Existence and Non-Existence Result for Singular Quasilinear Elliptic Equations*

Mingzhu Wu¹, Zuodong Yang¹,²

¹Institute of Mathematics, School of Mathematical Science, Nanjing Normal University, Nanjing, China
²College of Zhongbei, Nanjing Normal University, Nanjing, China
E-mail: zdyang_jin@263.net

Received June 1, 2010; revised August 11, 2010; accepted August 14, 2010

Abstract

We prove the existence of a ground state solution for the quasilinear elliptic equation

\[-\text{div}(|u|^{p-2} \nabla u) = f(x,u),\quad x \in \Omega,\]

\[u > 0, \quad x \in \Omega\]

\[u(x) = 0, \quad x \in \partial\Omega\]

in which \(\Omega \subset \mathbb{R}^N\), \(u(x) \to 0\) when \(|x| \to \infty\). Let \(f : \Omega \times [0,\infty) \to [0,\infty)\) be a locally Holder continuous function which may be singular at \(t = 0\).

The problem (1) appears in the study of non-Newtonian fluids [1,2] and non-Newtonian filtration [3]. The quantity \(p\) is a characteristic of the medium. Media with \(p > 2\) are called dilatant fluids and those with \(p < 2\) are called pseudoplastics. If \(p = 2\), they are Newtonian fluid. When \(p \neq 2\), the problem becomes more complicated since certain nice properties inherent to the case \(p = 2\) seem to be lose or at least difficult to verify. The main differences between \(p = 2\) and \(p \neq 2\) can be found in [4-6].

In recent years the study of ground state solutions for \(p = 2\) has received a lot of interest and gets numerous existence results (see [7-12]). For \(p\)-Laplacian equations, in most papers, the focus has been on separable nonlinearities like (2).

\[-\text{div}(|u|^{p-2} \nabla u) = b(x)g(u),\quad x \in \mathbb{R}^N,\]

\[u > 0, \quad x \in \mathbb{R}^N\]

\[u(x) \to 0, \quad as\ |x| \to \infty\]

We refer readers to the paper [13-19]. In this paper, we consider the situation of \(p > 1\).

In [20], the author extended results to the problem (1) for \(p = 2\) where \(f\) is not necessarily separable. For \(p \geq 2\) we can see [1-6]. Motivated by papers [4-6,20], we extend the results to \(p > 1\) and get two theorems. But we still have many difficulties to get entire ground state solution of (1).

The second purpose is to give a result for nonexistence of solution. To the best of our knowledge, there has been very less result for nonexistence of solution about singular elliptic equation. We solve an open problem in [13] for \(p > 1\), for the case when \(u\) is a radially symmetric solution.

The paper is organized as follows. In Section 2 we recall some facts and give many lemmas that will be needed in the paper. In Section 3, we give the proof of the main result of the paper.

Keywords: Quasilinear Elliptic Equations, Existence, Non-Existence, Singular

1. Introduction

In this paper, we are concerned with the existence of ground state solution for the following problem

\[-\text{div}(|u|^{p-2} \nabla u) = f(x,u),\quad x \in \Omega,\]

\[u > 0, \quad x \in \Omega\]

\[u(x) = 0, \quad x \in \partial\Omega\]

\(*\text{Project Supported by the National Natural Science Foundation of China (No.10871060); the Natural Science Foundation of Educational Department of Jiangsu Province (No.08KJB110005)}*

Copyright © 2010 SciRes.
2. Preliminaries

Firstly we list the following assumptions and results that needed below.

\[
\begin{align*}
-\text{div}(\nabla W |^{p-2} \nabla W) &= b(x), \quad x \in \mathbb{R}^N, \\
W &> 0, \quad x \in \mathbb{R}^N, \\
W(x) &\to 0, \quad \text{as } |x| \to \infty
\end{align*}
\]

(3) \[ [B_1] \] \( b > 0 \) is a locally Holder continuous function on \( \mathbb{R}^N \).

\[ [B_2] \int_0^\infty (t^{-N} \int_0^t s^{N-1} b^*(s) ds)^{\frac{1}{p-1}} dt < \infty, \quad \text{where} \]

\[
b^*(t) = \max \{ b(x) : |x| = t \} \quad \text{for } t > 0.
\]

\[ [B_3] \] Problem (3) has a solution \( W_b \) in \( \mathbb{R}^N \) when \( b \) satisfy some condition.

It can be proved that condition \( [B_2] \) implies \( [B_1] \). This follows from the observation that \( v(r) \to 0 \) as \( r \to \infty \) where \( v(r) = \int_0^\infty (t^{-N} \int_0^t s^{N-1} b^*(s) ds)^{\frac{1}{p-1}} dt \)

and satisfies the following equation

\[-\text{div}(\nabla v |^{p-2} \nabla v) = b^*(x). \]

Thus \( w(x) = v(|x|) \) is a supersolution of (3). Since \( 0 \leq w \), and zero is clearly a sub-solution it follows from standard results that (3) admits a solution \( W_b \) such that \( 0 < W_b \leq w \).

The following eigenvalue problem

\[
\begin{align*}
-\text{div}(\nabla \phi |^{p-2} \nabla \phi) &= \lambda(c(x)\phi |^{p-1}, \quad x \in \mathbb{R}^N, \\
\phi &\to 0, \quad \text{as } |x| \to \infty
\end{align*}
\]

(4) \where \( \Omega \subseteq \mathbb{R}^N \) is a bounded smooth domain, and \( c \in C^\alpha(\Omega \setminus (0,\infty)) \) for some \( 0 < \alpha < 1 \). The first eigenvalue of the problem (4) will be denoted by \( \lambda_{1\alpha} > 0 \). It is easily noted from the variational characterization of eigenvalues that \( \lambda_{1\alpha} \geq \lambda_1 \)

where \( \Omega_1 \subseteq \Omega_2 \) are smooth bounded domains in \( \mathbb{R}^N \).

Consider \( g : (0,\infty) \to (0,\infty) \) satisfies the following conditions:

\[ [G_1] g \] \( g \) is \( C^1 \).

\[ [G_2] \] \( \limsup_{t \to \infty} \frac{g(t)}{t} < \frac{1}{\|W_b\|_\infty} \), where

\[
\|W_b\|_\infty = \max_{0 < r < \infty} W_b(x).
\]

The nonlinearity \( f \) in problem (1) is assumed to be a real function that satisfies the following conditions:

\[ [F_1] \] \( f(x,t) \) is locally Holder continuous on \( \mathbb{R}^N \times (0,\infty) \) and continuously differentiable in the variable \( t \).

\[ [F_2] \] \( f(x,s) \leq b(x)g(s) \) for all \( (x,s) \in \mathbb{R}^N \times (0,\infty) \) and some functions \( b : \mathbb{R}^N \to (0,\infty) \) and \( g : (0,\infty) \to (0,\infty) \).

\[ [F_3] \] There is a continuous function \( a : \mathbb{R}^N \to (0,\infty) \) and some constants \( \beta > 0 \) and \( \lambda > \lambda_i \) such that \( f(x,s) \geq \lambda a(x) s^{p-1}, (x,s) \in \mathbb{R}^N \times (0,\infty) \)

where \( \lambda_i \) is the first eigenvalue of the problem (4) on the ball \( B(0,1) \) of radius one and centered at the origin and where \( a(x) = a(x) \).

Recall that the nonlinearity \( f(x,t) \) may exhibit singularity as \( t \to 0^+ \). We will consider the following Dirichlet problem for a given smooth bounded domain \( \Omega \subseteq \mathbb{R}^N \).

\[
\begin{align*}
-\text{div}(\nabla W |^{p-2} \nabla W) &= b(x), \quad x \in \Omega, \\
W &> 0, \quad x \in \Omega, \\
W(x) &= 0, \quad x \in \partial \Omega
\end{align*}
\]

(5)\To establish the main theorem, from reference [2] we give the following lemmas.

**Lemma 1.** (Weak Comparison Principle) Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) \((N \geq 2)\) with smooth boundary \( \partial \Omega \) and \( \theta : (0,\infty) \to (0,\infty) \) is continuous and non-decreasing. Let \( u_1, u_2 \in W^{1,p}_0(\Omega) \cap C(\Omega) \) satisfy

\[
\begin{align*}
\int_\Omega |\nabla u_1 |^{p-2} \nabla u_1 \psi dx + &\int_\Omega \theta(u_1) \psi dx \leq \int_\Omega |\nabla u_2 |^{p-2} \nabla u_2 \psi dx + &\int_\Omega \theta(u_2) \psi dx
\end{align*}
\]

for all non-negative \( \psi \in W^{1,p}_0(\Omega) \). Then the inequality

\[
\limsup_{t \to \infty} (u_1(x) - u_2(x)) \leq 0
\]

implies that \( u_1 \leq u_2 \) in \( \Omega \).

Before Lemma 2, we give the following equation

\[-\text{div}(\nabla u |^{p-2} \nabla u) + f(x,u) = 0, \quad x \in \mathbb{R}^N \]

(6)

**Lemma 2.** Suppose that \( f(x,u) \) is defined on \( \mathbb{R}^{N+1} \)

and is locally Holder continuous (with exponent \( \lambda \in (0,1) \)) in \( x \). Suppose moreover that there exist functions \( v,w \in C^{1,\lambda}(\mathbb{R}^N) \) such that

\[
\begin{align*}
-\text{div}(\nabla v |^{p-2} \nabla v) &\pm f(x,v) \leq 0 \quad \text{and} \quad -\text{div}(\nabla w |^{p-2} \nabla w) \pm f(x,w) \geq 0
\end{align*}
\]

and

\[
\begin{align*}
v(x) &\leq w(x) \quad \text{and that} \quad f(x,u) \quad \text{is locally Lipschitz continuous in} \quad u \quad \text{on the set} \quad \{(x,u) : x \in \mathbb{R}^N, v(x) \leq u \leq w(x)\}
\end{align*}
\]

Then Equation (6) possesses an entire solution \( u(x) \) satisfying

\[
\begin{align*}
v(x) &\leq u(x) \leq w(x), \quad x \in \mathbb{R}^N
\end{align*}
\]

**Lemma 3.** Let \( b \) satisfy \([B_1], [B_2]\) and \( g \) satisfy both \([G_1], [G_2]\). Then there is \( v \in C^1(\mathbb{R}^N) \) such
that
\[ - \text{div} (|\nabla v|^{p-2} \nabla v) \geq b(x)g^{p-1}(v(x)), \]
\[ v > 0, v(x) \to 0 \quad \text{as} \quad |x| \to \infty. \]

**Proof.** Since \( g \) satisfies \([G_2]\), we define
\[ \hat{g}(t) := \sup_{s > t} \left\{ \frac{g(s)}{s} \right\}, t > 0 \quad (7) \]

Note that \( \hat{g}(t) \) is non-increasing, positive and \( \hat{g}(t) \geq g(t)t^{-1} \). Furthermore, by \([G_2]\) we have
\[ \hat{g}(t) < \|W_s\|_\infty \] for sufficiently large \( t \).

Let
\[ h(t) = \frac{2}{t^2} \int_0^t \hat{g}(s)ds, t > 0 \quad (8) \]

Then \( h \) is \( C^1 \), non-increasing and
\[ \hat{g}(t) \leq h(t) \leq \frac{\hat{g}(t)}{2} \] for all \( t \in (0, \infty) \). Since \( h \) is non-increasing, we note that \( h(t) \to \alpha < \|W_s\|_\infty \) as \( t \to \infty \) for some \( \alpha \in [0, \infty) \). Now, let us set
\[ \eta(t) := \int_0^t \frac{1}{h(s)}ds, t > 0 \quad (9) \]

On using \( \hat{g}(t) < \|W_s\|_\infty \) in (8) for sufficiently large \( t > 0 \), we see from (9) that
\[ \eta(\gamma) > \gamma \|W_s\|_\infty, \quad (10) \]
for a sufficiently large \( \gamma \geq 1 \). Let \( \psi = \eta^{-1} \) be the inverse function of \( \eta \).

By direct calculation, we see that
\[ \psi(t) = h(\psi(t)) \]
\[ \psi(t) > 0 \quad \text{for} \quad t > 0 \quad \text{and} \quad \psi(0) = 0. \]
Furthermore
\[ \psi'(t) = h(\psi(t))h'(\psi(t)), t > 0. \]

By condition \([B_1]\), we take a solution \( W_s \) of (3) with \( \Omega = \mathbb{R}^N \). Let us now set \( v(x) := \psi(yW_s(x)) \) for all \( x \in \Omega \). We note from (10) that
\[ v(x) = \psi(yW_s(x)) \leq \psi(y\|W_s\|_\infty) < \gamma \quad (11) \]

A simple computation shows that \( v \) has the desired properties.

Indeed, on recalling \(-\text{div}(|\nabla W_s|^{p-2} \nabla W_s) = b \), we see that
\[ -\text{div} (|\nabla v|^{p-2} \nabla v)
\[ = -\gamma^p (p-1)(\psi')^{p-2} \psi'' |\nabla W_s|^{p-2} \nabla W_s)
\[ = -(\psi')^{p-1} \gamma^{p-1} \text{div} (|\nabla W_s|^{p-2} \nabla W_s)
\[ = -\gamma^p (p-1)(h(\psi(t))^{p-1} h'(\psi(t)) |\nabla W_s|^{p-2} \nabla W_s) + \gamma^{p-1} bh^{p-1}(v) \geq \gamma^{p-1} bh^{p-1}(v) \]
\[ \geq \frac{1}{\gamma} \gamma^{p-1} bh^{p-1}(v) \geq bh^{p-1}(v). \]

We have used (11) in the last inequality. Since \( \psi(0) = 0 \), it is clear that \( v(x) \to 0 \) as \( |x| \to \infty \).

Since \( \gamma \geq 1 \), observe that the solution \( v \) constructed in Lemma 3 also satisfies
\[ -\text{div} (|\nabla v|^{p-2} \nabla v) \geq b \hat{g}^{p-1}(v). \]

\[ (\hat{G}_1) \hat{g} \in C'((0, \infty), (0, \infty)); \]
\[ (\hat{G}_2) \lim_{u \to +\infty} \frac{g(u)}{u^{p-1}} = +\infty; \]
\[ (\hat{G}_3) \lim_{u \to +\infty} \frac{g(u)}{u^{p-1}} = 0. \]

**Lemma 4.** Assume that \( g \) satisfies \([\hat{G}_1, -\hat{G}_3]\). Then there exist functions \( g_1, \hat{g}_1 \) such that
(i) \( g_1, \hat{g}_1 \in C'((0, \infty), (0, \infty)); \)
(ii) \( g_1(s) \leq \hat{g}(s) (s+1)^{p-1} \leq \hat{g}(s), \forall s > 0 \)
\[ \lim_{s \to 0^+} g_1(s) = \lim_{s \to 0^+} \hat{g}_1(s) = \infty; \]
(iii) \( g_1, \hat{g}_1 \) are non-increasing on \( (0, \infty); \)
(iv) \( \lim_{s \to 0^+} g_1(s) = \lim_{s \to 0^+} \hat{g}_1(s) = 0. \)

**Proof.**
\[ \lim_{s \to 0^+} \frac{g(s)}{(s+1)^{p-1}} = \lim_{s \to 0^+} \frac{g(s)}{s^{p-1}} = +\infty, \]
\[ \lim_{s \to +\infty} \frac{g(s)}{(s+1)^{p-1}} = \lim_{s \to +\infty} \frac{g(s)}{s^{p-1}} = 0. \]

We set \( \bar{g}(s) = \sup_{t \geq 0} \frac{g(t)}{(t+1)^{p-1}}, t \geq 0 \), then we have
\[ \bar{g}(s) \geq \frac{g(t)}{(t+1)^{p-1}}, \forall s > 0 \quad \text{and} \quad t \geq s \]
and \( \bar{g}(s) \) is non-increasing on \( (0, \infty) \). Moreover,
\[ \lim_{s \to 0^+} \bar{g}(s) = \infty \quad \text{and} \quad \lim_{s \to +\infty} \bar{g}(s) = 0. \]

Now we can assume \( \bar{g} \in C'(0, \infty) \). If not, we can replace it by
\[ \bar{g}(s) = \frac{2}{s^2} \int_0^s \bar{g}(t)dt, \quad s > 0 \]

Obviously,
\[ \bar{g}(s) \leq \bar{g}_1(s) \leq \bar{g}(\frac{s}{2}), \quad \forall s > 0; \]
and
\[
\mathcal{G}'(s) = \frac{2}{s} \left( \frac{1}{2} \mathcal{G}(s) - \frac{1}{2} \mathcal{G} \left( \frac{s^2}{2} \right) \right) - \frac{2}{s} \int_0^s \mathcal{G}(t) \, dt
\]

\[
\leq \frac{2}{s} \left( \frac{1}{2} \mathcal{G}(s) - \frac{1}{2} \mathcal{G} \left( \frac{s^2}{2} \right) - \frac{s}{2} \mathcal{G}(s) = \frac{1}{s}(\mathcal{G}(s) - \mathcal{G} \left( \frac{s^2}{2} \right)) \right) \leq 0.
\]

\[i.e., \mathcal{G}_1 \in C^1((0,\infty),(0,\infty)).\]

Observe that

\[0 \leq \mathcal{G}(s) \leq \frac{\mathcal{G}(t)}{(t+1)^{p-1}}, \quad \forall s > 0;\]

and \(\mathcal{G}(s)\) is non-increasing on \((0,\infty)\). Moreover,

\[\lim_{s \to 0^+} \mathcal{G}(s) = \infty \text{ and } \lim_{s \to \infty} \mathcal{G}(s) = 0.\]

Now we can assume \(g \in C^1(0,\infty)\). If not, we can replace it by

\[\mathcal{G}_2(s) = \frac{2}{s} \int_{s+1}^{s+1} \mathcal{G}(t) \, dt, s > 0\]

Obviously,

\[\mathcal{G}(s+1) \leq \mathcal{G}_2(s) \leq \mathcal{G}(s), \quad \forall s > 0\]

and

\[\mathcal{G}_2(s) = \mathcal{G}(s+1) - \mathcal{G}(s) \leq 0, \quad \forall s > 0\]

\[i.e., \mathcal{G}_2 \in C^1((0,\infty),(0,\infty)).\]

### 3. Proof of Main Theorems

In this section, we prove our main results.

**Theorem 1.** Let \(\Omega \subseteq \mathbb{R}^N\) be a bounded smooth domain that contains \(B(0,1)\), the ball of radius one centered at the origin, and let \(f\) satisfies \([F_1]\), \([F_2]\) and \([F_3]\), where \(b\) satisfies \([B_1]\), \([B_2]\) and \(g\) satisfies \([G_1]\), \([G_2]\). Then the problem

\[
\begin{cases}
-\text{div}(|\nabla u|^{p-2} \nabla u) = f(x,u), \quad x \in \Omega \\
u(x) = 0, \quad x \in \partial \Omega
\end{cases}
\]

has a positive solution \(u \in C^{1,\alpha}(\Omega) \cap C(\overline{\Omega})\) such that \(u \geq \phi_0\) where \(\phi_0\) is an eigenfunction of the eigenvalue problem (4) on \(\Omega\) with \(c(x)=a(x)\) normalized such that \(0 < \phi_0 \leq \gamma\) on \(\Omega\). Here \(\gamma\) is the constant in condition \([F_3]\).

**Proof.** Let \(\Omega_1\) be the solution of (5) and set \(\nu_1(x) = \eta^{-1}(\rho \Omega_1)\) where \(\eta\) and \(\gamma\) are defined as in (9) and (10) respectively. Then \(\nu_1 = 0\) on \(\partial \Omega\), and proceeding as in the proof of Lemma 3, we note that

\[-\text{div}(|\nabla \nu_1|^{p-2} \nabla \nu_1) \geq b(x)g^{p-1}(\nu_1), x \in \Omega.\]  

Therefore, by condition \([F_3]\), we see that

\[-\text{div}(|\nabla v_\Omega|^{p-2} \nabla v_\Omega) \geq f(x,v_\Omega), x \in \Omega.\]  

We recall, by the above, that \(v\) also satisfies

\[-\text{div}(|\nabla \nu_\Omega|^{p-2} \nabla \nu_\Omega) \geq b(x)g^{p-1}(\nu_\Omega), x \in \Omega.\]  

Let \(\Omega\) be a smooth bounded domain that contains \(B(0,1)\) the unit ball centered at the origin. Now, let \(\phi_0\) be the first eigenfunction of the problem (4) with \(c(x)=a(x)\) such that \(0 < \phi_0 \leq \gamma\), where \(\gamma\) is the positive constant in \([F_1]\). Invoking conditions \([F_2]\) and \([F_3]\), we get

\[-\text{div}(|\nabla \phi_0|^{p-2} \nabla \phi_0) = \lambda_0(a(x))\phi_0^{p-1} \leq \lambda(a(x))\phi_0^{p-1} \leq f(x,\phi_0), \quad x \in \Omega\]

Moreover, since \(0 < \phi_0 \leq 1\), we also note that,

\[-\text{div}(|\nabla \phi_0|^{p-2} \nabla \phi_0) \leq b(x)g^{p-1}(\phi_0) \leq b(x)g^{p-1}(\phi_0), \quad x \in \Omega\]

Therefore, we get

\[-\text{div}(|\nabla \phi_0|^{p-2} \nabla \phi_0) \leq b(x)g^{p-1}(\phi_0), \quad x \in \Omega\]

Recalling that \(g\) is non-increasing, by Lemma 1 we note, from (15) and (17), that \(\phi_0 \leq \nu_\Omega\). Then by the elliptic regularity theory and Lemma 2, (14) and (16), we conclude that (12) has a solution \(u\) such that

\[\phi_1 \leq u \leq \nu_\Omega, \quad \text{and } u \in C^{1,\alpha}(\Omega) \cap C(\overline{\Omega})\]

Let \(\Omega_1\) be as in the proof of the above Theorem. Then we note that \(\Omega_1 \subseteq \Omega_2\), and hence \(\nu_\Omega \leq v\) where \(v\) is as in Lemma 3. Then we deduce a non-singular case.

**Theorem 2.** If \(f\) satisfies \([F_1]\), \(f(x,0) = 0\), where \(b\) satisfies \([B_1]\), \([B_2]\) and \(g\) satisfies \([G_1]\), \([G_2]\), then problem (1) has a solution \(u \in C^{1,\alpha}(\Omega) \cap C(\overline{\Omega})\).

**Proof.** For each positive integer \(k\), let \(B_k = B(0,k)\) be the ball of radius \(k\) centered at the origin. By Theorem 1, for each positive integer \(k\) we let \(u_k \in C^{1,\alpha}(B_k)\) be a solution of

\[
\begin{cases}
-\text{div}(|\nabla u|^{p-2} \nabla u) = f(x,u), \quad x \in B_k \\
u(x) = 0, \quad x \in \partial B_k
\end{cases}
\]

Then \(0 \leq u_k(x) \leq v_k(x)\) in \(B_k\), where \(v_k\) are as in Theorem 1. Corresponding to the ball \(B_k\). It is easy to see that \(0\) is a subsolution.

Recall the above, \(v \geq v_k\) on \(B_k\) for all \(k \geq 1\), and hence \(0 \leq u_k(x) \leq v(x)\) for all \(x \in B_1\).

By a standard procedure, we conclude that \(\{u_k\}_{k}^{\infty}\) has a subsequence that converges uniformly to a function in \(C^{1,\alpha}(\Omega)\). By a diagonalization process it follows that \(\{u_k\}_{k}^{\infty}\) has a subsequence that converges uniformly on open bounded subsets of \(\mathbb{R}^N\) to \(u \in C^{1,\alpha}(\mathbb{R}^N)\) and that \(u\) is a solution of (1). Since \(0 \leq u \leq v\), it follows
that \( u(x) \to 0 \) as \( |x| \to \infty \).

In the last part of the paper, we prove a nonexistence result for the following problem,
\[
\begin{aligned}
-\text{div}(|\nabla u|^{p-2} \nabla u) \\
= d(x)[g(u) + r(u) + |\nabla u|^{q}], \quad u > 0, \quad \text{in } \mathbb{R}^N
\end{aligned}
\] (19)

\( u(x) \to 0, \) as \( |x| \to \infty \)

The result solves an open problem in [4] for \( p > 1, q \geq 0 \) for the case when \( u \) is a radially symmetric solution. Before the proof, we state some conditions which we needed at the below.

\[
G_1 \in C^1((0, \infty), (0, \infty)); \\
G_2 \lim_{u \to 0} \frac{g(u)}{u} = +\infty; \\
\tilde{G}_1 \lim_{u \to \infty} \frac{g(u)}{u} = 0.
\]

**Theorem 3.** Suppose \( \tilde{G}_1 = G_1 \) are fulfilled and \( d \) is a positive radial function, \( r \) is a nonnegative radial function that is continuous on \( \mathbb{R}^N \) and satisfies
\[
\int_0^\infty (\xi^{p-1})^{\frac{p}{2}} d(\sigma)\sigma^{\frac{1}{2}} d\xi = \infty,
\]
then the problem (19) has no positive radial solution that decays to zero near infinity.

**Proof.** Suppose (19) has such a solution \( u(r) \). Then
\[
-\text{div}(|\nabla u(r)|^{p-2} \nabla u(r)) \\
= d(r)[g(u(r)) + r(u(r)) + |\nabla u(r)|^{q}],
\]
or, equivalently, \( u(r) \) is a solution to the problem
\[
-(r^{N-1} u' - u'') \geq r^{N-1} d(r)[g(u(r)) + r(u(r))]
\] (20)

Integrating (20) \( r \) to 0, we have
\[
-(r^{N-1} u' - u'' \geq r^{N-1} d(r)[g(u(r)) + r(u(r))] d\sigma.
\]

Hence \( u'(r) < 0 \); i.e., \( u(r) \) is non-increasing. We put \( \ln (u(r) + 1) = \tilde{u}(r) > 0 \) for all \( r > 0 \). Then we have
\[
div(|\nabla \tilde{u}(r)|^{p-2} \nabla \tilde{u}(r)) \\
= \frac{1}{u+1} \text{div}(|\nabla u(r)|^{p-2} \nabla u(r)) - (p-1) \frac{1}{(u+1)^p} |\nabla u|^{p},
\]

and \( \tilde{u}(r) \) satisfies
\[
(\tilde{u}' - \tilde{u}'') + \frac{N-1}{r} \tilde{u}' \leq -(p-1) \frac{1}{(u+1)^p} |\nabla u|^{p},
\]

Multiplying Equation (21) by \( r^{N-1} \) and integrating on \( (0, \xi) \) yields
\[
r^{N-1} \xi^{p-2} \tilde{u}' + \int_0^\xi (p-1) \frac{1}{(u+1)^p} |\nabla u|^{p} d\sigma \\
\leq -\int_0^\xi \sigma^{N-1} d(\sigma) \frac{g(u(\sigma)) + r(u(\sigma))}{(u+1)^p} d\sigma
\]
\[\leq 0 \quad (22)\]

If \( p \) is even, we can deduce that
\[
\tilde{u}(r) - \tilde{u}(0) + \int_0^\xi (p-1) \frac{1}{(u+1)^p} |\nabla u|^{p} d\sigma \leq 0.
\]

We observe that \( u(r) < u(0) \) for all \( r > 0 \) and \( \tilde{u}(r) > \tilde{u}(0) \) for all \( r > 0 \)

Since \( \tilde{u}(r) \) is positive, then (23) implies
\[
\int_0^\xi \int_0^\sigma \frac{g(u(\sigma)) + r(u(\sigma))}{(u+1)^p} d\sigma d\xi \leq \tilde{u}(0)
\]
\[\leq 0 \quad (24)\]

for all \( r > 0 \). Now, using Lemma 4 in (24), we have
\[
\int_0^\xi \int_0^\sigma \frac{g(u(\sigma)) + r(u(\sigma))}{(u+1)^p} d\sigma d\xi \leq \tilde{u}(0).
\]

If \( p \) is odd, we deduce that
\[
\tilde{u}(r) - \tilde{u}(0) + \int_0^\xi (p-1) \frac{1}{(u+1)^p} |\nabla u|^{p} d\sigma \leq 0.
\]

\[
\tilde{u}(r) < \tilde{u}(0) \quad (25)
\]
then (25) implies
\[
\int_0^\xi \int_0^\sigma \frac{g(u(\sigma)) + r(u(\sigma))}{(u+1)^p} d\sigma d\xi \leq \tilde{u}(0)
\]
\[\leq 0 \quad (26)\]

for all \( r > 0 \). As the same as the above we get
\[
\int_0^\xi \int_0^\sigma \frac{g(u(\sigma)) + r(u(\sigma))}{(u+1)^p} d\sigma d\xi < \tilde{u}(0).
\]

But, since \( g(u) \) is non-increasing on \( (0, \infty) \), we have
\[ \infty = \int_0^1 \left( \int_0^{\infty} \sum_{i=0}^{n-1} d(\sigma) d\sigma \right) \frac{1}{g^{\frac{n}{n-1}}(u(0))} < \infty, \]

which is a contradiction.

4. References