Efficiency and Duality in Nondifferentiable Multiobjective Programming Involving Directional Derivative

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Abstract

In this paper, we introduce a new class of generalized d_I-univexity in which each component of the objective and constraint functions is directionally differentiable in its own direction d_i for a nondifferentiable multiobjective programming problem. Based upon these generalized functions, sufficient optimality conditions are established for a feasible point to be efficient and properly efficient under the generalised d_I-univexity requirements. Moreover, weak, strong and strict converse duality theorems are also derived for Mond-Weir type dual programs.

Keywords: Multiobjective Programming, Nondifferentiable Programming, Generalized d_I-Univexity, Sufficiency, Duality

1. Introduction

The field of multiobjective programming, also known as vector programming, has grown remarkably in different directions in the setting of optimality conditions and duality theory. It has been enriched by the applications of various types of generalizations of convexity theory, with and without differentiability assumptions, and in the framework of continuous time programming, fractional programming, inverse vector optimization, saddle point theory, symmetric duality and vector variational inequalities etc.

Hanson [1] introduced a class of functions by generalizing the difference vector \( x - \bar{x} \) in the definition of a convex function to any vector function \( \eta(x, \bar{x}) \). These functions were named invex by Craven [2] and \( \eta \)-convex by Kaul and Kaur [3]. Hanson and Mond [4] defined two new classes of functions called Type I and Type II functions, which were further generalized to pseudo Type I and quasi Type I functions by Rueda and Hanson [5]. Zhao [6] established optimality conditions and duality in nonsmooth scalar programming problems assuming Clarke [7] generalized subgradients under Type I functions.

Kaul et al. [8] extended the concept of type I and its generalizations for a multiobjective programming problem. They investigated optimality conditions and derived Wolfe type and Mond-Weir type duality results. Sunuja and Srivastava [9] introduced generalized d-type I functions in terms of directional derivative for a multiobjective programming problem and discussed Wolfe type and Mond-Weir type duality results. In [10], Kuk and Tanino derived optimality conditions and duality theorems for non-smooth multiobjective programming problems involving generalized Type I vector valued functions. Gulati and Agarwal [11] discussed sufficiency and duality results for nonsmooth multiobjective problems under \((F, \alpha, \rho, d)\)-type I functions. Agarwal et al. [12] established sufficient conditions and duality theorems for nonsmooth multiobjective problems under V-type I functions. Recently, Jayswal et al. [13] obtained some optimality conditions and duality results for nonsmooth multiobjective problems involving generalized \((F, \alpha, \rho, \theta) - d - V\)-univexity.

Antczak [14] studied d-invexity is one of the generalization of invex function, which is introduced by [15]. In [14], Antczak established, under weaker assumptions than Ye, the Fritz John type and Karush-Kuhn-Tucker type necessary optimality conditions for weak Pareto optimality and duality results which have been stated in terms of the right differentials of functions involved in the considered multiobjective programming problem. Some authors [16-18] proved that the Karush-Kuhn-Tucker type necessary conditions [14] are sufficient under various generalized d-invex functions. Antczak [19]...
corrected the Karush-Kuhn-Tucker necessary conditions in [14] and discussed the sufficiency and duality under \( d - r \)-type I functions. Recently, Silmani and Radjef [20] introduced generalized \( d_i \)-invexity in which each component of the objective and constraint functions is directionally differentiable in its own direction and established the necessary and sufficient conditions for efficient and properly efficient solutions. The duality results for a Mond-Weir type dual are also derived in [20]. They also observed that the Karush-Kuhn-Tucker sufficient conditions discussed in [16-18] are not applicable. More recently, Agarwal et al. [21] introduced a new class of generalized \( d - r \)-(\( \eta, \theta \))-type I for a non-smooth multiobjective programming problem and discussed optimality conditions and duality results.

In this paper, we introduce \( d_i \)-\( V \)-univexity and generalized \( d_i \)-\( V \)-univexity in which each component of the objective and constraint functions of a multiobjective programming problem is semidirectionally differentiable in its own direction \( d_i \). Various Karush-Kuhn-Tucker sufficient optimality conditions for efficient and properly efficient solutions to the problem are established involving new classes of semidirectionally differentiable generalized type I functions. Moreover, usual duality theorems are discussed for a Mond-Weir type dual involving aforesaid assumptions. The results in this paper extend many earlier work appeared in the literature [9,10,12,14-16, 19].

2. Preliminaries and Definitions

The following conventions for equalities and inequalities will be used. If \( x = (x_1, \cdots, x_n), \ y = (y_1, \cdots, y_n) \in \mathbb{R}^n \), then \( x = y \iff x_i = y_i, i = 1, \cdots, n \); \( x < y \iff x_i < y_i, i = 1, \cdots, n \); \( x \leq y \iff x_i \leq y_i, i = 1, \cdots, n \); \( x \leq y \iff x \leq y \) and \( x \neq y \).

We also note \( \mathbb{R}^n_+ \) (resp. \( \mathbb{R}^n_+ \) or \( \mathbb{R}^n_{++} \)) the set of vectors \( y \in \mathbb{R}^n \) with \( y \geq 0 \) (resp. \( y \geq 0 \) or \( y > 0 \)).

**Definition 1** [22]. Let \( D \) be a nonempty subset of \( \mathbb{R}^n \), \( \eta : D \times D \to \mathbb{R}^n \) and let \( x_0 \) be an arbitrary point of \( D \). The set \( D \) is said to be inexact at \( x_0 \) with respect to \( \eta \), if for each \( x \in D \),

\[
x_0 + \lambda \eta(x, x_0) \in D, \forall \lambda \in [0,1].
\]

\( D \) is said to be an inexact set with respect to \( \eta \), if \( D \) is inexact at each \( x_0 \in D \) with respect to the same \( \eta \).

**Definition 2** [23]. Let \( D \subseteq \mathbb{R}^n \) be an inexact set with respect to \( \eta : D \times D \to \mathbb{R}^n \). A function \( f : D \to \mathbb{R} \) is called in-pre-invex on \( D \) with respect to \( \eta \), if for all \( x, x_0 \in D \),

\[
\lambda f(x) + (1-\lambda) f(x_0) \geq f(x_0 + \lambda \eta(x, x_0)), \forall \lambda \in [0,1].
\]

**Definition 3** [14]. Let \( D \subseteq \mathbb{R}^n \) be an inexact set with respect to \( \eta : D \times D \to \mathbb{R}^n \). A m-dimensional vector valued function \( \Psi : D \to \mathbb{R}^m \) is pre-invex with respect to \( \eta \), if each of its components is pre-invex on \( D \) with respect to the same function \( \eta \).

**Definition 4** [7]. Let \( D \) be a nonempty open set in \( \mathbb{R}^n \). A function \( f : D \to \mathbb{R} \) is said to be locally Lipschitz at \( x_0 \in D \), if there exist a neighborhood \( \mathcal{U}(x_0) \) of \( x_0 \) and a constant \( K > 0 \) such that

\[
|f(y) - f(x)| \geq K \|y - x\|, \forall x, y \in \mathcal{U}(x_0),
\]

where \( \| \cdot \| \) denotes the Euclidean norm. We say that \( f \) is locally Lipschitz on \( D \) if its locally Lipschitz at any point of \( D \).

**Definition 5** [7]. If \( f : D \subseteq \mathbb{R}^n \to R \) is locally Lipschitz at \( x_0 \in D \), the Clarke generalized directional derivative of \( f \) at \( x_0 \) in the direction \( d \in \mathbb{R}^n \), denoted by

\[
f^\circ(x_0;d) = \limsup_{t \to x_0} \frac{f(y + td) - f(y)}{t},
\]

And the usual one-sided directional derivative of \( f \) at \( x_0 \) in the direction \( d \) is defined by

\[
f'(x_0;d) = \lim_{\lambda \to 0^+} \frac{f(x_0 + \lambda d) - f(x_0)}{\lambda},
\]

whenever this limit exists. Obviously,

\[
f^0(x_0;d) \geq f^\circ(x_0;d).
\]

We say that \( f \) is directionally differentiable at \( x_0 \) if its directional derivative \( f'(x_0;d) \) exists for all \( d \in \mathbb{R}^n \).

**Definition 6** [15]. Let \( f : D \to \mathbb{R}^n \) be a function defined on a nonempty open set \( D \subset \mathbb{R}^n \) and directionally differentiable at \( x_0 \in D \). \( f \) is called d-invex at \( x_0 \) on \( D \) with respect to \( \eta \), if there exists a vector function \( \eta : D \times D \to \mathbb{R}^n \), such that for any \( x \in D \),

\[
|f(y) - f(x)| \geq K \|y - x\|, \forall x, y \in \mathcal{U}(x_0), f_i(x) - f_i(x_0) \geq f_i'(x_0) \eta_i(x, x_0),
\]

for all \( i = 1, \cdots, N \),

where \( f_i'(x_0) \eta_i(x, x_0) \) denotes the directional derivative of \( f_i \) at \( x_0 \) in the direction

\[
\eta(x, x_0); f_i'(x_0) \eta_i(x, x_0) = \lim_{\lambda \to 0^+} \frac{f_i(x_0 + \lambda \eta_i(x, x_0)) - f_i(x_0)}{\lambda}.
\]

If Inequalities (1) are satisfied at any point \( x_0 \in D \), then \( f \) is said to be d-invex on \( D \) with respect to \( \eta \).

**Definition 7** [20]. Let \( D \) be a nonempty set in \( \mathbb{R}^n \) and \( \phi : D \times D \to \mathbb{R}^n \) a function.

- We say that \( f : D \to \mathbb{R} \) is a semi-directionally
differentiable at \( x_0 \in D \), if there exist a nonempty subset \( S \subseteq \mathbb{R}^n \) such that \( f'(x_0; d) \) exists finite for all \( d \in S \).

- We say that \( f \) is a semi-directionally differentiable at \( x_0 \in D \) in the direction \( \phi(x, x_0) \), if its directional derivative \( f'(x_0; \phi(x, x_0)) \) exists finite for all \( x \in D \).

**Definition 8** [20]. Let \( f : D \to \mathbb{R}^n \) be a function defined on a nonempty open set \( D \subseteq \mathbb{R}^n \) and for all \( i = 1, \ldots, N, f_i \) is semi-directionally differentiable at \( x_0 \in D \) in the direction \( \eta_i : \mathbb{R}^n \to \mathbb{R}^n \). \( f \) is called \( \eta_i \)-invex at \( x_0 \) on \( D \) with respect to \( (\eta_i)_{i=1, \ldots, N} \), if for any \( x \in D \),

\[
 f_i(x) - f_i(x_0) \leq f'_i(x_0; \eta_i(x, x_0)), \quad \text{for all } i, 2, \ldots, N,
\]

where \( f'_i(x_0; \eta_i(x, x_0)) \) denotes the directional derivative of \( f_i \) at \( x_0 \) in the direction \( \eta_i(x, x_0) \) defined as

\[
 \eta_i(x, x_0) = \lim_{\lambda \to 0^+} \frac{f_i(x_0 + \lambda \eta_i(x, x_0)) - f_i(x_0)}{\lambda}.
\]

If Inequalities (2) are satisfied at any point \( x_0 \in D \), then \( f \) is said to be \( \eta_i \)-invex on \( D \) with respect to \( (\eta_i)_{i=1, \ldots, N} \).

Consider the following multiobjective programming problem

**Definition 9.** A point \( x_0 \in X \) is said to be a local weakly efficient solution of the problem (MP), if there exists a neighborhood \( N(x_0) \) around \( x_0 \) such that \( f(x) \nless f(x_0) \) for all \( x \in N(x_0) \cap X \).

**Definition 10.** A point \( x_0 \in X \) is said to be a weakly efficient (an efficient) solution of the problem (MP), if there exists no \( x \in X \) such that

\[
 f(x) < f(x_0) \quad \text{or} \quad f(x) \leq f(x_0).
\]

**Definition 11.** An efficient solution \( x_0 \in X \) of (MP) is said to be properly efficient, if there exists a positive real number \( M \) such that inequality

\[
 f_i(x_0) - f_i(x) \leq M \left( f_j(x) - f_j(x_0) \right)
\]

is verified for all \( i \in \{1, \ldots, N\} \) and \( x \in X \) such that \( f_i(x) < f_i(x_0) \), and for a certain \( j \in \{1, \ldots, N\} \) such that \( f_j(x) > f_j(x_0) \).

Following Jeyakumar and Mond [24], Kaul et al. [8] and Simlani and Radjef [20], we give the following definitions.

**Definition 12.** \((f, g)\) is \( d_i - V \)-univex type I at \( x_0 \in D \) if there exist positive real valued functions \( \alpha_i \) and \( \beta_i \) defined on \( X \times D \), nonnegative functions \( b_0 \) and \( b_1 \), also defined on \( X \times D \), \( \phi_i : R \to R \), \( \eta_i : X \times D \to R^+ \), and \( \theta_j : X \times D \to R^+ \) such that

\[
 b_0(x, x_0) \phi_i \left( f_i(x) - f_i(x_0) \right) \leq \alpha_i(x, x_0) f'_i(x_0; \eta_i(x, x_0))
\]

and

\[
 -b_1(x, x_0) \phi_i \left( g_j(x) - g_j(x_0) \right) \leq \beta_i(x, x_0) g'_j(x_0; \theta_j(x, x_0))
\]

for every \( x \in X \) and for all \( i = 1, 2, \ldots, N \) and \( j = 1, 2, \ldots, k \).

If the inequality in (3) is strict (whenever \( x \neq x_0 \)), we say that (MP) is of semistrictly \( d_i - V \)-univex type I at \( x_0 \) with respect to \( (\eta_i)_{i=1, \ldots, N} \) and \( (\theta_j)_{j=1, \ldots, N} \).

**Definition 13.** \((f, g)\) is quasi-\( d_i - V \)-univex type I at \( x_0 \in D \) if there exist positive real valued functions \( \alpha_i \) and \( \beta_i \), defined on \( X \times D \), nonnegative functions \( b_0 \) and \( b_1 \), also defined on \( X \times D \), \( \phi_i : R \to R \), \( \eta_i : X \times D \to R^+ \), and \( \theta_j : X \times D \to R^+ \) dimensional vector functions \( \phi_i : R \to R \) and \( \theta_j : X \times D \to R^+ \) such that for some vectors \( \lambda \in R^n_+ \) and \( \mu \in R_+^k \):

\[
 b_j(x_0, x_0) \phi_i \left( \sum_{i=1}^N \lambda_i \alpha_i(x_0, x_0) (f_i(x) - f_i(x_0)) \right)
\]

\[
 \leq 0 = \sum_{i=1}^N \lambda_i f_i(x_0; \eta_i(x, x_0)) \leq 0 \quad \forall x \in X
\]

and

\[
 \beta_i(x_0, x_0) \phi_i \left( \sum_{j=1}^k \mu_j \beta_j(x_0, x_0) g_j(x_0) \right) \geq 0
\]

\[
 \geq \sum_{j=1}^k \mu_j g'_j(x_0; \theta_j(x, x_0)) \geq 0 \quad \forall x \in X.
\]

If the second inequality in (5) is strict \((x \neq x_0)\), we say that (MP) is of semi-strictly quasi-\( d_i - V \)-univex type I at \( x \) with respect to \( (\eta_i)_{i=1, \ldots, N} \) and \( (\theta_j)_{j=1, \ldots, N} \).
Definition 14. \((f,g)\) is pseudo-\(d_i - V\)-univex type I at \(x_0 \in D\) if there exist positive real valued functions \(\alpha_i\) and \(\beta_j\), defined on \(X \times D\), nonnegative functions \(b_i\) and \(b_j\), also defined on \(X \times D\), \(\phi_0 : R \rightarrow R\), \(\phi_i : R \rightarrow R\) and \((N + k)\) dimensions vector functions \(\eta_i : X \times D \rightarrow R^i, i = 1, N\) and \(\theta_j : X \times D \rightarrow R^j, j = 1, k\) such that for some vectors \(\lambda \in R^N\) and \(\mu \in R^k\):

\[
\sum_{i=1}^{N} \lambda_i f_i'(x_0; \eta_i(x, x_0)) + \sum_{j=1}^{k} \mu_j g_j'(x_0; \theta_j(x, x_0)) \geq 0
\]

for \(\forall x \in X\).

Definition 15. \((f,g)\) is quasi pseudo-\(d_i - V\)-univex type I at \(x_0 \in D\) if there exist positive real valued functions \(\alpha_i\) and \(\beta_j\), defined on \(X \times D\), nonnegative functions \(b_i\) and \(b_j\), also defined on \(X \times D\), \(\phi_0 : R \rightarrow R\), \(\phi_i : R \rightarrow R\) and \((N + k)\) dimensions vector functions \(\eta_i : X \times D \rightarrow R^i, i = 1, N\) and \(\theta_j : X \times D \rightarrow R^j, j = 1, k\) such that the relation (5) and (8) are satisfied. If the second inequality in (8) is strict \((x \neq x_0)\), we say that \((VP)\) is of quasi strictly pseudo \(d_i - V\)-type I at \(x_0\) with respect to \((\eta_{i,j})_{i=1,N}\) and \((\theta_{j})_{j=1,k}\).

Definition 16. \((f,g)\) is pseudoquasi-\(d_i - V\)-univex type I at \(x_0 \in D\) if there exist positive real valued functions \(\alpha_i\) and \(\beta_j\), defined on \(X \times D\), nonnegative functions \(b_i\) and \(b_j\), also defined on \(X \times D\), \(\phi_0 : R \rightarrow R\), \(\phi_i : R \rightarrow R\) and \((N + k)\) dimensions vector functions \(\eta_i : X \times D \rightarrow R^i, i = 1, N\) and \(\theta_j : X \times D \rightarrow R^j, j = 1, k\), such that \(\mu \in R^k\) the relations (7) and (6) are satisfied. If the second inequality in (7) is strict \((x \neq x_0)\), we say that \((MP)\) is of strictly pseudo quasi \(d_i - V\)-type I at \(x_0\) with respect to \((\eta_{i,j})_{i=1,N}\) and \((\theta_{j})_{j=1,k}\).

3. Optimality Conditions

In this section, we discuss some sufficient conditions for a point to be an efficient or properly efficient for \((MP)\) under generalized \(d_i - V\)-univex type I assumptions.

Theorem 3.1. Let \(x_0\) be a feasible solution for \((MP)\) and suppose that there exist \((N + J)\) vector functions \(\eta_i : X \times X \rightarrow R^i, i = 1, N\), \(\theta_j : X \times X \rightarrow R^j, j \in J(x_0)\) and scalars \(\lambda_i \geq 0, i = 1, N\), \(\sum_{i=1}^{N} \lambda_i = 1\), \(\mu_j \geq 0, j \in J(x_0)\). 

\[
\sum_{i=1}^{N} \lambda_i f_i'(x_0; \eta_i(x, x_0)) + \sum_{j \in J(x_0)} \mu_j g_i'(x_0; \theta_j(x, x_0)) \geq 0, \quad \forall x \in X.
\]

Further, assume that one of the following conditions is satisfied:

a) \((f,g)\) is quasi strictly-pseudo \(d_i - V\)-univex type I at \(x_0\) with respect to \((\eta_{i,j})_{i=1,N}\) and \((\theta_{j})_{j=1,k}\).

b) \((f,g)\) is quasi pseudo-\(d_i - V\)-univex type I at \(x_0\) with respect to \((\eta_{i,j})_{i=1,N}\) and \((\theta_{j})_{j=1,k}\).

Then \(x_0\) is an efficient solution for \((MP)\).

Proof: Condition a). Suppose that \(x_0\) is not an efficient solution for \((MP)\). Then there exists \(x \in X\) such that

\[
f(x) \leq f(x_0),
\]

which implies that

\[
\sum_{i=1}^{N} \lambda_i f_i'(x_0; \eta_i(x, x_0)) + \sum_{j \in J(x_0)} \mu_j g_i'(x_0; \theta_j(x, x_0)) \leq 0.
\]

Since \(b_i(x, x_0) > 0; u \geq 0 \Rightarrow \phi_i(u) \geq 0\), the above inequality gives

\[
b_i(x, x_0) \phi_i \left[ \sum_{i=1}^{N} \lambda_i \alpha_i (x, x_0) f_i'(x_0; \eta_i(x, x_0)) \right] \leq 0.
\]

From the above inequality and Hypothesis i) of a), we have

\[
\sum_{i=1}^{N} \lambda_i f_i'(x_0; \eta_i(x, x_0)) \leq 0.
\]

By using the Inequality (9) we deduce that

\[
\sum_{j \in J(x_0)} \mu_j g_i'(x_0; \theta_j(x, x_0)) \geq 0,
\]

which implies from the condition part ii) of a) that

\[
b_i(x, x_0) \phi_i \left[ \sum_{j \in J(x_0)} \mu_j \beta_j (x, x_0) g_j'(x_0) \right] < 0.
\]

Since \(b_i(x, x_0) > 0; \phi_i(u) < 0 \Rightarrow u < 0\), we get
\[
\sum_{j \in J(x_0)} \bar{\mu}_j \beta_j(x, x_0) g_j(x_0) < 0. \tag{11}
\]

As \( \lambda \geq 0 \) and \( g_j(x_0) = 0; \forall j \in J(x_0) \), it follows that 

\[
\sum_{j \in J(x_0)} \bar{\mu}_j \beta_j(x, x_0) g_j(x_0) = 0, \quad \forall j \in J(x_0),
\]

which implies that

\[
\sum_{j \in J(x_0)} \bar{\mu}_j \beta_j(x, x_0) g_j(x_0) = 0. \tag{12}
\]

The above equation contradicts Inequality (11) and hence the conclusion of the theorem follows:

**Condition b):** Since \( g_j(x_0) = 0, \bar{\mu}_j \geq 0, \forall j \in J(x_0) \), and \( \beta_j(x, x_0) > 0, j \in J(x_0) \), we obtain

\[
\sum_{j \in J(x_0)} \bar{\mu}_j \beta_j(x, x_0) g_j(x_0) = 0, \quad \forall x \in X.
\]

By Hypothesis ii) of b), we get

\[
b_i(x, x_0) \phi \left[ \sum_{j \in J(x_0)} \bar{\mu}_j \beta_j(x, x_0) \right] \geq 0.
\]

From the above inequality and the Hypothesis i) of b) (in view of reverse implication in (8), if follows that

\[
\sum_{j \in J(x_0)} \bar{\mu}_j g_j(x_0) < 0, \quad \forall x \in X \setminus \{x_0\}.
\]

By using Inequality (9), we deduce that

\[
\sum_{i=1}^{N} \lambda_i f_i(x_0; \eta_i(x_0)) > 0, \quad \forall x \in X \setminus \{x_0\}, \tag{13}
\]

which by virtue of relation (7) implies that

\[
b_i(x, x_0) \phi \left[ \sum_{i=1}^{N} \lambda_i \alpha_i(x, x_0) (f_i(x) - f_i(x_0)) \right] > 0, \quad \forall x \in X \setminus \{x_0\}.
\]

The above inequality along with Hypothesis ii) of b) gives

\[
\sum_{i=1}^{N} \lambda_i \alpha_i(x, x_0) (f_i(x) - f_i(x_0)) > 0, \quad \forall x \in X \setminus \{x_0\}.
\]

Since (10) and (13) contradict each other, and hence the conclusion follows:

**Theorem 3.2.** Let \( x_0 \) be a feasible solution for (MP) and suppose that there exist \((N + J)\) vector functions \( \eta_i : X \times X \rightarrow R^*, i = 1, N, \theta_j : X \times X \rightarrow R^*, j \in J(x_0) \) and scalars \( \lambda_i \geq 0, i = 1, N, \sum_{i=1}^{N} \lambda_i = 1, \bar{\mu}_j \geq 0, j \in J(x_0) \) such that Inequality (9) of Theorem 3.1 is satisfied.

Moreover, assume that one of the following conditions is satisfied.

a) i) \((f, g)\) is pseudo quasi \( d_i - V - \) univex type I at \( x_0 \) with respect to \( (\eta_i), \tau_{i \in J(x_0)}, \lambda_i, \bar{\mu} \) and for some positive functions \( \alpha_i, i = 1, N \) and \( \beta_j, j \in J(x_0) \).

ii) for any \( u \in R \),

\[
u \geq 0 \Rightarrow \phi(u) \geq 0, \phi(u) \geq 0 \Rightarrow u \geq 0, \]

\[
b_h(x, x_0) > 0, h(x, x_0) \geq 0;
\]

b) i) \((f, g)\) is strictly pseudo \( d_i - V - \) univex type I at \( x_0 \) with respect to \( (\eta_i), \tau_{i \in J(x_0)}, \lambda_i, \bar{\mu} \) and for positive functions \( \alpha_i = \frac{1}{N} \) and \( \beta_j, j \in J(x_0) \).

ii) for any \( u \in R \)

\[
u \leq 0 \Rightarrow \phi(u) \leq 0; \, u \geq 0 \Rightarrow \phi(u) \geq 0;
\]

\[
b_h(x, x_0) > 0, h(x, x_0) \geq 0.
\]

Then \( x_0 \) is an efficient solution for (MP). Further suppose that these exist positive real numbers \( n_i, m_i \) such that \( n_i < \alpha_i(x, x_0) < m_i, i = 1, N \) for all feasible \( x \). Then \( x_0 \) is a properly efficient solution for (MP)

**Proof:** Condition a). Suppose that \( x_0 \) is not an efficient solution of (MP). Then there exists an \( x \in X \) such that \( f(x) \leq f(x_0) \) which implies that

\[
\sum_{i=1}^{N} \lambda_i f(x_0; \eta_i(x_0)) (f_i(x) - f_i(x_0)) < 0. \tag{14}
\]

Since \( g_j(x_0) = 0, \bar{\mu}_j \geq 0 \) and

\[
\beta_j(x, x_0) > 0, \forall j \in J(x_0)
\]

we obtain

\[
\sum_{j \in J(x_0)} \mu_j \beta_j(x, x_0) g_j(x_0) = 0.
\]

From the above inequality and Hypothesis ii) of a), we have

\[
b_i(x, x_0) \phi \left[ \sum_{j \in J(x_0)} \bar{\mu}_j \beta_j(x, x_0) g_j(x_0) \right] \geq 0.
\]

Using Hypothesis i) of a), we deduce that

\[
\sum_{j \in J(x_0)} \bar{\mu}_j \beta_j(x, x_0) g_j(x_0) \leq 0. \tag{15}
\]

The Inequalities (9) and (14) yield that

\[
\sum_{i=1}^{N} \lambda_i f(x_0; \eta_i(x_0)) \geq 0,
\]

which by Hypothesis i) of a), we obtain

\[
b_h(x, x_0) \phi \left[ \sum_{i=1}^{N} \lambda_i f(x_0; \eta_i(x_0)) (f_i(x) - f_i(x_0)) \right] \geq 0. \tag{16}
\]

The Inequality (16) and Hypothesis ii) of a) give

\[
\sum_{i=1}^{N} \lambda_i f(x_0; \eta_i(x_0)) (f_i(x) - f_i(x_0)) \geq 0. \tag{17}
\]

Since (14) and (17) contradict each other, we conclude...
that \( x_0 \) is not an efficient solution of (MP). The properly efficient solution follows as in Hanson et al. [25]. For the proof of part b), we proceed as in part b) of Theorem 3.1, we get Inequality (17). Thus, complete the proof.

4. Mond-Weir Type Duality

Consider the following multiobjective dual to problem (MP)

\[(MD) \quad \text{Maximize } f(y) = (f_1(y), f_2(y), \ldots, f_N(y))\]

subject to

\[
\sum_{i=1}^{N} \lambda_i f_i'(y; \eta_i(x,y)) + \sum_{j=1}^{k} \mu_j g_j'(y; \theta_j(x,y)) \geq 0, \forall x \in X
\]

\[
\mu_j g_j'(y) \geq 0, j = 1, 2, \ldots, k, y \in D, \lambda \in R^N_{\geq}, \mu \in R^k, \eta : X \times D \rightarrow R^n, \forall i = 1, 2, \ldots, N,
\]

\[
\theta_j : X \times D \rightarrow R^n, j = 1, 2, \ldots, k.
\]

Let \( Y \) be the set of feasible solutions of problem (MD); that is,

\[
Y = \left\{ (y, \lambda, \mu, (\eta_i)_{i=1}^{N}, (\theta_j)_{j=1}^{k}) \right\}:
\]

\[
\sum_{i=1}^{N} \lambda_i f_i'(y; \eta_i(x,y)) + \sum_{j=1}^{k} \mu_j g_j'(y; \theta_j(x,y)) \geq 0,
\]

\[
\mu_j g_j'(y) \geq 0, \forall x \in X; y \in D, \lambda \in R^N_{\geq}, \mu \in R^k; \eta_i : X \times D \rightarrow R^n, \forall i = 1, 2, \ldots, N;
\]

\[
\theta_j : X \times D \rightarrow R^n, \forall j = 1, 2, \ldots, k.
\]

We denote by \( P_{\alpha} Y \), the projection of set \( Y \) on \( D \).

Theorem 4.1. (Weak Duality). Let \( x \) and \( (y, \lambda, \mu, (\eta_i)_{i=1}^{N}, (\theta_j)_{j=1}^{k}) \) be feasible solution for (MP) and (MD) respectively. Moreover, assume that one of the following conditions is satisfied:

a) i) \((f, g)\) is pseudo quasi \( d_i - V\)-univex type I at \( y \) with respect to \( \lambda > 0, \mu, (\eta_i)_{i=1}^{N}, (\theta_j)_{j=1}^{k} \) and for some positive functions \( \alpha_i, \beta_j \) for \( i = 1, 2, \ldots, k \)

ii) for any \( u \in R \)

\[
\phi(u) \geq 0 \Rightarrow u \geq 0; \quad u_0 \Rightarrow \phi(u) \geq 0;
\]

\[
b(u, x) > 0, b(u, x) \geq 0\]

b) i) \((f, g)\) is strictly-pseudo quasi \( d_i - V\)-univex type I at \( y \) with respect to \( \lambda, \mu, (\eta_i)_{i=1}^{N}, (\theta_j)_{j=1}^{k} \) and for some positive function \( \alpha_i, \beta_j \) for \( i = 1, 2, \ldots, k \)

ii) for any \( u \in R \)

\[
\phi(u) \geq 0 \Rightarrow u > 0; \quad u \geq 0 \Rightarrow \phi(u) \geq 0;
\]

\[
b(u, x) > 0, b(u, x) > 0\]

c) i) \((f, g)\) is quasi strictly-pseudo \( d_i - V\)-univex type I at \( y \) with respect to \( \lambda, \mu, (\eta_i)_{i=1}^{N}, (\theta_j)_{j=1}^{k} \) and for some positive functions \( \alpha_i, \beta_j \) for \( i = 1, 2, \ldots, k \)

ii) for any \( u \in R \)

\[
\phi(u) > 0 \Rightarrow u > 0; \quad u > 0 \Rightarrow \phi(u) > 0;
\]

\[
b(u, x) > 0, b(u, x) > 0\]

Then \( f(x) \leq f(y) \).

Proof: Since

\[
\mu_j g_j'(y) \geq 0, j = 1, 2, \ldots, k,
\]

\[
\mu_j g_j'(y) \geq 0, j = 1, 2, \ldots, k,
\]

we have

\[
b(u, x) \phi(u) \geq 0.
\]

By Condition a) (in view of definition 16), it follows that

\[
\sum_{j=1}^{k} \mu_j g_j'(y) \leq \sum_{j=1}^{k} \mu_j g_j'(y) = 0.
\]

Since \( y, \lambda, \mu, (\eta_i)_{i=1}^{N}, (\theta_j)_{j=1}^{k} \) is a feasible solution for (MD), the first dual constraint with (18) implies that

\[
\sum_{i=1}^{N} \lambda_i f_i'(y; \eta_i(x,y)) \geq 0.
\]

From (19) and Hypothesis i) of a), we obtain

\[
b(u, x) \phi(u) \sum_{i=1}^{N} \lambda_i \alpha_i(x,y) \geq 0.
\]

Condition ii) of a) and Inequality (20) give

\[
\sum_{i=1}^{N} \lambda_i \alpha_i(x,y) \geq 0.
\]

Assume that \( f(x) \leq f(y) \). Since

\[
\alpha_i > 0, i = 1, 2, \ldots, N \quad \text{and} \quad \lambda > 0,
\]

we obtain

\[
\sum_{i=1}^{N} \lambda_i \alpha_i(x,y) < 0,
\]

which contradicts (21). Therefore, the conclusion follows:

The proof of part b) and c) are very similar to proof of part a), except that: for part b), the Inequality (21) becomes strict (>) and Inequality (22) becomes non strict (\( \leq \)). For part c), the Inequality (18) becomes strict (\( < \)),

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it follows that the Inequalities (20) and (21) become strict (>). Since \( \lambda \geq 0 \), then the Inequality (22) becomes non strict (\( \leq \)). In this cases, the Inequalities (21) and (22) contradicts each other always.

**Remark 1:** If we omit the assumption \( \lambda > 0 \) in the condition i) or the word “strictly” in the condition b), we obtain, for this part of theorem, \( f(x) \leq f(y) \).

**Theorem 4.2.** (Weak Duality). Let \( x \) and 
\[
\begin{align*}
\{y, \lambda, \mu, (\eta_j)_{j=1,N}, \theta_j \}_{j=1,K} \end{align*}
\]
be feasible solutions for (MP) and (MD) respectively, assume that

1. \((f,g)\) is semi-strictly \( d_j - V\)-univex type I at \( y \) with respect to \( \lambda > 0, \mu, (\eta_j)_{j=1,N}, (\theta_j)_{j=1,K} \) and for some positive functions \( (\alpha^*_j)_{j=1,N}, (\beta^*_j)_{j=1,K} \),

2. for any \( u \in R \),
\[
\phi(u) > 0 \Rightarrow a > 0, \ (u) \geq 0 \Rightarrow \phi(u) \geq 0,
\]

Then \( f(x) \leq f(y) \).

**Proof:** Since \( \mu_j g_j(x) \geq 0, j = 1,2,\ldots,k \), which implies that
\[
\sum_{j=1}^{k} \mu_j g_j(x) \geq 0.
\] (23)

By (23) and Hypothesis i) (with \( b_j(x,y) / \beta_j(x,y) \)) in Definition 12 replaced by \( \beta_j(x,y) \) it follows that
\[
\sum_{j=1}^{k} \mu_j g_j(y, \theta_j(x,y)) \leq 0.
\] (24)

The first dual constraint and (24) give
\[
\sum_{i=1}^{N} \lambda_i f_i(x) \geq 0.
\] (25)

Dividing both sides of (3) by \( \alpha_i(x,y) \) and taking \( x \neq y \) by Hypothesis i), we get
\[
b_i(x,y) \geq \frac{1}{\alpha_i(x,y)} f_i(x) - f_i(y) > f_i(y, \eta_i(x,y)),
\]
i = 1, 2, \ldots, \( N \).

On Multiplying by \( \lambda_i \) and taking \( \alpha^*_i = \frac{1}{\alpha_i(x,y)} \), we get
\[
b_i(x,y) \lambda_i \alpha^*_i \geq f_i(x) - f_i(y) \geq \lambda_i f_i(x, \theta_j(x,y)),
\]
i = 1, 2, \ldots, \( N \).

Adding with respect to \( i \), and applying (25) and Hypothesis ii), we have
\[
\sum_{i=1}^{N} \lambda_i \alpha^*_i (x,y)(f_i(x) - f_i(y)) > 0.
\] (26)

Assume that \( f(x) \leq f(y) \). Since \( \alpha^*_i > 0 \) and \( \lambda > 0 \), we have
\[
\sum_{i=1}^{N} \lambda_i \alpha^*_i (x,y)(f_i(x) - f_i(y)) < 0,
\]
which contradicts (26).

**Theorem 4.3.** (Strong Duality). Let \( x_0 \) be a weakly efficient solution for (MP). Assume that the function \( g \) satisfies the \( d_j \)-constraint qualification at \( x_0 \) with respect to \( \theta_j \). Then there exist \( \lambda \in R^N \) and \( \mu \in R^K \) such that \( (x_0, \lambda, \mu, (\eta_j)_{j=1,N}, (\theta_j)_{j=1,K}) \in Y \) and objective functions of (MP) and (MD) have the same values at \( x_0 \) and \( (x_0, \lambda, \mu, (\eta_j)_{j=1,N}, (\theta_j)_{j=1,K}) \), respectively. If, further, the weak duality between (MP) and (MD) in theorem holds with the condition a) without \( \lambda > 0 \) (resp. with the condition b) or c), then
\[
(x_0, \lambda, \mu, (\eta_j)_{j=1,N}, (\theta_j)_{j=1,K}) \in Y \text{ is a weakly efficient (resp. an efficient) solutions of (MD)}.
\]

**Proof:** By the Theorem 31 [20], there exists \( \mu \in R^k \) and \( \lambda \in R^{|\mathcal{F}|} \) such that
\[
\sum_{i=1}^{N} \lambda_i f_i(x_0, \eta_i(x_0)) + \sum_{j=1}^{k} \mu_j g_j(x_0, \theta_j(x_0)) \geq 0, \quad \forall \ x \in X.
\]

It follows that \( (x_0, \lambda, \mu, (\eta_j)_{j=1,N}, (\theta_j)_{j=1,K}) \in Y \). Trivially, the objective function values of (MP) and (MD) are equal.

Suppose that \( (x_0, \lambda, \mu, (\eta_j)_{j=1,N}, (\theta_j)_{j=1,K}) \in Y \) is not a weakly efficient solution of (MD). Then there exists \( (y^*, \lambda^*, \mu^*, (\eta_j)_{j=1,N}, (\theta_j)_{j=1,K}) \in Y \) such that
\[
f(x_0) < f(y^*) \text{ which violates the weak duality theorem. Hence } (x_0, \lambda, \mu, (\eta_j)_{j=1,N}, (\theta_j)_{j=1,K}) \in Y \text{ is indeed a weakly efficient solution of (MD)}.
\]

**Theorem 4.4.** (Strict Converse Duality). Let \( x_0 \) and 
\[
(y_0, \lambda, \mu, (\eta_j)_{j=1,N}, (\theta_j)_{j=1,K}) \text{ be feasible solutions for (MP) and (MD) respectively, such that}
\]
\[
\sum_{i=1}^{N} \lambda_i f_i(x_0) = \sum_{i=1}^{N} \lambda_i f_i(y_0).
\] (27)

Moreover, assume that \( (f,g) \) is strictly pseudo quasi \( d_j - V\)-type I at \( y_0 \) with respect to \( (\eta_j)_{j=1,N}, (\theta_j)_{j=1,K} \) and for \( \lambda \) and \( \mu \). Then \( x_0 = y_0 \).

**Proof:** Since \( \mu_j g_j(y_0) \geq 0, \forall j = 1,2,\ldots,k \), we have
Using the second part of the hypothesis, we get
\[ \sum_{j=1}^{k} g_j'(y_0; \theta_j(x_0, y_0)) \leq 0. \] (28)

The Inequality (28) and feasibility of $$y_0, \lambda, \mu, (\eta_j)_{j=1}^{N}$$ for (MD) give
\[ \sum_{i=1}^{N} \alpha_i(x_0, y_0)(f_i(x_0) - f_i(y_0)) \geq 0, \]
which by the first part of Hypothesis ii), we obtain
\[ b_0(x_0, y_0)b_{\theta}(\sum_{i=1}^{N} \alpha_i(x_0, y_0)(f_i(x_0) - f_i(y_0))) > 0, \]
\[ \forall x \in X. \]

The above inequality along with Hypothesis iii) gives
\[ \sum_{i=1}^{N} \alpha_i(x_0, y_0)(f_i(x_0) - f_i(y_0)) > 0. \] (29)

By Hypothesis i), iii) and \( \alpha_i(x_0, y_0) > 0, \)
\[ \sum_{i=1}^{N} \alpha_i(x_0, y_0)(f_i(x_0) - f_i(y_0)) = 0. \] (30)

Now (29) and (30) contradict each other. Hence the conclusion follows.

5. Conclusion and Future Developments

In this paper, generalized \( d_j - V \)-univex functions have been introduced. The sufficient optimality conditions are discussed for a point to be an efficient or properly efficient for (MP) under the introduced functions. Appropriate Mond-Weir type duality relations are established under these assumptions. Sufficiency and duality with generalized \( d_j - V \)-univex functions will be studied for nonsmooth variational and nonsmooth control problems, which will orient the future research of the author.

6. References


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