and lead time crashing costs) which, according to the basic assumptions of the EOQ model, is:

$$TC(D_r, Q_r, L_r) = \sum_{r=1}^{n} \left[ D_r C_{pw} + \frac{D_r}{Q_r} C_{or} \right]$$

$$+ \left( \frac{Q_r}{2} + K_0 \sqrt{L_r} \right) C_{hr} + \frac{D_r}{Q_r} R(L_r)$$ (1)

Substituting $C_{pw}$, $D_r$ and $R(L_r)$ into (1) yields:

$$TC(D_r, Q_r, L_r) = \sum_{r=1}^{n} \left[ D_r C_{pw} + \frac{D_r}{Q_r} C_{or} \right]$$

$$+ \left( \frac{Q_r}{2} + K_0 \sqrt{L_r} \right) C_{hr} + \frac{D_r}{Q_r} \alpha L_r^{-\beta}$$ (2)

To solve this primal objective function which is a convex programming problem, we can write it in the form:

$$\min \text{TC} = \sum_{r=1}^{n} \left[ C_{pw} D_r + \frac{D_r}{Q_r} C_{or} \right]$$

$$+ \left( \frac{Q_r}{2} + K_0 \sqrt{L_r} \right) C_{hr} + \frac{D_r}{Q_r} \alpha L_r^{-\beta}$$ (3)

Applying Duffin et al. [9] results of geometric programming technique to (3), the enlarged predual function could be written in the form:

$$G(W) = \prod_{r=1}^{n} \left( \frac{D_r}{W_{br}} \right)^{W_{br}} \left( \frac{D_r C_{or}}{Q_r W_{r}} \right)^{W_{r}}$$

$$\times \left( \frac{K_0 C_{hr}}{W_{br}} \right)^{W_{br}} \left( \frac{D_r \alpha L_r^{-\beta}}{Q_r W_{r}} \right)^{W_{r}}$$ (4)

$$= \prod_{r=1}^{n} \left( \frac{C_{or}}{W_{br}} \right)^{W_{br}} \left( \frac{C_{hr}}{2W_{r}} \right)^{W_{r}}$$

$$\times \left( \frac{K_0 C_{hr}}{W_{br}} \right)^{W_{br}} \left( \frac{\alpha}{W_{r}} \right)^{W_{r}}$$

$$\times \left( \frac{D_r}{W_{br}} \right)^{W_{br}} \left( \frac{D_r C_{or}}{Q_r W_{r}} \right)^{W_{r}}$$

Since the dual variable vector $W_{br}$, $j=1,2,3,4,5,r=1,2,3,\ldots,n$ is arbitrary and can be chosen according to convenience subject to:

$$W_{br} + W_{br} + W_{br} + W_{br} + W_{br} = 1, W_{br} > 0$$ (5)

We choose $W_{br}$ such that the exponents of $D_r$, $Q_r$ and $L_r$ are zero, thus making the right hand side of (4) independent of the decision variables. To do this we require:

$$\left( 1 - b \right) W_{br} + W_{br} + W_{br} = 0$$

$$W_{br} + W_{br} - W_{br} = 0$$

$$\frac{1}{2} W_{br} - \beta W_{br} = 0$$ (6)

These are called the orthogonality conditions which together with (5) are sufficient to determine the values of $W_{br}, j=1,2,3,4,5, r=1,2,3,\ldots,n$. Solving (5) and (6) for $W_{br}$, we get:

$$W_{br} = \frac{1}{2b-1} \left( 1 - 2 \beta b W_{br} \right)$$

$$W_{br} = \frac{b-1}{2b-1} \left( 1 + 2 \beta b (b-1) \right) \left( 1 - 2 \beta b W_{br} \right)$$

$$W_{br} = \frac{b-1}{2b-1} \left( 1 - 2 \beta b W_{br} \right)$$ and

$$W_{br} = 2 \beta W_{br}$$ (7)

Substituting $W_{br}, j=1,2,3,4,5, r=1,2,3,\ldots,n$ in (4), we get the dual function $g(W_{br})$. To find $W_{br}$, which maximize $g(W_{br})$, the logarithm of both side of $g(W_{br})$, and the partial derivatives were taken relative to $W_{br}$. Setting it to equal zero and simplifying, we get:

$$f(W_{br}) = W_{br}^b + A \left( 1 + 2 \beta b \right) W_{br} - 2 \beta b W_{br}^b$$

$$- B b \left( 1 - 2 \beta b W_{br} \right)^b = 0$$ (8)

where:

$$b_1 = \frac{b-1}{2b-1}, b_2 = \frac{2b \beta b}{(2b-1)(1+2 \beta b)}$$

and

$$A = \frac{\alpha}{C_{or}} \left( \frac{K_0 C_{hr}}{2 \beta} \right)^{b+1} \left( 1 + 2 \beta b \right)^b$$

$$\times \left( \frac{2b-1}{(2b-1)^b} \right)^b$$

It is clear that $f(0) < 0$ and $f(1) > 0$ which means that there exists a root $W_{br} \in (0,1)$. The trial and error method can be used to find this root. However, we shall
first verify the root $W_{r_s}$ calculated from (8) to maximize $g(W_{r_s})$. This is confirmed by the second derivative to $\ln g(W_{r_s})$ with respect to $W_{r_s}$, which is always negative. Thus, the root $W_{r_s}$ calculated from (8) maximize the dual function $g(W_{r_s})$. Hence, the optimal solution is $W_{r^*_r}$, $j = 1, 2, 3, 4, 5$, $r = 1, 2, 3, \ldots, n$, where $W_{r^*_r}$ is the solution of (8) and $W_{r^*_r}$, $j = 1, 2, 3, 4$ are evaluated by substituting value of $W_{r^*_r}$ in (7).

To find the optimal values $D^*_r, Q^*_r, L^*_r$, we apply Duffin et al. [9] of geometric programming as indicated below:

$$C_{pr} D^*_r W_{r^*_r}^{2b} = W_{r^*_r} g(W_{r^*_r}),$$

$$C_{pr} D^*_r = W_{r^*_r} g(W_{r^*_r}),$$

$$C_{hr} \frac{Q^*_r}{2} = W_{r^*_r} g(W_{r^*_r}),$$

$$C_{hr} K \sigma \sqrt{L^*_r} = W_{r^*_r} g(W_{r^*_r}).$$

By solving these relations, the optimal demand rate is given by:

$$D^*_r = \left[ \frac{C_{sr} C_{hr}}{2C_{pr}} \cdot \frac{W_{r^*_r}^{2b}}{W_{r^*_r}^2 W_{r^*_r}} \right]^{1-2b},$$

(9)

The optimal order quantity is:

$$Q^*_r = \left[ \frac{C_{sr} C_{hr}}{C_{pr} W_{r^*_r}^2} \cdot \frac{W_{r^*_r}}{2C_{pr}} \cdot \frac{W_{r^*_r}^{2b}}{W_{r^*_r}^2 W_{r^*_r}} \right]^{b},$$

(10)

The optimal lead time is:

$$L^*_r = \left[ \frac{C_{sr} C_{hr}}{2K \sigma C_{pr} W_{r^*_r}^2 W_{r^*_r}} \cdot \frac{W_{r^*_r}^{2b}}{W_{r^*_r}^2 W_{r^*_r}} \right]^{1-2b},$$

(11)

By substituting the values of $D^*_r, Q^*_r$ and $L^*_r$ in (3), we deduce the minimum total cost as:

$$\min TC(D^*_r, Q^*_r, L^*_r) = \sum_{r=1}^{n} \left[ C_{pr} \left( \frac{C_{sr} C_{hr}}{2C_{pr}} \cdot \frac{W_{r^*_r}^{2b}}{W_{r^*_r}^2 W_{r^*_r}} \right)^{2b} \right] \times \left[ 1 + W_{r^*_r} + \frac{C_{sr} C_{hr} W_{r^*_r}}{C_{pr} W_{r^*_r}} \right]^{2b} \times \left[ \frac{2C_{sr} C_{hr}}{C_{sr} C_{hr}} \cdot \frac{W_{r^*_r}^{2b}}{W_{r^*_r}^2 W_{r^*_r}} \right]$$

(12)

As a special case, we assume $L^*_r = 0$, $\beta = 0$ and $b \rightarrow \infty \Rightarrow R(L) = 0$, $W_{r^*_r} = W_{r^*_r} = 0$, and $W_{r^*_r} = W_{r^*_r} = \frac{1}{2}$. This is the classical EOQ inventory model.

4. An Illustrative Example

We shall compute the decision variables (optimal order quantity $Q^*_r$, optimal demand rate $D^*_r$ and optimal lead time $L^*_r$) whose values are to be determined to minimize the annual relevant total cost for three items ($n = 3$). The parameters of the model are shown in Table 1.

Assume that the standard deviation $\sigma = 6$ unit/year and $K = 2$.

For some different values of $\beta$ and $b$, we use equation (8) to determine $W_{r^*_r}$, whose value is to be determined to obtain $W_{r^*_r}$, $j = 1, 2, 3, 4$, $r = 1, 2, 3$ from (6).

It follows that the optimal values of the production batch quantity $Q^*_r$, demand rate $D^*_r$, lead time $L^*_r$ and minimum annual total cost are given in Tables 2-7.

Table 1. The parameters of the model.

<table>
<thead>
<tr>
<th>r</th>
<th>$C_{sr}$</th>
<th>$C_{pr}$</th>
<th>$C_{hr}$</th>
<th>$K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$200$</td>
<td>$10$</td>
<td>$0.8$</td>
<td>$1$</td>
</tr>
<tr>
<td>2</td>
<td>$140$</td>
<td>$8$</td>
<td>$0.5$</td>
<td>$2$</td>
</tr>
<tr>
<td>3</td>
<td>$100$</td>
<td>$5$</td>
<td>$0.3$</td>
<td>$3$</td>
</tr>
</tbody>
</table>

Table 2. The optimal solution of $Q^*_r$ and $D^*_r$ as a function of $b$ (for all $\beta$).

<table>
<thead>
<tr>
<th>b</th>
<th>$Q^*_1$</th>
<th>$Q^*_2$</th>
<th>$Q^*_3$</th>
<th>$D^*_1$</th>
<th>$D^*_2$</th>
<th>$D^*_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$28.21$</td>
<td>$31.17$</td>
<td>$33.10$</td>
<td>$1.450$</td>
<td>$1.550$</td>
<td>$1.540$</td>
</tr>
<tr>
<td>5</td>
<td>$26.41$</td>
<td>$28.55$</td>
<td>$31.00$</td>
<td>$1.395$</td>
<td>$1.455$</td>
<td>$1.440$</td>
</tr>
<tr>
<td>8</td>
<td>$25.65$</td>
<td>$27.49$</td>
<td>$29.90$</td>
<td>$1.316$</td>
<td>$1.350$</td>
<td>$1.340$</td>
</tr>
<tr>
<td>10</td>
<td>$25.25$</td>
<td>$26.99$</td>
<td>$29.40$</td>
<td>$1.280$</td>
<td>$1.300$</td>
<td>$1.295$</td>
</tr>
<tr>
<td>20</td>
<td>$24.20$</td>
<td>$25.72$</td>
<td>$28.03$</td>
<td>$1.170$</td>
<td>$1.180$</td>
<td>$1.178$</td>
</tr>
</tbody>
</table>

Table 3. The optimal solution of $L^*_r$ and min $TC$ as a function of $b$ ($\beta = 0.1$).

<table>
<thead>
<tr>
<th>b</th>
<th>$L^*_1$</th>
<th>$L^*_2$</th>
<th>$L^*_3$</th>
<th>min $TC$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$8.4 \times 10^{-4}$</td>
<td>$8.9 \times 10^{-3}$</td>
<td>$1.3 \times 10^{-3}$</td>
<td>$59.6932$</td>
</tr>
<tr>
<td>5</td>
<td>$4.2 \times 10^{-1}$</td>
<td>$2.4 \times 10^{-1}$</td>
<td>$4.9 \times 10^{-1}$</td>
<td>$66.0704$</td>
</tr>
<tr>
<td>8</td>
<td>$2.1 \times 10^{-1}$</td>
<td>$2.4 \times 10^{-1}$</td>
<td>$1.0 \times 10^{-1}$</td>
<td>$71.6195$</td>
</tr>
<tr>
<td>10</td>
<td>$1.2 \times 10^{-3}$</td>
<td>$8.1 \times 10^{-3}$</td>
<td>$2.7 \times 10^{-3}$</td>
<td>$83.8121$</td>
</tr>
<tr>
<td>20</td>
<td>$3.4 \times 10^{-3}$</td>
<td>$6.9 \times 10^{-3}$</td>
<td>$1.5 \times 10^{-3}$</td>
<td>$422.838$</td>
</tr>
</tbody>
</table>

Table 4. The optimal solution of $L^*_r$ and min $TC$ as a function of $b$ ($\beta = 0.2$).

<table>
<thead>
<tr>
<th>b</th>
<th>$L^*_1$</th>
<th>$L^*_2$</th>
<th>$L^*_3$</th>
<th>min $TC$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$2.7 \times 10^{-3}$</td>
<td>$5.1 \times 10^{-3}$</td>
<td>$6.5 \times 10^{-3}$</td>
<td>$77.539200$</td>
</tr>
<tr>
<td>5</td>
<td>$2.5 \times 10^{-3}$</td>
<td>$2.2 \times 10^{-3}$</td>
<td>$2.3 \times 10^{-3}$</td>
<td>$243.99000$</td>
</tr>
<tr>
<td>8</td>
<td>$2.2 \times 10^{-4}$</td>
<td>$4.3 \times 10^{-4}$</td>
<td>$3.7 \times 10^{-4}$</td>
<td>$2906.5400$</td>
</tr>
<tr>
<td>10</td>
<td>$7.1 \times 10^{-8}$</td>
<td>$1.4 \times 10^{-7}$</td>
<td>$1.7 \times 10^{-7}$</td>
<td>$15175.300$</td>
</tr>
<tr>
<td>20</td>
<td>$6.8 \times 10^{-4}$</td>
<td>$7.7 \times 10^{-3}$</td>
<td>$9.2 \times 10^{-3}$</td>
<td>$1.3203 \times 10^{7}$</td>
</tr>
</tbody>
</table>

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Table 5. The optimal solution of $L_r$ and $\min TC$ as a function of $b$ ($\beta = 0.3$).

<table>
<thead>
<tr>
<th>$b$</th>
<th>$L_1^*$</th>
<th>$L_2^*$</th>
<th>$L_3^*$</th>
<th>$\min TC$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$2.6 \times 10^{-2}$</td>
<td>$2.5 \times 10^{-8}$</td>
<td>$5.5 \times 10^{-7}$</td>
<td>112.62400</td>
</tr>
<tr>
<td>5</td>
<td>$2.4 \times 10^{-12}$</td>
<td>$2.1 \times 10^{-14}$</td>
<td>$1.2 \times 10^{-13}$</td>
<td>6729.1300</td>
</tr>
<tr>
<td>8</td>
<td>$1.7 \times 10^{-24}$</td>
<td>$8.7 \times 10^{-20}$</td>
<td>$4.1 \times 10^{-18}$</td>
<td>75148.000</td>
</tr>
<tr>
<td>10</td>
<td>$6.4 \times 10^{-20}$</td>
<td>$5.4 \times 10^{-23}$</td>
<td>$2.3 \times 10^{-22}$</td>
<td>1.495 $\times 10^7$</td>
</tr>
<tr>
<td>20</td>
<td>$1.2 \times 10^{-26}$</td>
<td>$2.8 \times 10^{-35}$</td>
<td>$1.9 \times 10^{-28}$</td>
<td>2.858 $\times 10^{20}$</td>
</tr>
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</table>

Table 6. The optimal solution of $L_r$ and $\min TC$ as a function of $b$ ($\beta = 0.4$).

<table>
<thead>
<tr>
<th>$b$</th>
<th>$L_1^*$</th>
<th>$L_2^*$</th>
<th>$L_3^*$</th>
<th>$\min TC$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$5.9 \times 10^{-10}$</td>
<td>$1.2 \times 10^{-8}$</td>
<td>$2.0 \times 10^{-7}$</td>
<td>309.50100</td>
</tr>
<tr>
<td>5</td>
<td>$3.9 \times 10^{-17}$</td>
<td>$1.6 \times 10^{-14}$</td>
<td>$5.4 \times 10^{-12}$</td>
<td>99640.100</td>
</tr>
<tr>
<td>8</td>
<td>$3.4 \times 10^{-25}$</td>
<td>$3.9 \times 10^{-20}$</td>
<td>$4.6 \times 10^{-18}$</td>
<td>1.317 $\times 10^6$</td>
</tr>
<tr>
<td>10</td>
<td>$7.9 \times 10^{-20}$</td>
<td>$2.9 \times 10^{-24}$</td>
<td>$8.7 \times 10^{-22}$</td>
<td>2.050 $\times 10^9$</td>
</tr>
<tr>
<td>20</td>
<td>$6.8 \times 10^{-41}$</td>
<td>$5.0 \times 10^{-31}$</td>
<td>$1.5 \times 10^{-22}$</td>
<td>2.3888 $\times 10^{24}$</td>
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</table>

Table 7. The optimal solution of $L_r$ and $\min TC$ as a function of $b$ ($\beta = 0.5$).

<table>
<thead>
<tr>
<th>$b$</th>
<th>$L_1^*$</th>
<th>$L_2^*$</th>
<th>$L_3^*$</th>
<th>$\min TC$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$1.2 \times 10^{-12}$</td>
<td>$3.6 \times 10^{-9}$</td>
<td>$7.4 \times 10^{-8}$</td>
<td>1013.00000</td>
</tr>
<tr>
<td>5</td>
<td>$6.1 \times 10^{-16}$</td>
<td>$7.2 \times 10^{-13}$</td>
<td>$8.4 \times 10^{-12}$</td>
<td>2.5485 $\times 10^6$</td>
</tr>
<tr>
<td>8</td>
<td>$5.2 \times 10^{-26}$</td>
<td>$6.0 \times 10^{-21}$</td>
<td>$7.1 \times 10^{-17}$</td>
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<td>10</td>
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<td>$5.6 \times 10^{-23}$</td>
<td>$6.7 \times 10^{-22}$</td>
<td>2.31429 $\times 10^{12}$</td>
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<td>$4.8 \times 10^{-44}$</td>
<td>$5.7 \times 10^{-38}$</td>
<td>2.3888 $\times 10^{24}$</td>
</tr>
</tbody>
</table>

Figure 1. The optimal order quantity against $b$ (for all $\beta$).

Figure 2. The optimal demand rate against $b$ (for all $\beta$).

Figure 3. The minimum total cost against $b$ (for all $\beta$).

Figure 4. The minimum total cost against $\beta$ (for all $b$).

Solution of the problem may be determined more readily by plotting $(Q_r^*, D_r^*, \min TC)$ against $b$ and $\min TC$ against $\beta$, for each values of $\beta$.

5. Conclusions

This paper is devoted to study multi-item inventory model that consider the order quantity, the demand rate and the leading time as three decision variables. These decision variables, $r=1, 2, 3, \cdots, n$ are evaluated and the minimum annual total cost $\min TC$ is deduced. The classical system is derived as special case and a numerical example is solved.

The smallest value of the minimum total cost is found at the smallest values of $b$ and $\beta$.

6. References


579-582.


