An Extension of the Black-Scholes and Margrabe Formulas to a Multiple Risk Economy

Werner Hürlimann
FRS Global Switzerland, Zürich, Switzerland
E-mail: werner.huerlimann@frsglobal.com
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Abstract

We consider an economic model with a deterministic money market account and a finite set of basic economic risks. The real-world prices of the risks are represented by continuous time stochastic processes satisfying a stochastic differential equation of diffusion type. For the simple class of log-normally distributed instantaneous rates of return, we construct an explicit state-price deflator. Since this includes the Black-Scholes and the Vasicek (Ornstein-Uhlenbeck) return models, the considered deflator is called Black-Scholes-Vasicek (BSV) deflator. Besides a new elementary proof of the Black-Scholes and Margrabe option pricing formulas a validation of these in a multiple risk economy is achieved.

Keywords: State-Price Deflator, Option Pricing, Black-Scholes Model, Vasicek Model, Margrabe Formula

1. Introduction

The first rigorous mathematical derivation of the Black-Scholes formula in [1] (see also [2]) relies on a dynamic delta-hedge portfolio and a risk-free argument of no-arbitrage. Later on [3] introduced state-price deflators, which led to the insight that deflator based market valuation using the real-world probability measure is equivalent to market valuation based on a risk-neutral martingale measure.

The present contribution focuses entirely on state-price deflators, which are summarized in Section 2. We consider in Section 3 an economic model that contains a money market account with deterministic continuous-compounded spot rates and a finite set of basic economic risks (interest rates, stocks and equity, property, commodities, inflation, currency, etc.). The real-world prices of these risks are represented by continuous time stochastic processes satisfying a stochastic differential equation of diffusion type. In the simplest situation of log-normally distributed instantaneous rates of return, which includes the Black-Scholes and the Vasicek (Ornstein-Uhlenbeck) return models, we construct in Proposition 3.2 the so-called Black-Scholes-Vasicek (BSV) deflator.

The application of the BSV deflator to option pricing follows in Section 4. Besides a new elementary proof of the (slightly extended) Black-Scholes formula it provides a validation of it in a financial market with multiple economic risks. The same holds true for Margrabe’s formula for a European option to exchange one risky asset for another one.

2. Valuation with State-Price Deflators

Let \((\Omega, F, P)\) be a probability space such that \(\Omega\) is the sample space, which describes the states of the world, \(F\) is the \(\sigma\)-field of events, and \(P\) is the probability measure assigning to any event \(E\) in \(F\) its probability \(P(E)\). At each time \(t \geq 0\), the \(\sigma\)-field \(F_t \subset F\) denotes the set of events, which describes the information available at time \(t\). An adapted process \(X\) is a set \(\{X_t\}_{t \geq 0}\) such that \(X_t\) is a random variable with respect to the measurable space \((\Omega, F)\).

In continuous time finance one considers adapted price processes \(S = \{S_t\}_{t \geq 0}\) such that \(S_t\) represents the random value at time \(t\) of a financial instrument. To place a market value or price on any financial instrument, we consider a (state-price) deflator \(D = \{D_t\}_{t \geq 0}\), that is a strictly positive adapted process such that the stochastic value \(S_t\) payable at time \(t\) has value at time \(s < t\) given by the formula

\[ S_s = D_s^{-1} \cdot E_s[D_t S_t], \quad 0 \leq s < t, \]

where by convention \(E_s[X_t] = E_{F_t}[X_t|F_s]\) denotes the expected value (under the real-world probability measure

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of a stochastic integral as follows.

**Proposition 1** The real-world prices of the financial instruments \( I_k, k = 1, \ldots, m \), satisfy the following stochastic representations

\[
S^{(i)}_t = S^{(i)}_0 \exp \left\{ r_i - r_t - \frac{1}{2} \sigma_i^2 \left( u_i r_i^{(i)} \right) dt \right\},
\]

where \( 0 \leq s < t, \quad k = 1, \ldots, m \).

**Proof.** For fixed \( k \) set \( X_i = (t^{(i)}, S^{(i)}_t) \) to see that

\[
dX_i = M(t, X_i) dt + \sum (t, X_i) dW^{(i)}_t
\]

with

\[
M(t, X_i) = \left( \mu_i \left( t, r_t^{(i)} \right), \sigma_i \left( t, r_t^{(i)} \right) \right),
\]

\[
\sum (t, X_i) = \left( \sigma_k \left( t, r_t^{(i)} \right), \sigma_i \left( t, r_t^{(i)} \right) \right).
\]

The result follows through application of the bivariate version of Itô’s Lemma (e.g. [5], Section 2).

For simplicity, and to describe the main features in an analytical way, we restrict the attention to either Black-Scholes return processes \( d \tilde{r}_t^{(i)} = \mu_i dt + \sigma_i dW^{(i)}_t \) (stocks and equity) or Vasicek (Ornstein-Uhlenbeck) return processes \( d \tilde{r}_t^{(i)} = a_i \left( r_t^{(i)} - \tilde{r}_t^{(i)} \right) dt + \sigma_i dW^{(i)}_t \) (interest rates, property, commodities, inflation, currency, etc.). In both cases the return differences \( r_t^{(i)} - \tilde{r}_t^{(i)}, 0 \leq s < t \), are normally distributed, which implies that the prices (6) are lognormally distributed. For a unified analysis let \( m_k(s, t) \) and \( v_k(s, t) \) denote the mean and standard deviation per unit time of these return differences as given by Black-Scholes return model

\[
m_k(s, t) = \mu_k, \quad v_k(s, t) = \sigma_k, \quad 0 \leq s < t
\]

Vasicek model

\[
m_k(s, t) = \frac{b_k - \tilde{r}_t^{(i)}}{t - s}, \quad v_k(s, t) = \sigma_k \sqrt{\frac{1 - e^{-2a_k(t-s)}}{2a_k(t-s)}}
\]

In this situation Proposition 1 yields the following equalities in distribution

\[
S^{(i)}_t = S^{(i)}_0 \exp \left\{ m_k(s, t) - \frac{1}{2} v_k^2 (t-s) + v_k \sqrt{t-s} \cdot W^{(i)}_t \right\},
\]

where \( W^{(i)}_t \)'s are correlated standard Wiener processes such that \( E[dW^{(i)}_t dW^{(j)}_t] = \rho_{ij} dt \).

Following Section 2 consider now the Black-Scholes-Vasicek deflator in the multiple risk economy, for short BSV deflator, which has the same form as the price processes in (9), i.e.
for some parametric function \( \alpha^{(a)}(s,t) \) and vectors

\[
\mathbf{p}^{(a)}(s,t) = \begin{pmatrix}
\beta_1^{(a)}(s,t), \ldots, \beta_m^{(a)}(s,t)
\end{pmatrix}^T,
\]

\[
W_{i-s} = \begin{pmatrix}
W_{1-s}^{(1)}, \ldots, W_{m-s}^{(m)}
\end{pmatrix}^T.
\]

To define a state-price deflator the stochastic processes (9) and (10) must satisfy the martingale conditions

\[
E_s [D_t^{(a)}] = D_s^{(a)} e^{\int_t^s \sigma_i (v_i(s,t) - \beta^{(a)}(s,t)) ds},
\]

\[
E_s [D_t^{(a)} s_i^{(k)}] = D_s^{(a)} s_i^{(k)}, \quad 0 \leq s < t, \quad k = 1, \ldots, m.
\]

**Proposition 2** (BSV deflator) Given is a financial market with a risk-free money market account \( m \) and economic risks that have log-normal real-world prices (9). Assume a non-singular valid correlation matrix \( \mathbf{C} \) with non-vanishing determinant. Then, the BSV deflator (10) is determined by

\[
D_t^{(a)} = D_s^{(a)} \exp \left\{-R(s,t)(t-s) - \frac{1}{2} \sum_{i,j=1}^m \beta_{ij}^{(a)}(s,t) \beta_{ij}^{(a)}(s,t)(t-s) - \sum_{j=1}^m \beta_{j}^{(a)}(s,t) \sqrt{t-s \cdot W_{i-s}^{(j)}} \right\},
\]

\[0 \leq s < t,
\]

with

\[
\beta_{ij}^{(a)}(s,t) = \det(\mathbf{C})^{-1} \sum_{s=1}^m (-1)^{i+s} \det(\mathbf{C}^{(i)}) \cdot \lambda_i(s,t),
\]

\[
\lambda_i(s,t) = \frac{m_i(s,t) - R(s,t) - \frac{1}{2} \left( \sigma_i^2 - v_i^2(s,t) \right)}{v_i(s,t)},
\]

\[0 \leq s < t,
\]

where \( \mathbf{C}^{(i)} \) is the matrix formed by deleting the \( i \)-th row and \( j \)-th column of \( \mathbf{C} \). The quantity \( \lambda_i(s,t) \) is called market price of the \( i \)-th economic risk.

**Remark 1** In the Black-Scholes return model the market price of the \( i \)-th economic risk is given by \( \lambda_i(s,t) = (\mu_i - R(s,t))/\sigma_i \) (Sharpe ratio).

**Proof.** The martingale conditions (11) are equivalent with the system of equations

\[
\beta_{ij}^{(a)}(s,t) \cdot \det(\mathbf{C}) = \lambda_i(s,t) \left(1 - \rho_{ij}^2\right) - \lambda_j(s,t) \left(\rho_{ij} - \rho_{ij} \rho_{ij} + \rho_{ij} \rho_{ij} + \rho_{ij} \rho_{ij}\right),
\]

\[
\lambda_i(s,t) = \frac{m_i(s,t) - R(s,t) - \frac{1}{2} \left( \sigma_i^2 - v_i^2(s,t) \right)}{v_i(s,t)}, \quad 0 \leq s < t,
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\]
4. The Black-Scholes and Margrabe Formulas in a Multiple Risk Economy

We begin with an elementary result in probability theory. Suppose that the random vector \((S_t, S_t)\) has a bivariate lognormal distribution with parameter vector \((\mu_t, \nu_t, \mu_s, \nu_s, \rho)\) such that the standardized random vector

\[
(U_1, U_2) = \left( \frac{\ln S_t - \mu_t}{\nu_1}, \frac{\ln S_t - \mu_t}{\nu_2} \right)
\]

has a standard bivariate normal distribution with correlation coefficient \(\rho\).

Lemma 1. The expected positive difference of the bivariate lognormal spread is given by

\[
E_c \left[ (S_t - S_t)_+ \right] = \exp \left( \mu_t + \frac{1}{2} \nu_1^2 \right) \Phi \left( \frac{\mu_t - \mu_s - \nu_1^2 - \rho \nu_1 \nu_2}{\sqrt{\nu_1^2 + \nu_2^2 - 2 \rho \nu_1 \nu_2}} \right) - \exp \left( \mu_s + \frac{1}{2} \nu_2^2 \right) \Phi \left( \frac{\mu_s - \mu_t - \nu_2^2 + \rho \nu_1 \nu_2}{\sqrt{\nu_1^2 + \nu_2^2 - 2 \rho \nu_1 \nu_2}} \right),
\]

with \(\Phi(x)\) the standard normal distribution.

Proof. The derivation is left as exercise. \(\square\)

The formula (20) is the unifying mathematical content leading to the (slightly extended) European call option formula by [6] (Theorem 1) (see also [7-9]) and the (slightly extended) formula by [10] for pricing the exchange option (Theorem 2). Both formulas are validated within a multiple risk economy.

Theorem 1 (Black-Scholes in a multiple risk economy)

Under the assumptions of Proposition 2, the market value at time \(t_s \geq 0\) of a European call option on the financial instrument \(I_k, k \in [1, \ldots, m]\) with strike time \(t < t_s\) and strike price \(K\) is given by the formula

\[
E_c \left[ D_s^{(m)} \left( S_t^{(i)} - K \right) \right] = D_s^{(m)} \left( S_t^{(i)} \right) \Phi \left( d_s^{(i,m)}(s,t) \right) - K \cdot P(s,t) \cdot \Phi \left( d_s^{(i,m)}(s,t) \right),
\]

with

\[
d_i^{(k,m)}(s,t) = \frac{\ln \left( S_t^{(i)} / K \right) + \left( R(s,t) + \frac{1}{2} \nu_k^2(s,t) \right)(t-s)}{\nu_k(s,t) \sqrt{t-s}},
\]

\[
d_2^{(k,m)}(s,t) = d_1^{(k,m)}(s,t) - \nu_k(s,t) \sqrt{t-s},
\]

\(0 \leq s < t, \quad k = 1, \ldots, m.\)

Remark 2 If \(m = 1\) one specializes to the Black-Scholes single risk economy with constant risk-free return \(R(s,t) = r > 0\) and Black-Scholes return model with constant volatility \(\nu_1(s,t) = \sigma_1\), one recovers the original formula in [6].

Proof. This is an application of Lemma 1. We distinguish between two cases. If \(m = 1\) one writes using (14) that \(D_s^{(1)} \left( S_t^{(i)} - K \right) = \exp \left( \nu_1(s,t) \right) \), with

\[
\mu_1 = \ln D_s^{(1)} - R(s,t)(t-s) - \frac{1}{2} \nu_1^2(s,t)(t-s)
\]

and \(\lambda_1(s,t) = \nu_1(s,t) \sqrt{t-s}\),

\[
\mu_2 = \ln D_s^{(1)} - R(s,t)(t-s) - \frac{1}{2} \nu_1^2(s,t)(t-s) + \ln K,
\]

\[
u_1 = \nu_1(s,t) - \lambda_1(s,t) \nu_1(s,t) \sqrt{t-s}, \quad \nu_2 = \lambda_1(s,t) \nu_1(s,t) \sqrt{t-s},
\]

\(U_1 = \text{sgn}(\nu_1(s,t) - \lambda_1(s,t)) \cdot W^{(i)}_{t-s}, \quad U_2 = -W^{(i)}_{t-s},
\]

\(\rho = -\text{sgn}(\nu_1(s,t) - \lambda_1(s,t)).\)

Through elementary algebra one sees that (use the definition of \(\lambda_1(s,t)\) in (13))

\[
\nu_1^2(s,t) - \nu_2^2 + \rho \nu_1 \nu_2 = v_1^2(s,t)(t-s),
\]

\[
\mu_1 - \mu_2 + \nu_2^2 - \rho \nu_1 \nu_2 = \ln \left( S_t^{(i)} / K \right) + \left( R(s,t) + \frac{1}{2} \nu_1^2(s,t) \right)(t-s),
\]

\[
\mu_1 - \mu_2 + \nu_2^2 + \rho \nu_1 \nu_2 = \ln \left( S_t^{(i)} / K \right) + \left( R(s,t) - \frac{1}{2} \nu_1^2(s,t) \right)(t-s),
\]

\[
\mu_1 + \frac{1}{2} \nu_2^2 = \ln D_s^{(1)} + \ln S_t^{(i)},
\]

\[
\mu_2 + \frac{1}{2} \nu_2^2 = \ln D_s^{(1)} + \ln K - R(s,t)(t-s).
\]

Plugging into (20) one obtains (21). If \(m \geq 2\) it suffices (for reasons of symmetry) to show (22) for \(k = 1\). For this consider the quantities \(\overline{B}_s^{(m)}(s,t), \overline{P}_s^{(m)}(s,t)\) defined by

\[
\overline{B}_s^{(m)}(s,t) = \sum_{j=1}^{m} \beta_j^{(m)}(s,t)^2 + \sum_{j \neq k} \rho_{jk} \beta_j^{(m)}(s,t) \beta_k^{(m)}(s,t),
\]

\[
\overline{P}_s^{(m)}(s,t) = \sum_{j=1}^{m} \beta_j^{(m)}(s,t).
\]

From the proof of Proposition 2 one has

\(\lambda(s,t) = C \cdot \beta^{(m)}(s,t)\), which in particular implies the identity \(\lambda(s,t) = \beta^{(m)}(s,t) + \overline{B}_s^{(m)}(s,t)\), which is used several times below. Using (12) one has

\(D_s^{(m)} \left( S_t^{(i)} - K \right) = \exp \left( \nu_1(s,t) \right) \), with
\[ \mu_1 = \ln D_1^{(m)} - \left( R(s,t) + \frac{1}{2} A(s,t) \right)(t-s) + \ln S^{(i)}_t + \left( m_i(s,t) - \frac{1}{2} \sigma_i^2 \right)(t-s), \]
\[ \mu_2 = \ln D_2^{(m)} - \left( R(s,t) + \frac{1}{2} A(s,t) \right)(t-s) + \ln K, \]
\[ A(s,t) = \beta_1^{(m)}(s,t)^2 + \beta_2^{(m)}(s,t)^2 + 2 \beta_1^{(m)} \beta_2^{(m)}(s,t) \overline{m}_2^{(m)}(s,t), \]
\[ v_1^2 = \left( A(s,t) + v_1(s,t) \delta(t) \right)(t-s), \quad v_2^2 = A(s,t)(t-s), \]
\[ \nu_1 U_1 = \left( v_1(s,t) - \beta_1^{(m)}(s,t) \right) \sqrt{W_{t-s}^{(i)}} - \sum_{j=2}^m \beta_1^{(m)}(s,t) \sqrt{W_{t-s}^{(j)}}, \]
\[ \nu_2 U_2 = \left( v_1(s,t) - \beta_1^{(m)}(s,t) \right) \sqrt{W_{t-s}^{(i)}} - \sum_{j=2}^m \beta_1^{(m)}(s,t) \sqrt{W_{t-s}^{(j)}}, \]
\[ \rho_{v_1 v_2} = \text{Cov}[\nu_1 U_1, \nu_2 U_2] = \left( A(s,t) - v_1(s,t) \delta(t) \right)(t-s). \]

Elementary algebra shows the relations (use the definition of \( \lambda_i(s,t) \) in (13))
\[ v_1^2 + v_2^2 - 2 \rho_{v_1 v_2} = v_1^2(s,t)(t-s), \]
\[ \mu_1 - \mu_2 + v_1^2 = \ln \left( S^{(i)}_t / K \right) + \left( R(s,t) + \frac{1}{2} v_1^2(s,t) \right)(t-s), \]
\[ \mu_1 - \mu_2 - v_2^2 + \rho_{v_1 v_2} = \ln \left( S^{(i)}_t / K \right) + \left( R(s,t) - \frac{1}{2} v_1^2(s,t) \right)(t-s), \]
\[ \mu_1 + \frac{1}{2} v_1^2 = \ln D_1^{(m)} + \ln S^{(i)}_t, \]
\[ \mu_2 + \frac{1}{2} v_2^2 = \ln D_2^{(m)} + \ln K - R(s,t)(t-s). \]

Inserting into (20) one obtains (21) for \( k = 1 \).

**Theorem 2** *(Margrabe in a multiple risk economy)*
Under the assumptions of Proposition 2, the market value at time \( s \geq 0 \) of a European exchange option on the financial instruments \( I_k, I_\ell, k \neq \ell \in \{1, \ldots, m\} \) with strike time \( t > s \) is given by the formula
\[ E_s \left[ D_1^{(m)} \left( S^{(1)}_t - S^{(2)}_t \right) \right] = \]
\[ D_1^{(m)} \left[ S^{(1)}_t \Phi \left( \ln \left( S^{(1)}_t / S^{(i)}_t \right) + \frac{1}{2} v^2(s,t)(t-s) \right) - S^{(i)}_t \Phi \left( \ln \left( S^{(i)}_t / S^{(i)}_t \right) - \frac{1}{2} v^2(s,t)(t-s) \right) \right], \]
(23)
\[ v^2(s,t) = v_1^2(s,t) + v_2^2(s,t) - 2 \rho_{v_1 v_2} v_1(s,t) v_2(s,t), \]
\[ 0 \leq s < t, \quad k \neq \ell \in \{1, \ldots, m\}. \]

**Proof.** For reasons of symmetry it suffices to show (23) for \( k = 1, \ell = 2 \). Consider the quantities \( \overline{m}_1^{(m)}(s,t), \overline{m}_2^{(m)}(s,t) \) defined by (if \( m = 2 \) the sums are empty and the quantities zero)
\[ \overline{m}_1^{(m)}(s,t) = \sum_{j=3}^m \rho_{j} \beta_j^{(m)}(s,t) \overline{m}_1^{(m)}(s,t), \]
\[ \overline{m}_2^{(m)}(s,t) = \sum_{j=3}^m \rho_{j} \beta_j^{(m)}(s,t). \]

Since \( \lambda(s,t) = C \cdot \beta^{(m)}(s,t) \) (proof of Proposition 2) one has in particular the identities
\[ \lambda_1(s,t) = \beta_1^{(m)}(s,t) + \rho_{2} \beta_2^{(m)}(s,t) + \overline{m}_1^{(m)} \overline{m}_1^{(m)}(s,t), \]
\[ \lambda_2(s,t) = \rho_2 \beta_1^{(m)}(s,t) + \beta_2^{(m)}(s,t) + \overline{m}_2^{(m)} \overline{m}_2^{(m)}(s,t). \]

Using (12) one writes
\[ D_1^{(m)} \left( S^{(1)}_t - S^{(2)}_t \right) = \left( e^{\alpha_1 + \theta_1 U_1} - e^{\alpha_2 + \theta_2 U_2} \right), \]
with
\[ \mu_t = \ln D_1^{(m)} - \left( R(s,t) + \frac{1}{2} A(s,t) \right)(t-s) \]
\[ + \ln S^{(i)}_t + \left( m_i(s,t) - \frac{1}{2} \sigma_i^2 \right)(t-s), \quad i = 1, 2, \]
\[ A(s,t) = \beta_1^{(m)}(s,t)^2 + \beta_2^{(m)}(s,t)^2 + \overline{m}_2^{(m)}(s,t)^2 \]
\[ + 2 \rho_{12} \beta_1^{(m)}(s,t) \beta_2^{(m)}(s,t) \]
\[ + 2 \overline{m}_1^{(m)} \beta_1^{(m)}(s,t) \overline{m}_1^{(m)}(s,t), \]
\[ + 2 \overline{m}_2^{(m)} \beta_2^{(m)}(s,t) \overline{m}_2^{(m)}(s,t), \]
\[ + 2 \overline{m}_2^{(m)} \beta_2^{(m)}(s,t) \overline{m}_2^{(m)}(s,t). \]
\[ \nu_i^2 = \left( A(s,t) + v_i(s,t) \right)^2 - 2 v_i(s,t) \lambda_i(s,t) (t-s), \]
\[ \nu_2^2 = \left( A(s,t) + v_2(s,t) \right)^2 - 2 v_2(s,t) \lambda_2(s,t) (t-s), \]
\[ \nu_k U_1 = \left( v_1(s,t) - \beta^{(m)}_k(s,t) \right) \sqrt{t-s} \cdot W_{1,k}^{(1)} \]
\[ - \beta^{(m)}_2(s,t) \sqrt{t-s} \cdot W_{1,2}^{(1)} - \sum_{j=3}^{m} \beta^{(m)}_j(s,t) \sqrt{t-s} \cdot W_{1,j}^{(1)}, \]
\[ \nu_2 U_2 = - \beta^{(m)}_1(s,t) \sqrt{t-s} \cdot W_{1,2}^{(1)} \]
\[ + \left( v_2(s,t) - \beta^{(m)}_2(s,t) \right) \sqrt{t-s} \cdot W_{2,2}^{(1)} \]
\[ - \sum_{j=3}^{m} \beta^{(m)}_j(s,t) \sqrt{t-s} W_{2,j}^{(1)}, \]

\[ \rho_{\nu_1 \nu_2} = \text{Cov} \left[ \nu_1 U_1, \nu_2 U_2 \right] \]
\[ = \left( A(s,t) - v_1(s,t) \lambda_1(s,t) - v_2(s,t) \lambda_2(s,t) \right) (t-s), \]

One obtains the relations (use again the definition of \( \lambda_i(s,t), i=1,2 \) in (13))
\[ \nu_i^2 + \nu_2^2 - 2 \rho_{\nu_1 \nu_2} \nu_i = \left( v_i(s,t) + v_2(s,t) - 2 \rho_{\nu_1 \nu_2} v(s,t) \right)^2 (t-s), \]
\[ \mu_i - \mu_2 + \nu_i^2 - \rho_{\nu_1 \nu_2} \nu_2 \]
\[ = \ln \left( S_1^{(i)} / S_2^{(i)} \right) + \frac{1}{2} \nu_i^2 (s,t) (t-s), \]
\[ \mu_i - \mu_2 - \nu_i^2 + \rho_{\nu_1 \nu_2} \nu_2 \]
\[ = \ln \left( S_1^{(i)} / S_2^{(i)} \right) - \frac{1}{2} \nu_i^2 (s,t) (t-s), \]
\[ \mu_i + \frac{1}{2} \nu_i^2 = \ln D^{(m)}_i + \ln S_1^{(i)}, \quad i=1,2. \]

Inserting into (20) one obtains the Formula (23) for \( k=1, \ell=2 \). \( \square \)

It might be useful to conclude with a short summary. If one starts with the stochastic representation (9) of the real-world prices for the risks in the economy, the derivation of the formulas is rather elementary. It only uses introductory Probability Theory (including the notion of Martingale) and Linear Algebra. Therefore, the proof is accessible to any knowledgeable person in these mathematical areas. Moreover, the approach is different from the original one (hedging argument, use of Itô’s Lemma and solution of a partial differential equation). It leads to new insight in Option Pricing Theory. Besides a general validation in a multiple risk economy, the proposed derivation implies a risk-neutral property of independent interest, i.e. the formulas are invariant with respect to the market prices of the risk factors.

5. References

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