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Quasi-Rational Canonical Forms of a Matrix over a Number Field

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Abstract

A matrix is similar to Jordan canonical form over the complex field and the rational canonical form over a number field, respectively. In this paper, we further study the rational canonical form of a matrix over any number field. We firstly discuss the elementary divisors of a matrix over a number field. Then, we give the quasi-rational canonical forms of a matrix by combining Jordan and the rational canonical forms. Finally, we show that a matrix is similar to its quasi-rational canonical forms over a number field.

Keywords

Matrix, Jordan Canonical Form, Rational Canonical Form, Quasi-Rational Canonical Form

1. Introduction

A matrix is similar to Jordan canonical form over the complex field and the rational canonical form over a number field, respectively. Thus, Jordan and the rational canonical forms of a matrix over the complex field are similar.

Recently, Radjabalipour [1] studied the symmetrization of the Jordan canonical form, Abo *et al.* [2] and Barone *et al.* [3] discussed the relations between the eigenstructures and Jordan canonical form. Moreover, Li [4] discussed the property of the rational canonical form of a matrix, Liu [5] gave out a constructive proof of existence theorem for rational form, and Radjabalipour [6] investigated the rational canonical form via the splitting field.

In this paper, we further study the rational canonical form over any number field. We firstly discuss the concept of elementary divisors of a matrix over any number field. Then, we give the quasi-rational canonical forms of a matrix by combining Jordan and the rational canonical forms. Finally, we show that a matrix is similar to its quasi-rational canonical forms over any number field.

2. Jordan and Rational Canonical Forms

Given a matrix A, it is an interesting work to find a simple matrix that is similar to A. We know that such a simple matrix is Jordan canonical form or the rational canonical form of A.

Lemma 2.1. ([7], pp 244-247) An $n\times n$ matrix A is similar over the complex field to Jordan canonical form of its, such Jordan canonical form is unique up to a rearrangement of the order of its characteristic values, *i.e.*, A is similar to the quasi-diagonal matrix of order n

$$J=\begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_s \end{bmatrix}$$

where

$$J_i=\begin{bmatrix} c_i & 0 & \cdots & 0 & 0 \\ 1 & c_i & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c_i \end{bmatrix}_{m_i\times m_i}$$

is called an elementary Jordan matrix with characteristic value c_i, $i=1,2,\cdots,s$ and $m_1+m_2+\cdots+m_s=n$.

Definition 2.1. For a non-scalar monic polynomial $d(\lambda)=\lambda^n+a_1\lambda^{n-1}+\cdots+a_n$ over a number field P, the $n\times n$ matrix

$$B=\begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix}$$

is called the companion matrix or Frobenius matrix of the monic polynomial $d(\lambda)$.

A polynomial matrix, or λ-matrix, is a rectangular matrix $A(\lambda)$ whose elements are polynomials in λ

$$A(\lambda)=\left(a_{ij}(\lambda)\right)=\left(a_{ij}^k\lambda^k+a_{ij}^{k-1}\lambda^{k-1}+\cdots+a_{ij}^0\right).$$

Here k is the largest of the degrees of the polynomial $a_{ij}(\lambda)$.

Two polynomial matrices $A(\lambda)$ and $B(\lambda)$ are called equivalent if one of them can be obtained from the other by means of some elementary operations.

An arbitrary rectangular polynomial matrix is equivalent to a canonical matrix

$$\begin{bmatrix} a_1(\lambda) & 0 & \cdots & 0 & \cdots & 0 \\ 0 & a_2(\lambda) & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & a_s(\lambda) & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

where the polynomials $a_1(\lambda), a_2(\lambda), \cdots, a_s(\lambda)$ are not identically equal to zero and each is divisible by the preceding one.

Let $A(\lambda)$ be a polynomial matrix of rank *r*, *i.e.*, the matrix has minors of order *r* not identically equal to zero, but all the minors of order greater than *r* are identically equal to zero in λ. We denote by $D_j(\lambda)$ the greatest common divisor of all the minors of order *j* in $A(\lambda)(j = 1, 2, \cdots, r)$. It is easy to see that in the series

$$D_1(\lambda), D_2(\lambda), \cdots, D_r(\lambda)$$

each polynomial is divisible by the preceding one (see [8], pp 139-140).

An easy verification shows immediately that the elementary operations change neither the rank of $A(\lambda)$ nor the polynomials $D_1(\lambda), D_2(\lambda), \cdots, D_r(\lambda)$. Thus, $r = s$, the corresponding quotients will be denoted by

$$d_1(\lambda) = D_1(\lambda), d_2(\lambda) = \frac{D_2(\lambda)}{D_1(\lambda)}, \cdots, d_r(\lambda) = \frac{D_r(\lambda)}{D_{r-1}(\lambda)}$$

is invariant under elementary operations and

$$d_1(\lambda) = a_1(\lambda), d_2(\lambda) = a_2(\lambda), \cdots, d_r(\lambda) = a_r(\lambda).$$

The polynomials $d_1(\lambda), d_2(\lambda), \ldots, d_r(\lambda)$ are called the invariant polynomials of the λ-matrix $A(\lambda)$.

Definition 2.2. ([8], pp 144-145) Let $A = (a_{ij})$ be an $n \times n$ matrix. We form its characteristic matrix

$$\lambda E - A = \begin{bmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{bmatrix}.$$

The characteristic matrix is a λ-matrix of rank *n*. Its invariant polynomials

$$d_1(\lambda) = D_1(\lambda), d_2(\lambda) = \frac{D_2(\lambda)}{D_1(\lambda)}, \cdots, d_n(\lambda) = \frac{D_n(\lambda)}{D_{n-1}(\lambda)}$$

are called the invariant polynomials of the matrix *A*.

It is easy to see that the invariant polynomials of the companion matrix *B* of the monic polynomial $d(\lambda)$ are $1, \cdots, 1, d(\lambda)$.

Definition 2.3. The following quasi-diagonal matrix

$$C = \begin{bmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_s \end{bmatrix}$$

is called the direct sum of the companion matrices B_i of non-scalar monic polynomials $d_1(\lambda), d_2(\lambda), \cdots, d_s(\lambda)$ such that $d_{i+1}(\lambda)$ divides $d_i(\lambda)$ for $i = 1, \cdots, s-1$ and said to be in rational canonical form.

The invariant polynomials of the rational canonical form matrix *B* in Definition 2.3 are

$$1,\cdots,1,d_1(\lambda),d_2(\lambda),\cdots,d_s(\lambda).$$

Lemma 2.2. ([7], pp 238-239, Theorem 5) An $n\times n$ matrix A is similar over a number field P to one and only one matrix which is in rational canonical form, *i.e.*, A is similar to the quasi-diagonal matrix

$$C=\begin{bmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_s \end{bmatrix}$$

where $B_1,B_2,\cdots,B_s$ are the companion matrices of the non-scalar invariant polynomials $d_1(\lambda),d_2(\lambda),\cdots,d_s(\lambda)$ of matrix A.

3. The Elementary Divisors of a Matrix over a Number Field

Let P be a number field. Then a non-scalar monic polynomial in $P[x]$ can be factored as a product of monic irreducible polynomials in $P[x]$ in one and, except for order, only one way. In the factorization of a given non-scalar monic polynomial $f(x)$, some of the monic irreducible factors may be repeated. If

$$p_1(x),p_2(x),\cdots,p_s(x)$$

are the distinct monic irreducible polynomials occurring in this factorization of $f(x)$, then

$$f(x)=p_1^{r_1}(x)\cdot p_2^{r_2}(x)\cdots p_s^{r_s}(x)$$

the exponent r_i being the number of times the irreducible polynomial $p_i(x)$ occurs in the factorization. This decomposition is also clearly unique, and is called the primary decomposition of $f(x)$.

Theorem 3.1. If B is the companion matrix of the monic polynomial

$$p(\lambda)=\lambda^n+a_1\lambda^{n-1}+\cdots+a_n$$

$$C=\begin{bmatrix} B & & & \\ E_{1n} & B & & \\ & \ddots & \ddots & \\ & & E_{1n} & B \end{bmatrix}=(c_{ij})_{rn\times rn}$$

where

$$E_{1n}=\begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{n\times n}$$

i.e.,

$$c_{n+1,n}=c_{2n+1,2n}=\cdots=c_{(r-1)n+1,(r-1)n}=1$$

then the invariant polynomials of C are

$$1,1,\cdots,p^r(\lambda).$$

Proof. Let

$$C(\lambda)=\lambda E-C=\begin{bmatrix}\lambda E-B & & & \\ -E_{1n} & \lambda E-B & & \\ & \ddots & \ddots & \\ & & -E_{1n} & \lambda E-B\end{bmatrix}=\left(c_{ij}(\lambda)\right)_{rn\times rn}.$$

Then $c_{21}(\lambda)=c_{32}(\lambda)=\cdots=c_{rn,rn-1}(\lambda)=-1$ and a minor of order $rn-1$

$$\begin{vmatrix}c_{21}(\lambda) & c_{22}(\lambda) & \cdots & c_{2,rn-1}(\lambda)\\ c_{31}(\lambda) & c_{32}(\lambda) & \cdots & c_{3,rn-1}(\lambda)\\ \vdots & \vdots & & \vdots\\ c_{rn,1}(\lambda) & c_{rn,2}(\lambda) & \cdots & c_{rn,rn-1}(\lambda)\end{vmatrix}=\pm 1.$$

Thus,

$$D_1(\lambda)=\cdots=D_{rn-1}(\lambda)=1,\ D_{rn}(\lambda)=|\lambda E-C|.$$

By Laplace Theorem, we have that

$$D_{rn}(\lambda)=|\lambda E-C|=|\lambda E-B|^r=p^r(\lambda).$$

Therefore, the invariant polynomials of C are

$$1,1,\cdots,p^r(\lambda).$$

The theorem is proved.

The matrix C is called the rational block of $p^r(\lambda)$ and the characteristic polynomial of C is precisely the last invariant polynomial $p^r(\lambda)$ of C.

We decompose the invariant polynomials

$$d_1(\lambda),d_2(\lambda),\cdots,d_r(\lambda)$$

of the λ-matrix $A(\lambda)$ into irreducible factors over the number field P

$$\begin{aligned}d_1(\lambda)&=p_1^{k_{11}}(\lambda)\,p_2^{k_{12}}(\lambda)\cdots p_s^{k_{1s}}(\lambda),\\ d_2(\lambda)&=p_1^{k_{21}}(\lambda)\,p_2^{k_{22}}(\lambda)\cdots p_s^{k_{2s}}(\lambda),\\ &\vdots\\ d_r(\lambda)&=p_1^{k_{r1}}(\lambda)\,p_2^{k_{r2}}(\lambda)\cdots p_s^{k_{rs}}(\lambda).\end{aligned}$$

Here, $p_1(\lambda),p_2(\lambda),\cdots,p_s(\lambda)$ are all the distinct irreducible polynomials over P (and with highest coefficient 1) that occur in $d_1(\lambda),d_2(\lambda),\cdots,d_r(\lambda)$ and

$$k_{1j}\le k_{2j}\le\cdots\le k_{rj},\quad j=1,\cdots,s.$$

All the power among

$$p_1^{k_{11}}(\lambda),\ p_2^{k_{12}}(\lambda),\cdots,\ p_s^{k_{rs}}(\lambda)$$

as far as they are distinct from 1, are called the elementary divisors of the λ-matrix $A(\lambda)$ in the number field P.

Theorem 3.2. Assume that

$$d(\lambda)=p_1^{k_1}(\lambda)\,p_2^{k_2}(\lambda)\cdots p_S^{k_S}(\lambda)$$

is a polynomial of degree n, where

$$p_1(\lambda),\ p_2(\lambda),\cdots,\ p_s(\lambda)$$

are the distinct monic irreducible polynomials and $k_1, k_2, \cdots, k_s$ are all positive integers. If $C_1, C_2, \cdots, C_s$ are, respectively, the rational blocks of

$$p_1^{k_1}(\lambda),\ p_2^{k_2}(\lambda),\cdots,\ p_s^{k_s}(\lambda)$$

then the invariant polynomials of the quasi-diagonal matrix

$$F=\begin{bmatrix} C_1 & & & \\ & C_2 & & \\ & & \ddots & \\ & & & C_s \end{bmatrix}$$

are $1,\cdots,1, d(\lambda)$.

Proof. It is easy to see that

$$\lambda E-F=\begin{bmatrix} \lambda E-C_1 & & & \\ & \lambda E-C_2 & & \\ & & \ddots & \\ & & & \lambda E-C_s \end{bmatrix}$$

and the invariant polynomials of $\lambda E-C_i\ (i=1,2,\cdots,s)$ is $1,\cdots,1, p_i^{k_i}(\lambda)$. Thus, by elementary operations of λ-matrices, $\lambda E-C_i$ can be transformed into the canonical form (see [8], pp 140-141)

$$C_i(\lambda)=\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & p_i^{k_i}(\lambda) \end{bmatrix}$$

and $\lambda E-F$ be further transformed into

$$F(\lambda)=\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & p_1^{k_1}(\lambda) & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & p_s^{k_s}(\lambda) \end{bmatrix}.$$

For λ-matrix $F(\lambda)$, we have that

$$\begin{aligned} &D_{n-1}(\lambda) \\ &=\left(p_2^{k_2}(\lambda)\cdots p_s^{k_s}(\lambda),\ p_1^{k_1}(\lambda)p_3^{k_3}(\lambda)\cdots p_s^{k_s}(\lambda),\cdots,\ p_1^{k_1}(\lambda)\cdots p_{s-1}^{k_{s-1}}(\lambda)\right)=1 \end{aligned}$$

$$D_n(\lambda)=p_1^{k_1}(\lambda)p_2^{k_2}(\lambda)\cdots p_s^{k_s}(\lambda)=d(\lambda).$$

Thus,

$$D_1(\lambda)=\cdots=D_{n-1}(\lambda)=1, D_n(\lambda)=d(\lambda)$$

and the invariant polynomials of F is

$$1, \frac{D_2(\lambda)}{D_1(\lambda)}=1,\cdots,\frac{D_{n-1}(\lambda)}{D_{n-2}(\lambda)}=1,\frac{D_n(\lambda)}{D_{n-1}(\lambda)}=d(\lambda).$$

The theorem is proved.

4. Quasi-Rational Form of a Matrix

By combining Jordan canonical form over the complex field and the rational canonical forms over a number field and using the rational blocks of $p^r(\lambda)$, we give the quasi-rational canonical forms of a matrix over a number field.

Theorem 4.1. If the invariant polynomials of a matrix A over a number field P are $1,\cdots,1,d_1(\lambda),\cdots,d_s(\lambda)$, $F_1,F_2,\cdots,F_s$ are the corresponding matrices of the non-scalar invariant polynomials $d_1(\lambda),\cdots,d_s(\lambda)$ in Theorem 3.2, then matrix A is similar over the number field P to the quasi-diagonal matrix

$$G=\begin{bmatrix} F_1 & & & \\ & F_2 & & \\ & & \ddots & \\ & & & F_s \end{bmatrix}.$$

Proof. It is easy to verify that

$$\lambda E-G=\begin{bmatrix} \lambda E-F_1 & & & \\ & \lambda E-F_2 & & \\ & & \ddots & \\ & & & \lambda E-F_s \end{bmatrix}$$

and the invariant polynomials of $\lambda E-F_i\ (i=1,2,\cdots,s)$ are $1,\cdots,1,d_i(\lambda)$. Thus, by elementary operations of λ-matrices, $\lambda E-F_i$ can be transformed into the canonical form

$$F_i(\lambda)=\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & d_i(\lambda) \end{bmatrix}$$

and $\lambda E-G$ be further transformed into

$$G(\lambda)=\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & d_1(\lambda) & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & d_s(\lambda) \end{bmatrix}.$$

By interchanging rows or columns of $G(\lambda)$, $G(\lambda)$ can be transformed into

$$\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & d_1(\lambda) & & \\ & & & & & \ddots & \\ & & & & & & d_s(\lambda) \end{bmatrix}.$$

Thus, $\lambda E - G$ and $\lambda E - A$ have same canonical forms, *i.e.*, $\lambda E - G$ and $\lambda E - A$ are equivalent. Therefore, A and G are similar.

The theorem is proved.

The quasi-diagonal matrix G in Theorem 4.1 is called the quasi-rational canonical form of matrix A.

Noting that $F_i\,(i = 1, 2, \cdots, s)$ in Theorem 4.1 is the direct sum of the rational blocks of some elementary divisors of matrix A, we see that these little block matrices appearing in the quasi-rational canonical form of A are precisely the rational blocks of all elementary divisors of A. Thus, if we find all elementary divisors

$$p_1^{k_1}(\lambda),\ p_2^{k_2}(\lambda), \cdots,\ p_m^{k_m}(\lambda)$$

of A and the corresponding rational blocks $F_1, F_2, \cdots, F_m$, then

$$\begin{bmatrix} \lambda E - F_1 & & & \\ & \lambda E - F_2 & & \\ & & \ddots & \\ & & & \lambda E - F_m \end{bmatrix}$$

is equivalent to

$$F(\lambda) = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & p_1^{k_1}(\lambda) & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & p_m^{k_m}(\lambda) \end{bmatrix}.$$

By interchanging rows or columns, we know that $F(\lambda)$ is equivalent to

$$G(\lambda) = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & d_1(\lambda) & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & d_s(\lambda) \end{bmatrix}$$

$G(\lambda)$ is equivalent to

$$\begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & d_1(\lambda) & & \\ & & & & & \ddots & \\ & & & & & & d_s(\lambda) \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} \lambda E - F_1 & & & \\ & \lambda E - F_2 & & \\ & & \ddots & \\ & & & \lambda E - F_m \end{bmatrix}$$

is equivalent to

$$\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & d_1(\lambda) & & \\ & & & & & \ddots & \\ & & & & & & d_s(\lambda) \end{bmatrix}.$$

Therefore, A and the quasi-diagonal matrix

$$\begin{bmatrix} F_1 & & & \\ & F_2 & & \\ & & \ddots & \\ & & & F_s \end{bmatrix}$$

are similar.

Similar to Jordan canonical forms of a matrix over the complex field, if we find all elementary divisors of a matrix over a number field and rational blocks of these elementary divisors, then the direct sum of these rational blocks is precisely the quasi-rational canonical form of the matrix.

Of course, the quasi-rational canonical form of a matrix is not unique. But, the quasi-rational canonical form is unique up to a rearrangement of the order of rational blocks.

5. Conclusion

In this paper, we further study the rational canonical form over any number field and give the quasi-rational canonical forms of a matrix by combining Jordan and the rational canonical forms. Unlike the companion matrices in the rational canonical form of a matrix A in [4] [5] [7], these little block matrices in the quasi-rational canonical form of a matrix A are the rational blocks of elementary divisors of A and not the companion matrices of the non-scalar invariant polynomials of A.

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