RBFs Meshless Method of Lines for the Numerical Solution of Time-Dependent Nonlinear Coupled Partial Differential Equations

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Received December 25, 2010; revised February 3, 2011; accepted February 6, 2011

Abstract

In this paper a meshless method of lines is proposed for the numerical solution of time-dependent nonlinear coupled partial differential equations. Contrary to mesh oriented methods of lines using the finite-difference and finite element methods to approximate spatial derivatives, this new technique does not require a mesh in the problem domain, and a set of scattered nodes provided by initial data is required for the solution of the problem using some radial basis functions. Accuracy of the method is assessed in terms of the error norms $L_2$, $L_\infty$ and the three invariants $C_1$, $C_2$, $C_3$. Numerical experiments are performed to demonstrate the accuracy and easy implementation of this method for the three classes of time-dependent nonlinear coupled partial differential equations.

Keywords: RBFs, Meshless Method of Lines, Time-Dependent PDEs

1. Introduction

Nonlinear coupled partial differential equations have numerous applications in the field of science and engineering, including solid state physics, fluid mechanics, chemical physics, plasma physics etc. (see [1-3] and the references therein). In 1981 Hirota-Satsuma introduced the coupled KdV equations, [4] which has many applications in physical sciences. Coupled KdV equations describe an interaction of the two long waves with different dispersion relation. The Burgers’ equations describe phenomena such as a mathematical model of turbulence [5] and the nonlinear hyperbolic system [6] represents interaction of the two waves traveling in the opposite directions.

In the last decade many authors have studied the numerical and approximate solution of time-dependent nonlinear coupled partial differential equations by various numerical methods. These include Adomian decomposition method [7], the local discontinuous Galerkin method [8], the variational iteration method [9], the Chebyshev spectral collocation method [10] and the radial basis functions method [6,11,12].

In the last decade, the theory of radial basis functions (RBFs) has enjoyed a great success as scattered data interpolating technique. A radial basis function, $\phi(x-x_j) = \phi(||x-x_j||)$, is a continuous spline which depends upon the separation distances of a subset of data centers, $X \subset \mathbb{R}^n$, $\{x_j \in X, j=1,2,\ldots,N\}$. Due to spherical symmetry about the centers $x_j$, the RBFs are called radial. The distances, $||x-x_j||$, are usually taken to be the Euclidean metric.

Hardy [13] was the first to introduced a general scattered data interpolation method, called radial basis functions method for the approximation of two-dimensional geographical surfaces. In 1982 Franke [14] in a review paper made the comparison among all the interpolation methods for scattered data sets available at that time, and the radial basis functions outperformed all the other methods regarding efficiency, stability and ease of implementations. Franke found that Hardy’s multiquadrics (MQ) were ranked the best in accuracy, followed by thin plate splines (TPS). Despite MQ’s excellent performance, it contains a shape parameter $c$, and the accuracy of MQ is greatly affected by the choice of shape parameter $c$ whose optimal value is still unknown. Franke [15] used the formula $c^2 = (1.25)^2d^2$ where $d$ is the mean dis-
tance from each data point to its nearest neighbor. Hick-
ernell and Hon [16] and Golberg et al. [17] had success-
fully used the technique of cross-validation to obtain an
optimal value of the shape parameter. Various research-
ers have contributed recently to this field (see [18-25]
ect.)

The method of lines (MOL) [26] is a general pro-
dure for the solution of time dependent partial differen-
tial equations (PDEs). The basic idea of the MOL is to
replace the spatial (boundary value) derivatives in the
PDEs with algebraic approximations. Once this is done,
the spatial derivatives are no longer stated explicitly in
terms of the spatial independent variables. Thus only the
initial value variable, typically time in a physical prob-
lem, remains. In other words, we have a system of ODEs
that approximate the original PDE. Now we can apply
any integration algorithm for initial value ODEs to com-
pute an approximate numerical solution to the PDE. Thus,
one of the salient features of the MOL is the use of ex-
gisting, and generally well established, numerical methods
for ODEs. Very recently Quan Shen [27] use this ap-
proach for the numerical solution of KdV equation. In
this paper, we will use RBFs approximation method with
the method of lines (MOL) for the numerical solution of
time-dependent nonlinear partial differential equations
given as:

Class A: Coupled KdV equations

\[ u_t = -\alpha u_{xx} - 6\alpha uu_x + 2vvv_x, \]
\[ v_t = -\beta v_{xx} - 3\beta uv_x, \]

Class B: Coupled Burgers’ equations

\[ u_t = uu_x - 2uv_x - \alpha(\nu\nu)_x, \]
\[ v_t = v_{xx} - 2vv_x - \beta(\nu\nu)_x, \]

Class C: System of nonlinear hyperbolic equations

\[ u_t = -uu_x - \alpha uv, \]
\[ v_t = uu_x - \alpha uv, \]

where \( \alpha, \beta, \nu \) are positive parameters.

Rest of the paper is organized as follows: In Section 2,
The radial basis functions collocation method coupled
with MOL is presented. Section 3 is devoted to the nu-
merical tests of the method on the problems related to the
coupled KdV, the coupled Burgers’ equations and the
system of nonlinear hyperbolic equations. In Section 4,
the results are concluded.

2. RBFs Meshless Method of Lines

2.1. Coupled KdV Equations

Consider the nonlinear coupled KdV equations,

\[ u_t = -\alpha uu_x - 6\alpha uu_x + 2vvv_x, \]  
\[ v_t = -\beta v_{xx} - 3\beta uv_x, \]

with boundary conditions

\[ u(a,t) = f_1(t), \quad u(b,t) = f_2(t), \]
\[ v(a,t) = g_1(t), \quad v(b,t) = g_2(t), \quad t > 0, \]

and initial conditions

\[ u(x,0) = f(x), \]
\[ v(x,0) = g(x), \quad a \leq x \leq b, \]

where \( \alpha, \beta, \nu \) are positive real constants.

For a given set of \( N \) collocation points \( \{ x_j \}_{j=1}^N \) in
the domain \([a,b]\), the RBFs approximation for \( u \) and
\( v \) of (1) are given by

\[ U(x) = \sum_{j=1}^{N} \lambda_j \phi_j(r), \quad V(x) = \sum_{j=1}^{N} \lambda_j \phi_j(r), \]

(4)

where \( \{ \lambda_j \}_{j=1}^N \) are the unknown constants to be deter-
mained, \( \phi_j(r) = \phi(\|x-x_j\|) \) can be any well known radial
basis function and \( r_j = \|x-x_j\| \) is the Euclidean
norm between points \( x \) and \( x_j \). Here we are using two radial
basis functions, the multiquadric \( \phi(r) = \sqrt{r^2 + c^2} \)
and cubic \( \phi(r) = r^3 \). Now for each node
\( x_i, \quad i = 1,2,3,\ldots, N \) in the domain \([a,b]\), (4) can be written as

\[ U = A \lambda, V = A \lambda, \]

where

\[ U = \left[ U(x_1), U(x_2), U(x_3), \ldots, U(x_N) \right]^T, \]
\[ \lambda = \left[ \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_N \right]^T, \quad i = 1,2. \]

\[ A = \left[ \Phi^T(x_1), \Phi^T(x_2), \Phi^T(x_3), \ldots, \Phi^T(x_N) \right], \]
\[ \Phi^T(x_i) = \left[ \phi_1(x_i), \phi_2(x_i), \phi_3(x_i), \ldots, \phi_N(x_i) \right]. \]

where \( i = 1,2,3,\ldots, N \).

Equation (4) can also be written as

\[ U(x) = \Phi^T(x) A^{-1} U = D(x) U, \]
\[ V(x) = \Phi^T(x) A^{-1} V = D(x) V \]

(6)

where

\[ D(x) = \Phi^T(x) A^{-1} = [D_1(x), D_2(x), \ldots, D_N(x)]. \]

Using the approximations \( U_i(t) \) and \( V_i(t) \) of the
solutions \( u(x, t) \) and \( v(x, t) \) given in (6), (1) at each node \( x_i, \ i = 1, 2, 3, \ldots, N \) can be written as
\[
\frac{dU}{dt} = -\alpha D_{xx}(x_i) U - 6\alpha U_i(D_x(x_i) U)^2 + 2\nu r_i(D_x(x_i) V)
\]
\[
\frac{dV}{dt} = -\beta D_{xx}(x_i) V - 3\beta U_i(D_x(x_i) V)
\]
(7)
where
\[
D_x(x_i) = \left[D_{xx}(x_i), D_{xx}(x_i), \ldots, D_{xx}(x_i)\right],
\]
\[
D_{xx}(x_i) = \frac{\partial}{\partial x} D_x(x_i), \quad j = 1, 2, \ldots, N,
\]
\[
D_{xxx}(x_i) = \frac{\partial^3}{\partial x^3} D_x(x_i), \quad j = 1, 2, \ldots, N.
\]
In more compact form (7) can be written as
\[
\frac{dU}{dt} = -\alpha D_{xx} U - 6\alpha U \ast (D_x U) + 2\nu V \ast (D_x V),
\]
\[
\frac{dV}{dt} = -\beta D_{xx} V - 3\beta U \ast (D_x V),
\]
(8)
where the symbol \( \ast \) denotes component by component multiplication of two vectors and
\[
D_x = \left[D_{xx}(x_i)\right]_{i=1}^N, \quad D_{xxx} = \left[D_{xxx}(x_i)\right]_{i=1}^N.
\]
For simplicity we write (8) as
\[
\frac{dU}{dt} = F_1(U, V), \quad \frac{dV}{dt} = F_2(U, V),
\]
(9)
where
\[
F_1(U, V) = -\alpha D_{xx} U - 6\alpha U \ast (D_x U) + 2\nu V \ast (D_x V),
\]
\[
F_2(U, V) = -\beta D_{xx} V - 3\beta U \ast (D_x V).
\]
The corresponding boundary conditions are given by
\[
U_i = f_i(t), \quad U_N = f_2(t)
\]
\[
V_i = g_i(t), \quad V_N = g_2(t)
\]
(10)
and the initial conditions are as
\[
U(t_0) = [f(x_1), f(x_2), \ldots, f(x_N)]
\]
\[
V(t_0) = [g(x_1), g(x_2), \ldots, g(x_N)]
\]
(11)
Now we solve the system of ODEs (9)-(11) by using the well known ODE solvers.

1) The classical forth order Runge-Kutta method (RK4) given by
\[
U^{n+1} = U^n + \frac{\Delta t}{6} (K_1 + 2K_2 + 2K_3 + K_4),
\]
where
\[
K_1 = F_1(U^n, V^n), \quad K_2 = F_1(U^n + \delta t K_1/2, V^n),
\]
\[
K_3 = F_1(U^n + \delta t K_2/2, V^n), \quad K_4 = F_1(U^n + \delta t K_3, V^n),
\]
\[
J_1 = F_2(U^n, V^n), \quad J_2 = F_2(U^n, V^n + \delta t J_1/2),
\]
\[
J_3 = F_2(U^n, V^n + \delta t J_2/2), \quad J_4 = F_2(U^n, V^n + \delta t J_3).
\]

2) Low-storage third-order (TVD-RK3) scheme given by [28]
\[
U^{(1)} = U^n + \delta t F_1(U^n, V^n),
\]
\[
U^{(2)} = \frac{3}{4} U^n + \frac{1}{4} U^{(1)} + \frac{1}{4} \delta t F_2(U^{(1)}, V^{(1)}),
\]
\[
U^{(3)} = \frac{3}{4} U^n + \frac{1}{4} U^{(1)} + \frac{1}{4} \delta t F_2(U^{(1)}, V^{(1)}),
\]
\[
V^{(1)} = V^n + \delta t F_2(U^n, V^n),
\]
\[
V^{(2)} = \frac{3}{4} V^n + \frac{1}{4} V^{(1)} + \frac{1}{4} \delta t F_2(U^n, V^{(1)}),
\]
\[
V^{(3)} = \frac{3}{4} V^n + \frac{1}{4} V^{(1)} + \frac{1}{4} \delta t F_2(U^n, V^{(1)}).
\]

2.2. Nonlinear Coupled Burgers’ Equations
Consider the nonlinear coupled Burgers’ equations
\[
u_j = u_j - 2u_ju_i - \alpha (uv)_x,
\]
\[
u_j = v_j - 2v_ju_i - \beta (uv)_x,
\]
(12)
with the boundary conditions
\[
u_j = f_j(t), \quad u_j = f_j(t),
\]
\[
u_j = g_j(t), \quad v_j = g_j(t), \quad t > 0,
\]
(13)
and initial conditions
\[
u(x, 0) = f(x),
\]
\[
u(x, 0) = g(x), \quad a \leq x \leq b,
\]
(14)
where \( \alpha, \beta, \nu \) are positive parameters. The same procedure as discussed in Section 2.1 can be used for the solution of (12)-(14).

2.3. Nonlinear Coupled Hyperbolics
Consider the nonlinear hyperbolic system
\[
u_j = -u_j - \alpha uv, \quad v_j = v_j - \alpha uv,
\]
(15)
with boundary conditions
\[ u(a,t) = f_1(t), \quad u(b,t) = f_2(t), \quad v(a,t) = g_1(t), \quad v(b,t) = g_2(t), \quad t > 0, \]

and initial conditions
\[
\begin{align*}
\quad u(x,0) &= f(x), \\
\quad v(x,0) &= g(x), \quad a \leq x \leq b,
\end{align*}
\]

where \( \alpha \) is a positive real constant. The same procedure as discussed in Section 2.1 can be used for the solution of (15)-(17).

3. Numerical Examples

In this section, we apply the RBFs meshless method of lines for the numerical solution of three classes of partial differential equations, defined earlier. We use the \( L_2 \) and \( L_\infty \) error norms to measure the difference between the numerical and analytic solutions. The \( L_2 \) and \( L_\infty \) error norms of the solution are defined by
\[
L_2 = \| u - U \|_2 = \left[ \sum_{i=1}^{N} (u - U)^2 \right]^{1/2},
\]
\[
L_\infty = \| u - U \|_\infty = \max_i |u - U|.
\]

We examine our results by calculating the following three conservative laws. Hirota-Satsuma [4] proved that the coupled KdV equations defined in (1) possesses three conservative laws. Hirota-Satsuma [4] proved that the three invariants \( C_1, C_2 \) and \( C_3 \) as well as their normalized values,
\[
NC_1 = \frac{(C_1(t) - C_1(0))/C_1(0)}, \quad NC_2 = \frac{(C_2(t) - C_2(0))/C_2(0)}, \quad NC_3 = \frac{(C_3(t) - C_3(0))/C_3(0)},
\]

are absolutely conserved in time during the computations which demonstrates the accuracy of the schemes. We also noted that the value of MQ shape parameter for which the solution converges belongs to the interval \( 0.1 \leq c \leq 0.6 \) as shown in Table 3. The motion of solitary waves \( u \) and \( v \) is shown in Figure 1, which are initially centered at \( x = 0 \) moving from left to right with the constant speed \( \lambda \), having the amplitudes \( \lambda/\alpha \) and \( \sqrt{2}\lambda/\alpha \) respectively.

Problem 2 Consider the nonlinear coupled Burgers’ Equations (12), whose exact solution [10] is given by
\[
\begin{align*}
\quad u(x,t) &= a_0 - 2A \left( \frac{2\alpha - 1}{4\alpha\beta - 1} \right) \tanh \left[ A(x - 2At) \right],
\quad v(x,t) &= a_0 \left( \frac{2\beta - 1}{2\alpha - 1} \right) - 2A \left( \frac{2\alpha - 1}{4\alpha\beta - 1} \right) \tanh \left[ A(x - 2At) \right].
\end{align*}
\]

where \( A = (1/2)(a_0 (4\alpha\beta - 1)/(2\alpha - 1)), \quad a_0, \quad \alpha, \quad \beta \) are arbitrary constants.

The boundary conditions \( u(a,t), \quad u(b,t), \quad v(a,t), \quad v(b,t) \) and the initial conditions \( u(x,0), \quad v(x,0) \) are extracted from the exact solution (21). We solved the problem in the domain \(-10 \leq x \leq 10 \) by using MOL coupled with RBFs collocation method. The classical RK4 and TVD-RK3 scheme are used in our computations. The results are listed in Table 5 and Figure 2, and compare with earlier results [10]. It is observed that the results are comparable with [10] and well agreed with the exact solution.

Problem 3 Now we consider nonlinear coupled hyperbolic Equations (15). For the sake of comparison [6], we take \( a = -0.5, \quad b = 0.5, \quad \nu = 100 \) and the initial conditions
\[
\begin{align*}
\quad u(x,0) &= f(x), \\
\quad v(x,0) &= g(x),\quad a \leq x \leq b,
\end{align*}
\]
Figure 1. Plots of Problem 1 corresponding to $\alpha = 0.1$, $\beta = 0.1$, $\lambda = 0.5$, $\delta_x = 0.1$, $\delta_t = 0.001$, $c = 0.58$. Figures 1(a) and 1(b) show the motion of the solitary waves $u$ and $v$ moving from left to right, initially centered at $x = 0$ when $t = 0, 1, 2, 3, 4, 5$. Figures 1(c) and 1(d) represent numerical solutions $u$ and $v$ over time $[0, 5]$.

\[ u(x,0) = \begin{cases} 0.5\left[1 + \cos(10\pi x)\right], & x \in [-0.3, -0.1]; \\ 0, & \text{otherwise} \end{cases} \]  
\[ v(x,0) = \begin{cases} 0.5\left[1 + \cos(10\pi x)\right], & x \in [0.1, 0.3]; \\ 0, & \text{otherwise} \end{cases} \]

and the boundary conditions

\[ u(a,t) = u(b,t) = 0, \]
\[ v(a,t) = v(b,t) = 0. \]

This problem is solved by RBFs meshless method of lines using MQ with RK4 scheme. We take the initial solutions $v$ and $v(x,0)$ which are located at $x = -0.2$ and $x = 0.2$, respectively. When $t > 0$ the nonlinear term, $uv$, causes these waves to move without change in shape, $u$ to the right and $v$ to the left. The two waves collide when $t = 0.1$ which results in change of shapes of the waves. The two waves overlap each other near $t = 0.25$ and they separate again at $t = 0.3$ approximately. From this time onwards the linear term becomes dominant and the pulses lose their symmetry and experience a decrease in the amplitude due to nonlinear interaction as shown in the Figures 3(a)-3(f). The numerical results of the solutions $u$ and $v$ are presented graphically. Since the exact solution of this problem is not known, we use cubic radial basis function $r^3$ to find the numerical solution. These graphical results are agreed well with the results obtained by quasi-linear interpolation method [6].

4. Closure

We have applied the meshless method of lines using radial basis functions for the numerical solutions of time-dependent nonlinear coupled partial differential equa-
Table 1. Error norms and the three invariants for the solutions $u$, $v$ using MQ when $\delta_t = 0.001$, $N = 100$, $c = 0.53$, $\alpha = 0.01$, $\beta = 0.01$ and $\lambda = 0.01$ corresponding to Problem 1.

<table>
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<th>$L_\infty(v)$</th>
<th>$L_2(v)$</th>
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Table 2. The three invariants and its normalized invariants for the solutions $u$, $v$ using MQ, when, $\delta_t = 0.001$, $N = 100$, $c_1 = 0.53$, $c_2 = 0.53$, $\alpha = 0.05$, $\beta = 0.05$, $\lambda = 0.05$ corresponding to Problem 1.

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Table 3. Error norms and normalized invariants of the solutions $u$, $v$ for different values of MQ shape parameter $c$ when $t = 1.0$, $\delta_t = 0.001$, $N = 100$, $\alpha = 0.1$, $\beta = 0.1$, $\lambda = 0.5$ corresponding to Problem 1.

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Table 4. Error norms and normalized invariants of the solutions $u, v$ for different values of time step size $\delta_t$ when $t = 1.0, c = 0.53, N = 100, \alpha = 0.1, \beta = 0.1, \lambda = 0.5$ corresponding to Problem 1.

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Table 5. Error norms of solutions $u$ and $v$ using MQ, corresponding to Problem 2.

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<th>$b$</th>
<th>$a_0$</th>
<th>$c$</th>
<th>$L_\infty(u)$</th>
<th>$L_\infty(v)$</th>
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Figure 2. Comparison of numerical solution using RK4 with MQ versus exact solutions $u, v$ at time $t = 1$ when $\alpha = 1, \beta = 2, a_0 = 0.1, c = 0.58$ and $\delta_t = 0.001$, corresponding to Problem 2.
Figure 3. Numerical solution using quintics. Figures 3(a)-3(f) show the motion and interaction of the waves $u$ and $v$. 
5. Acknowledgements

The work presented in this paper is supported by HEC Pakistan through grants 063-281079-Ps3-169, IRSIP 14 Ps3.

6. References


