Abstract

In this paper, we introduce the definition of $L$-fuzzy vector subspace, define its dimension by an $L$-fuzzy natural number. For a finite-dimensional $L$-fuzzy vector subspace, we prove that the equality \[ \dim(\tilde{E}_1 + \tilde{E}_2) + \dim(\tilde{E}_1 \cap \tilde{E}_2) = \dim \tilde{E}_1 + \dim \tilde{E}_2 \] holds without any restricted conditions. At the same time, we deduce that the formula \[ \dim(\text{int} f) + \dim(\text{ker} f) = \dim \tilde{E} \] holds.

Keywords

$L$-Fuzzy Sets, $L$-Fuzzy Vector Subspace, $L$-Fuzzy Dimension

1. Introduction

Firstly, fuzzy vector subspace was introduced by Katsaras and Liu [1]. Then its properties and characters were investigated (see [2] [3] [4] [5], etc). The dimension of a fuzzy vector space was defined as a $n$-tuple by Lowen [6]. Subsequently, it was defined as a non-negative real number or infinity by Lubczonok [5], and proved that the formula

\[ \dim(\tilde{E}_1 + \tilde{E}_2) + \dim(\tilde{E}_1 \cap \tilde{E}_2) = \dim \tilde{E}_1 + \dim \tilde{E}_2 \]  \hspace{1cm} (1)

is valid under certain conditions, where $\tilde{E}_1$ and $\tilde{E}_2$ are fuzzy vector spaces. Recently, basis and dimension of a fuzzy vector space were redefined as a fuzzy set and a fuzzy natural number by Shi and Huang [7], respectively. Under the definitions, more properties of (crisp) vector spaces were correct in fuzzy vector spaces.

In this paper, we generalize the results in [7] to $L$ lattice, and prove that some formulas still hold in the lattice $L$. In particular, we present the definition of $L$-fuzzy vector subspace and its $L$-fuzzy dimension. The $L$-fuzzy dimension of a finite dimensional fuzzy vector subspace is a fuzzy natural number. We prove that (1) holds without any re-
stricted conditions and \( \dim(\text{ker} f) + \dim(\text{im} f) = \dim E \) holds.

2. Preliminaries

Given a set \( X \) and a completely distributive lattice \( L \), we denote the power set of \( X \) and the set of all \( L \)-fuzzy sets on \( X \) (or \( L \)-sets for short) by \( 2^X \) and \( \mathcal{L}(X) \), respectively. For any \( A \subseteq X \), we denote the cardinality of \( A \) by \( |A| \).

An element \( a \) in \( L \) is called a prime element if \( a \geq b \land c \) implies \( a \geq b \) or \( a \geq c \). \( a \) in \( L \) is called co-prime if \( a \leq b \lor c \) implies \( a \leq b \) or \( a \leq c \). The set of non-unit prime elements in \( L \) is denoted by \( \mathcal{P}(L) \). The set of non-zero co-prime elements in \( L \) is denoted by \( \mathcal{J}(L) \).

The binary relation \( < \) in \( L \) is defined as follows: for \( a, b \in L \), \( a < b \) if and only if for every subset \( D \subseteq L \) with \( b \leq \sup D \) always implies the existence of \( d \in D \) with \( a \leq d \). \( \{ a \in L : a < b \} \) is called the greatest minimal family of \( b \) in the sense of [10], denoted by \( \beta(b) \), and \( \beta'(b) = \beta(b) \cap J(L) \). Moreover, for \( b \in L \), we define \( \alpha(b) = \{ a \in L : a \leq \sup \{ b \} \} \) and \( \alpha'(b) = \alpha(b) \cap P(L) \). In a completely distributive lattice \( L \), there exist \( \alpha(b) \) and \( \beta(b) \) for each \( b \in L \), and \( b = \lor \beta(b) = \land \alpha(b) \) (see [10]).

In [10], Wang thought that \( \beta(0) = \{ 0 \} \) and \( \alpha(1) = \{ 1 \} \). In fact, it should be that \( \beta(0) = \emptyset \) and \( \alpha(1) = \emptyset \).

Throughout this paper, \( L \) denotes a completely distributive lattice, and \( E \) is a crisp vector space. We often do not distinguish a crisp subset \( A \) of \( E \) and its characteristic function \( \chi_A \).

If \( A \in L^X \) and \( a \in L \), we can define

\[
\begin{align*}
A_{\geq a} &= \{ x \in X : A(x) \geq a \}, \quad A_{a} = \{ x \in X : a \in \beta(A(x)) \}, \\
A_{\leq a} &= \{ x \in X : a \notin \alpha(A(x)) \}, \quad A_{a} = \{ x \in X : A(x) \notin a \}.
\end{align*}
\]

Some properties of these cut sets can be found in [11]-[16].

In [17] Shi introduced the concept of \( L \)-fuzzy natural numbers (denoted by \( \mathbb{N}(L) \)), defined their operations and discussed the relation of \( \alpha \)-cut sets. We simply recall as follows: for any \( \lambda, \mu \in \mathbb{N}(L) \), \( a \in L \),

1. \( (\lambda + \mu)[a] = \lambda[a] + \mu[a] \subseteq \lambda + \mu[a] ; \)
2. \( (\lambda + \mu)[a] = \lambda[a] + \mu[a] \subseteq \lambda + \mu[a] ; \)
3. For any \( \lambda, \mu \in \mathbb{N}(L) \) and \( a \in P(L) \), it follows that \( (\lambda + \mu)[a] = \lambda[a] + \mu[a] \).

3. \( L \)-Fuzzy Vector Subspaces

**Definition 3.1.** \( L \)-fuzzy vector subspace is a pair \( \tilde{E} = (E, \mu) \) where \( E \) is a vector space on field \( F \), \( \mu : E \to L \) is a map with the property that for any \( x, y \in E, k, l \in F \), we have \( \mu(kx + ly) \geq \mu(x) \land \mu(y) \).

In this definition, when \( L = [0,1] \), \( L \)-fuzzy vector subspace is exactly the fuzzy vector subspace defined in [1]. We denote the family of \( L \)-fuzzy vector subspaces by \( \mathcal{LFVS} \).

Let \( \tilde{E} = (E, \mu) \) be a member of \( \mathcal{LFVS} \), we denote
We can obtain some properties of LFVS analogous to fuzzy vector subspaces as follows.

**Theorem 3.2.** Let \( \tilde{E} = (E, \mu) \) be a member of LFVS, then

1. \( \mu(0) = \sup_{x \in E} \mu(x) \).

2. For any \( k \in F \setminus \{0\} \) and \( x \in E, \mu(kx) = \mu(x) \).

The prove is trivial and omitted.

**Remark:** Since \( L \) is a completely distributive lattice, the property that if \( \mu(x) \neq \mu(y) \), then \( \mu(x + y) = \mu(x) \land \mu(y) \) not holds for LFVS. This can be seen from the following example.

**Example 3.3.** Let \( L \) be a completely distributive lattice with four elements as follows.

Let \( \tilde{E} = (R^2, \mu) \) be an \( L \)-fuzzy vector subspace on \( R^2 \) where \( \mu \) is defined by

\[
\mu(x) = \begin{cases} 
1, & x = (0, 0) \\
a, & x \in \{(y, 0) : y \in R \setminus \{0\}\} \\
b, & x \in \{(0, y) : y \in R \setminus \{0\}\} \\
0, & \text{otherwise.}
\end{cases}
\]

We can easily check \( \tilde{E} \) is an \( L \)-fuzzy vector subspace on \( R^2 \). Suppose that \( x = (3, 2) \) and \( y = (0, -2) \), then \( \mu(x + y) = \mu(3, 0) = a > \mu(x) \land \mu(y) = 0 \land b = 0 \). This example illustrates for \( L \)-fuzzy vector subspace \( \mu(x) \neq \mu(y) \), \( \mu(x + y) > \mu(x) \land \mu(y) \).

**Theorem 3.4.** Let \( E \) be a vector space, \( \mu \in L^E \) and \( \tilde{E} = (E, \mu) \). Then the following statements are equivalent:

1. \( \tilde{E} \) is an \( L \)-fuzzy vector subspace.

2. (a) For all \( x \in E \) and \( k \in F, \mu(kx) \geq \mu(x) \).

   (b) For any \( x, y \in E, \mu(x + y) \geq \mu(x) \land \mu(y) \).

3. For any \( x_1, \cdots, x_r \in E \) and \( k_1, \cdots, k_r \in F \), where \( r \) is a finite natural number, we have
\[ \mu \left( \sum_{i=1}^{r} k_i x_i \right) \geq \bigwedge_{i=1}^{r} \mu(x_i). \]

The prove is trivial and omitted.

In the following paper, the vector spaces we discuss are finite-dimensional. For their \(L\)-fuzzy vector subspaces, the following observation will be useful.

**Remark:** Let \( \hat{E} = (E, \mu) \) be a member of LFVS. Suppose that \( \mu(E) = \{ \mu(x) : x \in E \} \). Since \( E \) is finite-dimensional vector space, denotes \( \dim E = n \), then \( \mu(E) \) is a finite subset of \( L \).

In the fact, let \( B \) be a basis of \( E \), then \( |B| = n \). Suppose that \( \mu(E) \) is infinite, then for all \( a \in L \), the total number of \( \hat{E}_{[a]} \) is infinite. Since \( B \cap \hat{E}_{[a]} \) is a basis of \( \hat{E}_{[a]} \), we have \( \hat{E}_{[a]} = \{ B \cap \hat{E}_{[a]} \} \). Again since \( B \) is finite, the total number of \( \hat{E}_{[a]} \) is also finite. It contradicts with the hypothesis. Therefore \( \mu(E) \) is a finite subset of \( L \) with at most \( 2^n + 1 \) values; \( 2^n \) values which can be attained at the vectors of \( E \setminus \{0\} \) and the maximum which is attained at \( 0 \).

**Theorem 3.5.** Let \( E \) be a vector space, \( \mu \in L^E \) and \( \hat{E} = (E, \mu) \). Then the following statements equivalent:

1. \( \hat{E} \) is an \( L \)-fuzzy vector subspace.
2. For all \( a \in L \), \( \hat{E}_{[a]} \) is a vector space.
3. For all \( a \in J(L) \), \( \hat{E}_{[a]} \) is a vector space.
4. For all \( a \in L \), \( \hat{E}^{[a]} \) is a vector space.
5. For all \( a \in P(L) \), \( \hat{E}^{[a]} \) is a vector space.
6. For all \( a \in P(L) \), \( \hat{E}^{(a)} \) is a vector space.

Proof. We prove \((1) \Leftrightarrow (4)\) and \((1) \Leftrightarrow (6)\), the others can be proved analogously.

\((1) \Leftrightarrow (4)\) We show that \( \hat{E}^{[a]} \) is a vector space as follows. Suppose that \( x, y \in \hat{E}^{[a]} \), then \( a \notin \alpha(\mu(x)) \) and \( a \notin \alpha(\mu(y)) \), i.e. \( a \notin \alpha(\mu(x) \cup \alpha(\mu(y))) = \alpha(\mu(x) \land \mu(y)) \).

Since \( \hat{E} = (E, \mu) \) be an \( L \)-fuzzy vector subspace, then \( \alpha(\mu(x) \land \mu(y)) \supseteq \alpha(\mu(kx + ly)) \), we have \( a \notin \alpha(\mu(kx + ly)) \), this means \( kx + ly \in \hat{E}^{[a]} \). Therefore \( \hat{E}^{[a]} \) is a vector space.

\((4) \Leftrightarrow (1)\) Suppose that for all \( a \in L \), \( \hat{E}^{[a]} \) is a vector space. Let \( x, y \in E \) and \( k, l \in F \).

Since \( \hat{E}^{[a]} \) is a vector space, then \( kx + ly \in \hat{E}^{[a]} \) if and only if \( x \in \hat{E}^{[a]} \) and \( y \in \hat{E}^{[a]} \). We have

\[
\mu(kx + ly) = \bigwedge_{a \in L} \left( a \land \hat{E}^{[a]}(kx + ly) \right) \\
= \bigwedge_{a \in L} \left( a \lor \left( \hat{E}^{[a]}(x) \land \hat{E}^{[a]}(y) \right) \right) \\
= \left( \bigwedge_{a \in L} \left( a \lor \hat{E}^{[a]}(x) \right) \right) \land \left( \bigwedge_{a \in L} \left( a \lor \hat{E}^{[a]}(y) \right) \right) \\
= \mu(x) \land \mu(y). 
\]

Therefore \( \hat{E} \) is an \( L \)-fuzzy vector subspace.

\((1) \Leftrightarrow (6)\) Suppose that \( x, y \in \hat{E}^{(a)}, \) then \( \mu(x) \notin a \) and \( \mu(y) \notin a \). Since \( a \in P(L) \), then \( \mu(x) \land \mu(y) \notin a \). Because \( \hat{E} = (E, \mu) \) is an \( L \)-fuzzy vector subspace, we can have \( \mu(kx + ly) \notin a \), this implies \( kx + ly \in \hat{E}^{(a)} \). Thus \( \hat{E}^{(a)} \) is a vector space.

\((6) \Leftrightarrow (1)\) Let \( x, y \in E \) and \( k, l \in F \). Since \( \hat{E}^{(a)} \) is a vector space, then
\( kx + ly \in \bar{E}^{(a)} \) if and only if \( x \in \bar{E}^{(a)} \) and \( y \in \bar{E}^{(a)} \). We have the following implications.

\[
\mu(kx + ly) = \bigwedge_{a \in P(L)} \left( a \vee \bar{E}^{(a)}(x) \right) (kx + ly)
\]

\[
= \bigwedge_{a \in P(L)} \left( a \vee \left( \bar{E}^{(a)}(x) \wedge \bar{E}^{(a)}(y) \right) \right)
\]

\[
= \left( \bigwedge_{a \in P(L)} \left( a \vee \bar{E}^{(a)}(x) \right) \right) \wedge \left( \bigwedge_{a \in P(L)} \left( a \vee \bar{E}^{(a)}(y) \right) \right)
\]

\[
\mu(x) \wedge \mu(y).
\]

Therefore \( \bar{E} \) is an \( L \)-fuzzy vector subspace.

**Theorem 3.6.** Let \( E \) be a vector space, \( \mu : E \to L \) be a map, \( \bar{E} = (E, \mu) \), and for all \( a, b \in L \), \( \beta(a \wedge b) = \beta(a) \cap \beta(b) \). Then the following statements equivalent:

1. \( \bar{E} \) is an \( L \)-fuzzy vector subspace.
2. For all \( a \in L \), \( \bar{E}(a) \) is a vector space.

**Proof.**

(1) \( \Rightarrow \) (2) Suppose that \( x, y \in \bar{E}(a) \), then \( a \in \beta(\mu(x)) \) and \( a \in \beta(\mu(y)) \), i.e. \( a \in \beta(\mu(x)) \cap \beta(\mu(y)) \). Since for all \( a, b \in L, \beta(a \wedge b) = \beta(a) \cap \beta(b) \) and \( \bar{E} \) is an \( L \)-fuzzy vector subspace, we can know \( a \in \beta(\mu(x) \wedge \mu(y)) \subseteq \beta(\mu(ax + by)) \), this implies \( ax + by \in \bar{E}(a) \). Therefore \( \bar{E}(a) \) is a vector space.

(2) \( \Rightarrow \) (1) Suppose that for all \( a \in L \), \( \bar{E}(a) \) is a vector space. Let \( x, y \in E \) and \( k, l \in F \). Since \( \bar{E}(a) \) is a vector space, then \( kx + ly \in \bar{E}(a) \) if and only if \( x \in \bar{E}(a) \) and \( y \in \bar{E}(a) \). We have

\[
\mu(kx + ly) = \bigvee_{a \in L} \left( a \wedge \bar{E}(a) \right) (kx + ly)
\]

\[
= \bigvee_{a \in L} \left( a \wedge \left( \bar{E}(a) \left( x \wedge \bar{E}(a)(y) \right) \right) \right)
\]

\[
= \left( \bigvee_{a \in L} \left( a \wedge \bar{E}(a)(x) \right) \right) \wedge \left( \bigvee_{a \in L} \left( a \wedge \bar{E}(a)(y) \right) \right)
\]

\[
= \mu(x) \wedge \mu(y).
\]

Therefore \( \bar{E} \) is an \( L \)-fuzzy vector subspace.

We can define the operations between two \( L \)-fuzzy vector subspaces analogous to fuzzy vector subspaces.

**Definition 3.7.** Let \( \bar{E}_1 = (E, \mu_1) \) and \( \bar{E}_2 = (E, \mu_2) \) be two \( L \)-fuzzy vector subspaces on \( E \). Define the intersection of \( \bar{E}_1 \) and \( \bar{E}_2 \) to be \( \bar{E}_1 \cap \bar{E}_2 = (E, \mu_1 \land \mu_2) \). Define the sum of \( \bar{E}_1 \) and \( \bar{E}_2 \) to be \( \bar{E}_1 \oplus \bar{E}_2 = (E, \mu_1 + \mu_2) \) where \( \mu_1 + \mu_2 \) is defined by for all \( x \in E \)

\[
(\mu_1 + \mu_2)(x) = \bigvee_{x = x_1 \wedge x_2} \left( \mu_1(x_1) \wedge \mu_2(x_2) \right)
\]

\[
= \bigvee_{x_1 \in E} \left( \mu_1(x_1) \wedge \mu_2(x - x_1) \right).
\]

**Definition 3.8.** Let \( \bar{E}_1 = (E_1, \mu_1) \) and \( \bar{E}_2 = (E_2, \mu_2) \) be two members of \( \text{LFVS} \) and \( E = E_1 \oplus E_2 \). We define the direct sum of \( \bar{E}_1 \) and \( \bar{E}_2 \) to be \( \bar{E}_1 \oplus \bar{E}_2 = (E, \mu_1 \oplus \mu_2) \) where \( \mu_1 \oplus \mu_2 \) is defined by for all \( x \in E, x = x_1 \oplus x_2, x_i \in E_i, i = 1, 2 \)

\[
(\mu_1 \oplus \mu_2)(x) = (\mu_1 \oplus \mu_2)(x_1 \oplus x_2) = \mu_1(x_1) \wedge \mu_2(x_2).
\]
Theorem 3.9. Let $\tilde{E}_1 = (E, \mu_1)$ and $\tilde{E}_2 = (E, \mu_2)$ be two members of LFVS on $E$. We have

1. $\tilde{E}_1 \cap \tilde{E}_2$ is a member of LFVS on $E$.
2. $\tilde{E}_1 + \tilde{E}_2$ is a member of LFVS on $E$.

The proof of the theorem is trivial and it is omitted.

Theorem 3.10. Let $\tilde{E}_1 = (E, \mu_1)$ and $\tilde{E}_2 = (E, \mu_2)$ be members of LFVS. We have

1. For all $a \in L$, $(\tilde{E}_1 \cap \tilde{E}_2)(a) = (\tilde{E}_1)(a) \cap (\tilde{E}_2)(a)$.
2. For all $a \in L$, $(\tilde{E}_1 + \tilde{E}_2)(a) = (\tilde{E}_1)(a) + (\tilde{E}_2)(a)$.

Proof. The proofs of (1) and (2) are easy by the definition of $\tilde{E}_1 \cap \tilde{E}_2$ and the properties of L-fuzzy sets.

3. For any $a \in P(L)$, we have $(\tilde{E}_1 \cap \tilde{E}_2)(a) = (\tilde{E}_1)(a) \cap (\tilde{E}_2)(a)$.
4. By the definition of the sum of L-fuzzy vector subspaces, for any $a \in P(L)$ we have

$$x \in (\tilde{E}_1 + \tilde{E}_2)(a) \iff \bigvee_{x_1, x_2 \in E} \left( \mu_1(x_1) \land \mu_2(x_2) \leq a \right)$$

Suppose that for any $a, b \in L$, we have $\beta(a \land b) = \beta(a) \land \beta(b)$. Then

1. $(\tilde{E}_1 \cap \tilde{E}_2)(a) = (\tilde{E}_1)(a) \cap (\tilde{E}_2)(a)$,
2. $(\tilde{E}_1 + \tilde{E}_2)(a) = (\tilde{E}_1)(a) + (\tilde{E}_2)(a)$.

The proof is trivial and omitted.

4. Fuzzy Dimension of L-Fuzzy Vector Subspaces

Definition 4.1. Let $\mathbb{N}(L)$ be the family of L-fuzzy natural number. The map $\dim : \text{LFVS} \to \mathbb{N}(L)$ is defined by

$$\dim \tilde{E}(n) = \bigvee_{a \in L} \left( a \land \dim \tilde{E}(a) \right)(n)$$

is called the L-fuzzy dimensional function of the L-fuzzy vector subspace $\tilde{E}$, and $\dim \tilde{E}$ is called the L-fuzzy dimension of $\tilde{E}$, it is an L-fuzzy natural number. We
usually use another form of \( \dim \tilde{E} \) as follows.

\[
\dim \tilde{E}(n) = \lor \{ a \in L : \dim \tilde{E}_{[a]} \geq n \}.
\]

**Theorem 4.2.** For each \( \tilde{E} \in \text{LFVS} \) and \( n \in \mathbb{N} \), we have

\[
\dim \tilde{E}(n) = \lor_{a \in \tilde{L}} (a \land \dim \tilde{E}_{[a]})(n) = \lor \{ a \in L : \dim \tilde{E}_{[a]} \geq n \}.
\]

*Proof.* For any \( n \in \mathbb{N} \), let \( \lambda = \lor_{a \in \tilde{L}} (a \land \dim \tilde{E}_{[a]})(n) \). Obviously \( \lambda \leq \dim \tilde{E}(n) \). Next we show that \( \lambda \geq \dim \tilde{E}(n) \). Suppose that \( b \in L \) and \( b \in \beta\{ \dim \tilde{E}(n) \} \), then there exists \( a \in L \) and \( \dim \tilde{E}_{[a]} \geq n \) such that \( b \in \beta(a) \). In this case, \( n \leq \dim \tilde{E}_{[a]} \leq \dim \tilde{E}_{(b)} \leq \dim \tilde{E}_{[b]} \) which implies \( \lambda = \lor \{ a \in L : \dim \tilde{E}_{[a]} \geq n \} \geq b \). Thus we have

\[
\lambda \geq \lor \{ b | b \in \beta(\dim \tilde{E}(n)) \} = \dim \tilde{E}(n).
\]

This completes the proof.

**Theorem 4.3.** Let the pair \( \tilde{E} = (E, \mu) \) be a member of \( \text{LFVS} \). Then for any \( a \in L \),

\[
(\dim \tilde{E})_{[a]} \leq \dim \tilde{E}_{[a]} \leq (\dim \tilde{E})_{[a]}.
\]

If \( \beta(a \land b) = \beta(a) \cap \beta(b) \) for all \( a, b \in L \), then

\[
(\dim \tilde{E})_{[a]} \leq \dim \tilde{E}_{[a]} \leq (\dim \tilde{E})_{[a]}.
\]

In particular, \( (\dim \tilde{E})_{[a]} = \dim \tilde{E}_{[a]} \) for any \( a \in J(L) \).

*Proof.* In order to prove \( (\dim \tilde{E})_{[a]} \leq \dim \tilde{E}_{[a]} \), Suppose that \( n \leq (\dim \tilde{E})_{[a]} \), then \( a \in \beta(\dim \tilde{E}(n)) \). Since \( \beta \) is a preserve-union map, there is \( b \in L \) and \( n \leq \dim \tilde{E}_{[a]} \) such that \( a \in \beta(b) \). Because \( \tilde{E}_{[a]} \subseteq \tilde{E}_{[a]} \subseteq \tilde{E}_{[a]} \), thus \( n \leq \dim \tilde{E}_{[a]} \). Therefore \( (\dim \tilde{E})_{[a]} \leq \dim \tilde{E}_{[a]} \).

\( \dim \tilde{E}_{[a]} \leq \dim \tilde{E}_{[a]} \) is obvious. Moreover, we can obtain that \( \dim \tilde{E}_{[a]} \leq (\dim \tilde{E})_{[a]} \) from the definition of \( \dim (\tilde{E}) \).

In order to prove for any \( a \in J(L) \), \( (\dim \tilde{E})_{[a]} = \dim \tilde{E}_{[a]} \), we only need to show

\[
(\dim \tilde{E})_{[a]} \leq \dim \tilde{E}_{[a]}.
\]

Since the set \( \mu(E) \) is finite, for any \( a \in J(L) \) we have

\[
n \leq (\dim \tilde{E})_{[a]} \Rightarrow \dim \tilde{E}(n) \geq a
\]

\[
\Rightarrow \lor \{ b \in L : \dim \tilde{E}_{[a]} \geq n \} \geq a
\]

\[
\Rightarrow \exists a \leq b, \text{ such that } n \leq \dim \tilde{E}_{[b]}
\]

\[
\Rightarrow n \leq \dim \tilde{E}_{[a]}
\]

Therefore \( (\dim \tilde{E})_{[a]} = \dim \tilde{E}_{[a]} \).
**Theorem 4.4.** Let \( \tilde{E} = (E, \mu) \) be a member of LFVS. Then
\[
\left( \dim \tilde{E} \right)^{(a)} \leq \dim \tilde{E}^{(a)} \leq \dim \tilde{E}^{[a]} \leq \left( \dim \tilde{E} \right)^{[a]}.
\]
In particular, \( \left( \dim \tilde{E} \right)^{(a)} = \dim \tilde{E}^{(a)} \) for any \( a \in P(L) \).

**Proof.** \( \left( \dim \tilde{E} \right)^{(a)} \leq \dim \tilde{E}^{(a)} \) can be proved from the following implications.
\[
n \leq \left( \dim \tilde{E} \right)^{(a)} \iff \dim \tilde{E} (n) \not\subset a \\

\iff \vee \{ b \in L : \dim \tilde{E}^{(b)} \geq n \} \not\subset a \\

\iff \exists b \not\subset a, \text{ such that } n \leq \dim \tilde{E}^{(b)} \\

\implies \dim \tilde{E}^{(a)} = \dim \left( \bigcup_{b \in a} \tilde{E}^{(b)} \right) \geq n.
\]

Let \( a \in P(L) \). In order to show \( \dim \tilde{E}^{(a)} = \left( \dim \tilde{E} \right)^{(a)} \), we need to show that
\[
\dim \left( \bigcup_{b \in a} \tilde{E}^{(b)} \right) \leq \left( \dim \tilde{E} \right)^{(a)}.
\]
Suppose that \( n \leq \dim \left( \bigcup_{b \in a} \tilde{E}^{(b)} \right) \). Since the number of \( \tilde{E}^{(b)} \) is finite, then when \( b \not\subset a \), the number of \( \tilde{E}^{(b)} \) is finite, denotes \( \tilde{E}^{(b)} \leq \tilde{E}^{(b)} \), where \( a_i \not\subset a \) for any \( i \in \{ 1, 2, \cdots, r \} \). Thus \( \bigcup_{b \in a} \tilde{E}^{(b)} = \bigcup_{i=1}^{r} \tilde{E}^{(a_i)} \). Since \( a \in P(L) \), then we have \( c = a_1 \land a_2 \land \cdots \land a_r \not\subset a \). Further we have \( \bigcup_{i=1}^{r} \tilde{E}^{(a_i)} \subseteq \tilde{E}^{(c)} \). Thus for any
\[
n \leq \dim \left( \bigcup_{b \in a} \tilde{E}^{(b)} \right) = \dim \left( \bigcup_{i=1}^{r} \tilde{E}^{(a_i)} \right) \leq \dim \tilde{E}^{(a)} \leq \bigvee_{b \in a} \left( \dim \tilde{E}^{(b)} \right) = \left( \dim \tilde{E} \right)^{(a)}.
\]
Therefore for any \( a \in P(L) \), \( \left( \dim \tilde{E} \right)^{(a)} = \dim \tilde{E}^{(a)} \). 
\( \dim \tilde{E}^{(a)} \leq \dim \tilde{E}^{[a]} \) is obvious. We show that \( \dim \tilde{E}^{[a]} \leq \left( \dim \tilde{E} \right)^{[a]} \) in the following implications.
\[
\dim \tilde{E}^{[a]} = \dim \bigcap_{b \in P(L)} \tilde{E}^{(b)} \leq \bigland_{b \in P(L)} \dim \tilde{E}^{(b)}

= \bigland_{b \in P(L)} \left( \dim \tilde{E}^{(b)} \right) = \left( \dim \tilde{E} \right)^{[a]}.
\]

**Theorem 4.5.** Let \( \tilde{E}_1 = (E, \mu_1) \) and \( \tilde{E}_2 = (E, \mu_2) \) be two L-fuzzy vector subspaces. Then the following equality holds
\[
\dim (\tilde{E}_1 + \tilde{E}_2) + \dim (\tilde{E}_1 \cap \tilde{E}_2) = \dim \tilde{E}_1 + \dim \tilde{E}_2.
\]

**Proof.** We denote the sum of \( \tilde{E}_1 \) and \( \tilde{E}_2 \) by \( \tilde{E}_1 + \tilde{E}_2 = (E, \mu) \). From Theorem 11, we know that \( \tilde{E}_1 + \tilde{E}_2 \) is a L-fuzzy vector subspace. By the properties of L-fuzzy natural numbers, Theorem 12 and the dimensional formulation of vector spaces, we know for any \( a \in P(L) \),
Therefore \( \dim(\tilde{E}_1 + \tilde{E}_2) + \dim(\tilde{E}_1 \cap \tilde{E}_2) = \dim \tilde{E}_1 + \dim \tilde{E}_2 \).

**Definition 4.6.** Suppose that \( \tilde{E} = (E, \mu) \) is an \( L \)-fuzzy vector subspace. A map \( f : E \rightarrow E \) is called an \( L \)-fuzzy linear transformation, if it satisfies the following conditions:

1. \( f \) is a linear map on \( E \).
2. For all \( x \in E \), \( \mu(f(x)) \geq \mu(x) \).

**Theorem 4.7.** Suppose that \( \tilde{E} = (E, \mu) \) is an \( L \)-fuzzy vector subspace, \( f \) is an \( L \)-fuzzy linear transformation on \( E \), then \( \ker f = (\ker f, \mu_{\ker f}) \) and \( \text{im} f = (\text{im} f, \mu_{\text{im} f}) \) are \( L \)-fuzzy vector subspaces.

The proof is trivial and omitted.

**Theorem 4.8.** Suppose that \( \tilde{E} = (E, \mu) \) is an \( L \)-fuzzy vector subspace, \( f : E \rightarrow E \) is an \( L \)-fuzzy linear transformation, then
\[
\dim(\ker f) + \dim(\text{im} f) = \dim \tilde{E}
\]

**Proof.** Suppose that \( \varphi \) is a linear transformation on (crisp) vector spaces \( V \), then the equality \( \dim(\text{im} \varphi) + \dim(\ker \varphi) = \dim V \) holds. Hence, for all \( a \in P(L) \), we have
\[
\left( \dim(\tilde{\text{im}} f) + \dim(\tilde{\ker} f) \right)^{(a)} = \left( \dim(\text{im} f) \right)^{(a)} + \left( \dim(\ker f) \right)^{(a)} = \dim(\tilde{\text{im}} f)^{(a)} + \dim(\tilde{\ker} f)^{(a)} = \dim(\tilde{E}^{(a)} \cap \text{im} f) + \dim(\tilde{E}^{(a)} \cap \ker f)
\]
Since \( f|_{\tilde{E}^{(a)}} \) is a linear transformation on \( \tilde{E}^{(a)} \), we have
\[
\left( \dim(\tilde{\text{im}} f) + \dim(\tilde{\ker} f) \right)^{(a)} = \dim(\text{im} f|_{\tilde{E}^{(a)}}) + \dim(\ker f|_{\tilde{E}^{(a)}}) = \dim \tilde{E}^{(a)} = (\dim \tilde{E})^{(a)}.
\]
Therefore \( \dim(\tilde{\ker} f) + \dim(\tilde{\text{im}} f) = \dim \tilde{E} \).

5. Conclusion

In this paper, \( L \)-fuzzy vector subspace is defined and showed that its dimension is an \( L \)-fuzzy natural number. Based on the definitions, some good properties of crisp vector spaces are hold in a finite-dimensional \( L \)-fuzzy vector subspace. In particular, the equality \( \dim(\tilde{E}_1 + \tilde{E}_2) + \dim(\tilde{E}_1 \cap \tilde{E}_2) = \dim \tilde{E}_1 + \dim \tilde{E}_2 \) holds without any restricted
conditions. At the same time, \( \dim(\overline{\text{im} f}) + \dim(\overline{\text{ker} f}) = \dim \overline{E} \) holds.

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