Statistically Convergent Double Sequence Spaces in 2-Normed Spaces Defined by Orlicz Function

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Abstract

The concept of statistical convergence was introduced by Stinhaus [1] in 1951. In this paper, we study convergence of double sequence spaces in 2-normed spaces and obtained a criteria for double sequences in 2-normed spaces to be statistically Cauchy sequence in 2-normed spaces.

Keywords: Double Sequence Spaces, Natural Density, Statistical Convergence, 2-Norm, Orlicz Function

1. Introduction

In order to extend the notion of convergence of sequences, statistical convergence was introduced by Fast [2] and Schoenberg [3] independently. Later on it was further investigated by Fridy and Orhan [4]. The idea depends on the notion of density of subset of \( \mathbb{N} \).

The concept of 2-normed spaces was initially introduced by G. Ahler [5–7] in the mid of 1960’s. Since then, many researchers have studied this concept and obtained various results, see for instance [8].

Let \( X \) be a real vector space of dimension \( d \), where \( 2 \leq d \leq \infty \). A 2-norm on \( X \) is a function \( \| \cdot \| : X \times X \rightarrow \mathbb{R} \) which satisfies the following four conditions:

1) \( \| x, y \| = 0 \) if and only if \( x, y \) are linearly dependent;
2) \( \| x, y \| = \| y, x \| \);
3) \( \| \alpha x, y \| = \| x, y \| \) for any \( \alpha \in \mathbb{R} \);
4) \( \| x + y, z \| \leq \| x, z \| + \| y, z \| \)

The pair \( (X, \| \cdot \|) \) is then called a 2-normed space (see [9]).

Example 1.1. A standard example of a 2-normed space is \( \mathbb{R}^2 \) equipped with the following 2-norm

\( \| x, y \| := \text{the area of the triangle having vertices } 0, x, y. \)

Example 1.2. Let \( Y \) be a space of all bounded real-valued functions on \( \mathbb{R} \). For \( f, g \) in \( Y \), define

\[ \| f, g \| = \sup_{x \in \mathbb{R}} |f(x)| \cdot |g(x)|, \text{ if } f, g \text{ are linearly independent.} \]

Then \( \| \cdot \| \) is a 2-norm on \( Y \).

We recall some facts connecting with statistical convergence. If \( K \) is subset of positive integers \( \mathbb{N} \), then \( K_n \) denotes the set \( \{ k \in K : k \leq n \} \). The natural density of \( K \) is given by \( \delta(K) = \lim_{n \to \infty} \frac{|K_n|}{n} \), where \( |K_n| \) denotes the number of elements in \( K_n \), provided this limit exists. Finite subsets have natural density zero and \( \delta(K^c) = 1 - \delta(K) \) where \( K^c = \mathbb{N} \setminus K \), that is the complement of \( K \). If \( K_1 \subseteq K_2 \) and \( K_1 \) and \( K_2 \) have natural densities then \( \delta(K_1) \leq \delta(K_2) \). Moreover, if \( \delta(K_1) = \delta(K_2) = 1 \), then \( \delta(K_1 \cap K_2) = 1 \) (see [10]).

A real number sequence \( x = (x_i) \) is statistically convergent to \( L \) provided that for every \( \varepsilon > 0 \) the set \( \{ n \in \mathbb{N} : |x_i - L| \leq \varepsilon \} \) has natural density zero. The sequence \( x = (x_i) \) is statistically Cauchy sequence if for each \( \varepsilon > 0 \) there is positive integer \( N = N(\varepsilon) \) such that

\[ \delta(\{ n \in \mathbb{N} : |x_i - x_n(\varepsilon)| \}) = 0 \] (see [11]).

If \( x = (x_i) \) is a sequence that satisfies some property \( P \) for all \( n \) except a set of natural density zero, then we say that \( (x_i) \) satisfies some property \( P \) for “almost all \( n \)”.

An Orlicz Function is a function \( M : [0, \infty) \rightarrow [0, \infty) \) which is continuous, nondecreasing and convex with \( M(0) = 0 \), \( M(x) > 0 \) for \( x > 0 \) and \( M(x) \rightarrow \infty \) as \( x \rightarrow \infty \).

If convexity of \( M \) is replaced by \( M(x + y) \leq M(x) + M(y) \), then it is called a Modulus function (see Maddox [12]). An Orlicz function may be bounded or un-
bounded. For example, \( M(x) = x^p \) \((0 < p \leq 1)\) is unbounded and \( M(x) = \frac{x}{x+1} \) is bounded.

Lindenstrauss and Tzafriri [13] used the idea of Orlicz sequence space;

\[
l_M := \left\{ x \in W : \sum_{k=1}^\infty M \left( \frac{|x_k|}{\rho} \right) < \infty \text{ for some } \rho > 0 \right\}
\]

which is Banach space with the norm

\[
\|x\|_M = \inf \left\{ \rho > 0 : \sum_{k=1}^\infty \left( \frac{|x_k|}{\rho} \right) \leq 1 \right\}.
\]

The space \( l_M \) is closely related to the space \( l_p \), which is an Orlicz sequence space with \( M(x) = x^p \) for \( 1 < p \leq \infty \).

An Orlicz function \( M \) satisfies the \( \Delta_2 \) condition \( (M \in \Delta_2 \text{ for short}) \) if there exist constant \( K \geq 2 \) and \( u_0 > 0 \) such that

\[
M(2u) \leq KM(u)
\]

whenever \( |u| \leq u_0 \).

Note that an Orlicz function satisfies the inequality

\[
M(\lambda x) \leq \lambda M(x) \text{ for all } \lambda > 0 \text{ with } 0 < \lambda < 1.
\]

Orlicz function has been studied by V. A. Khan [14-17] and many others.

Throughout a double sequence \( x = (x_{k,l}) \) is a double infinite array of elements \( x_{k,l} \) for \( k,l \in \mathbb{N} \).

Double sequences have been studied by V. A. Khan [18-20], Moricz and Rhoades [21] and many others.

A double sequence \( x = (x_{j,k}) \) called statistically convergent to \( L \) if for every \( \epsilon > 0 \), the set

\[
\left\{ j \in \mathbb{N} : \|x_j - L, z\| \geq \epsilon \right\}
\]

has natural density zero for each nonzero \( z \) in \( X \), in other words \( x_j \) statistically convergent to \( L \) in 2-normed space \( (X,\|\cdot\|) \) if

\[
\lim_{n \to \infty} \frac{1}{n} \left| \left\{ j : \|x_j - L, z\| \geq \epsilon \right\} \right| = 0
\]

for each nonzero \( z \) in \( X \). It means that for every \( z \in X \),

\[
\|x_j - L, z\| < \epsilon \text{ a.a.n.}
\]

In this case we write

\[
st - \lim_{n \to \infty} \|x_j - L, z\| := \|L, z\|.
\]

**Example 2.1** Let \( X = \mathbb{R}^2 \) be equipped with the 2-norm by the formula

\[
\|x, y\| = |x_1 y_2 - x_2 y_1|, x = (x_1, x_2), y = (y_1, y_2).
\]

Define the \( (x_j) \) in 2-normed space \( (X,\|\cdot\|) \) by

\[
x_j = \begin{cases} (1, n) & \text{if } n = k^2, k \in \mathbb{N}, \\ (1, \frac{n-1}{n}) & \text{otherwise.} \end{cases}
\]

and let \( L = (1,1) \) and \( z = (z_1, z_2) \). If \( z_j = 0 \) then

\[
K = \left\{ j \in \mathbb{N} : \|x_j - L, z\| \geq \epsilon \right\} = \emptyset
\]

for each \( z \in X \), \( j \in \mathbb{N} : n \neq k^2, k \geq \frac{|z_1|}{\epsilon} \) is a finite set, so

\[
\left\{ j \in \mathbb{N} : \|x_j - L, z\| \geq \epsilon \right\} = \emptyset
\]

For each \( z \) in \( X \), \( \left\{ j \in \mathbb{N} : n \neq k^2, k \geq \frac{|z_1|}{\epsilon} \right\} \) is a finite set, so

\[
\left\{ j \in \mathbb{N} : \|x_j - L, z\| \geq \epsilon \right\} = \emptyset
\]

for each \( z \) in \( X \). Hence, \( \delta \left( \left\{ j \in \mathbb{N} : \|x_j - L, z\| \geq \epsilon \right\} \right) = 0 \)

for every \( \epsilon > 0 \) and \( z \in X \).

V. A. Khan and Sabiha Tabassum [20] defined a double sequence \( (x_{j,k}) \) in 2-normed space \( (X,\|\cdot\|) \) to be Cauchy with respect to the 2-norm if

\[
\lim_{j,p \to \infty} \|x_{j,k} - x_{p,q}, z\| = 0 \text{ for every } z \in X \text{ and } k,q \in \mathbb{N}.
\]

If every Cauchy sequence in \( X \) converges to some \( L \in X \), then \( X \) is said to be complete with respect to the 2-norm. Any complete 2-normed space is said to be 2-Banach space.

**Example 2.2** Define the \( x_j \) in 2-normed space \( (X,\|\cdot\|) \) by

\[
x_j = \begin{cases} (0, j) & \text{if } j = k^2, k \in \mathbb{N}, \\ (0,0) & \text{otherwise.} \end{cases}
\]
and let $L = (0,0)$ and $z = (z_1, z_2)$. If $z_i = 0$ then
$$\{ j \in \mathbb{N} : \|x_j - L, z\| \geq \varepsilon \} \subseteq \{1, 4, 9, 16, \ldots, j^2 \}.$$ 

We have that $\delta \left( \{ j \in \mathbb{N} : \|x_j - L, z\| \geq \varepsilon \} \right) = 0$ for every $\varepsilon > 0$ and $z \in X$. This implies that $st - \lim_{n \to \infty} j_k = \|L, z\|$. But the sequence $x_j$ is not convergent to $L$.

A sequence which converges statistically need not be bounded. This fact can be seen from Example [2.1] and Example [2.2].

3. Main Results

In this paper we define a double sequence $(x_{jk})$ in 2-normed space $(X, \|\cdot\|)$ to be statistically Cauchy with respect to the 2-norm if for every $\varepsilon > 0$ and every nonzero $z \in X$ there exists a number $p = p(\varepsilon, z)$ and $q = q(\varepsilon, z)$ such that
$$\lim_{n,m \to \infty} \frac{1}{mn} \# \left\{ (j,k) \in n \times n : \|x_{jk} - x_{jm}, z\| \geq \varepsilon, j \leq m, k \leq n \right\} = 0.$$

In this case we write $st - \lim_{n \to \infty} x_{jk} = \|L, z\|$. Let the double sequence $(x_{jk})$ be a double sequence in 2-normed space $(X, \|\cdot\|)$ and $L, L' \in X$ and $st_2 - \lim x_{jk}, z' = \|L, z\|$ and $st_2 - \lim x_{jk}, z'' = \|L', z\|$, then $L = L'$.

**Proof.** Assume $L \neq L'$. Then $L - L' \neq 0$, so there exists a $z \in X$, such that $L - L'$ and $z$ are linearly independent. Therefore
$$\|L - L', z\| = 2\varepsilon, \text{ with } \varepsilon > 0.$$

Now
$$2\varepsilon = \|L - x_{jk}, z + (x_{jk} - L, z')\| \leq \|x_{jk} - L, z\| + \|x_{jk} - L, z\|.$$

So $\{ (j,k) : \|x_{jk} - L', z\| \leq \varepsilon \} \subseteq \{ (j,k) : \|x_{jk} - L', z\| \geq \varepsilon \}$. But $\delta \left( \{ (j,k) : \|x_{jk} - L', z\| \geq \varepsilon \} \right) = 0$. Contradicting the fact that $x_{jk} \rightarrow L'(\text{stat})$.

**Theorem 3.2.** Let the double sequence $(x_{jk})$ and $(y_{jk})$ in 2-normed space $(X, \|\cdot\|)$ and $L, L' \in X$ and $a \in \mathbb{R}$.

If $st_2 - \lim x_{jk}, z'' = \|L, z\|$ and $st_2 - \lim y_{jk}, z'' = \|L', z\|$, for every nonzero $z \in X$, then
1) $st_2 - \lim x_{jk}, z'' = \|L + L', z\|$, for each nonzero $z \in X$ and
2) $st_2 - \lim a x_{jk}, z'' = \|aL, z\|$, for each nonzero $z \in X$.

**Proof 1) Assume that** $st_2 - \lim x_{jk}, z'' = \|L, z\|$ and $st_2 - \lim y_{jk}, z'' = \|L', z\|$, for every nonzero $z \in X$. Then
$$\delta(K_1) = 0 \text{ and } \delta(K_2) = 0$$

where
$$K_1 = K_1(\varepsilon) := \{ (j,k) \in n \times n : \|x_{jk} - L, z\| \geq \varepsilon \}$$
$$K_2 = K_2(\varepsilon) := \{ (j,k) \in n \times n : \|y_{jk} - L', z\| \geq \varepsilon \}$$

for every $\varepsilon > 0$ and $z \in X$. Let
$$K = K(\varepsilon) := \{ (j,k) \in n \times n : \|x_{jk} + y_{jk} - (L + L'), z\| \geq \varepsilon \}.$$

To prove that $\delta(K) = 0$, it is sufficient to prove that $K \subseteq K_1 \cup K_2$ Suppose $j_0, k_0 \in K$. Then
$$\|x_{j_0k_0} + y_{j_0k_0} - (L + L'), z\| \geq \varepsilon \quad (3.2)$$
Suppose to the contrary that \( j_0, k_0 \notin K_1 \cup K_2 \). Then \( j_0, k_0 \notin K_1 \) and \( j_0, k_0 \notin K_2 \). If \( j_0, k_0 \notin K_1 \) and \( j_0, k_0 \notin K_2 \), then
\[
\|x_{j_0 k_0} - L_z\| < \frac{\varepsilon}{2} \quad \text{and} \quad \|x_{j_0 k_0} - L_z\| < \frac{\varepsilon}{2}.
\]
Then, we get
\[
\left\| x_{j_0 k_0} + y_{j_0 k_0} - (L + L'), z \right\| \\
\leq \left\| x_{j_0 k_0} - L_z, z \right\| + \left\| y_{j_0 k_0} - L', z \right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]
which contradicts \([3.2]\). Hence \( K \subset K_1 \cup K_2 \).

2) Let \( st_z - \lim \|x_{j\ell}, z\| = \|L_z, z\|, a \in \mathbb{R} \) and \( a \neq 0 \).

Then
\[
\left\{ (j, k) \in N \times N : \|x_{j\ell} - L_z\| \geq \varepsilon \right\} = 0.
\]
Then we have
\[
\left\{ (j, k) \in N \times N : \|ax_{j\ell} - al, z\| \geq \varepsilon \right\} = \left\{ (j, k) \in N \times N : \|x_{j\ell} - L_z, z\| \geq \varepsilon \right\}
\]
Hence, the right handside of above equality equals 0. Hence, \( st_z - \lim \|x_{j\ell}, z\| = \|L_z, z\|, \) for every nonzero \( z \in X \).

From Theorem 1 of Fridy \([11]\) we have

**Theorem 3.4.** Let \( \{x_{j\ell}\} \) be statistically Cauchy sequence in a finite dimensional 2-normed space \( (X, \|\|) \).

Then there exists a convergent double sequence \( \{y_{j\ell}\} \) in \( (X, \|\|) \) such that \( x_{j\ell} = y_{j\ell} \) for almost all \( n \).

**Proof.** See proof of Theorem 2.9 \([9]\).

**Theorem 3.5.** Let \( \{x_{j\ell}\} \) be a double sequence in 2-normed space \( (X, \|\|) \). The double sequence \( \{x_{j\ell}\} \) is statistically convergent if and only if \( \{x_{j\ell}\} \) is a statistically Cauchy sequence.

**Proof.** Assume that \( st_z - \lim \|x_{j\ell}, z\| = \|L_z, z\| \) for every nonzero \( z \in X \) and \( \varepsilon > 0 \).

Then, for every \( z \in X \),
\[
\|x_{j\ell} - L_z\| < \frac{\varepsilon}{2} \quad \text{almost all } n,
\]
and if \( p = p(\varepsilon, z) \) and \( q = q(\varepsilon, z) \) is chosen so that \( \|x_{j\ell} - L_z\| < \frac{\varepsilon}{2} \), then we have
\[
\|x_{j\ell} - x_{j\ell q}, z\| \\
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{almost all } n,
\]
Hence, \( \{x_{j\ell}\} \) is statistically Cauchy sequence.

Conversely, assume that \( x_{j\ell} \) is a statistically Cauchy sequence. By Theorem 3.4, we have \( st_z - \lim \|x_{j\ell}, z\| = \|L_z, z\| \) for each \( z \in X \).

**4. References**


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