Least-Squares Solutions of Generalized Sylvester Equation with $X_i$ Satisfies Different Linear Constraint

Xuelin Zhou, Dandan Song, Qingle Yang, Jiaofen Li*

School of Mathematics and Computing Science, Guangxi Colleges and Universities Key Laboratory of Data Analysis and Computation, Guilin University of Electronic Technology, Guilin, China

Received 12 March 2016; accepted 11 June 2016; published 14 June 2016

Copyright © 2016 by authors and Scientific Research Publishing Inc.
This work is licensed under the Creative Commons Attribution International License (CC BY).

Abstract

In this paper, an iterative method is constructed to find the least-squares solutions of generalized Sylvester equation $A_1X_1B_1 + A_2X_2B_2 + A_3X_3B_3 + A_4X_4B_4 = C$, where $[X_1, X_2, X_3, X_4]$ is real matrices group, and $X_i$ satisfies different linear constraint. By this iterative method, for any initial matrix group $[X_1^0, X_2^0, X_3^0, X_4^0]$ within a special constrained matrix set, a least squares solution group $[X_1^\ast, X_2^\ast, X_3^\ast, X_4^\ast]$ with $X_i^\ast$ satisfying different linear constraint can be obtained within finite iteration steps in the absence of round off errors, and the unique least norm least-squares solution can be obtained by choosing a special kind of initial matrix group. In addition, a minimization property of this iterative method is characterized. Finally, numerical experiments are reported to show the efficiency of the proposed method.

Keywords

Least-Squares Problem, Centro-Symmetric Matrix, Bisymmetric Matrix, Iterative Method

1. Introduction

A matrix $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ is said to be a Centro-symmetric matrix if $a_{ij} = a_{n+1-j, n+1-i}$ for all $1 \leq i, j \leq n$. A matrix $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ is said to be a Bisymmetric matrix if $a_{ij} = a_{ji} = a_{n+1-j, n+1-i}$ for all $1 \leq i, j \leq n$. Let


http://dx.doi.org/10.4236/alamt.2016.62008
$R^{m\times n}, SR^{m\times n}, CSR^{m\times n}$ and $BSR^{m\times n}$ denote the set of $m \times n$ real matrices, $n \times n$ real symmetric matrices, $n \times n$ real Centro-symmetric matrices and $n \times n$ real Bisymmetric matrices, respectively. $S_n = \{e_1, e_{n+1}, \ldots, e_n\}$ where $e_i$ denotes $i$th column of $n \times n$ unit matrix. For a matrix $A \in R^{m\times n}$, we denote its transpose, traced by $A^T, tr(A)$ respectively. In space $R^{m\times n}$, we define inner product as: $\langle A, B \rangle = tr(B^T A)$ for all $A, B \in R^{m\times n}$; then the norm of a matrix $A$ generated by this inner product is, obviously, Frobenius norm and denoted by $\|A\|_F$.

Denote

$$K = \left\{ [X_1 , X_2 , X_3 , X_4] \mid X_1 \in R^{m\times n} , X_2 \in SR^{p\times q} , X_3 \in CSR^{n\times n} , X_4 \in BSR^{n\times n} \right\}.$$ 

Obviously, $K$, i.e. $R^{m\times n} \times SR^{p\times q} \times CSR^{n\times n} \times BSR^{n\times n}$, is a linear subspace of real number field.

In this paper, we mainly consider the following two problems:

**Problem I.** Given matrices $A_i \in R^{p\times n}$, $B_i \in R^{n\times q}$ ($i = 1, 2, 3, 4$), find matrix group $\left[ \tilde{X}_1 , \tilde{X}_2 , \tilde{X}_3 , \tilde{X}_4 \right] \in K$ such that

$$\min_{\left[ X_1 , X_2 , X_3 , X_4 \right] \in K} \left\| A_i \tilde{X}_i B_i + A_2 \tilde{X}_2 B_2 + A_3 \tilde{X}_3 B_3 + A_4 \tilde{X}_4 B_4 - C \right\|.$$

**Problem II.** Denote by $S_E$ the solution set of Problem I. Find matrix group $\left[ \tilde{X}_1 , \tilde{X}_2 , \tilde{X}_3 , \tilde{X}_4 \right] \in S_E$ such that

$$\|\tilde{X}_1\| + \|\tilde{X}_2\| + \|\tilde{X}_3\| + \|\tilde{X}_4\| = \min.$$ 

In fact, Problem II is to find the least norm solution of Problem I.

There are many valuable efforts on formulating solutions of various linear matrix equations with or without linear constraint. For example, Baksalary and Kala [1], Chu [2] [3], Peng [4], Liao, Bai and Lei [5] and Xu, Wei and Zheng [6] considered the nonsymmetric solution of the matrix equation

$$AXB + CYD = E,$$  

by using Moore-Penrose generalized inverse and the generalized singular value decomposition of matrices, while Chang and Wang [7] considered the symmetric conditions on the solution of the matrix equations

$$AXA^T + BYB^T = E, \quad AX + YA = C.$$  

Zietak [8] [9] discussed the $l_p$-solution and Chebyshev-solution of the matrix equation

$$AX + YB = C.$$  

Peng [10] researched the general linear matrix equation

$$A_1 X_1 B_1 + A_2 X_2 B_2 + \cdots + A_4 X_4 B_4 = C,$$  

with the bisymmetric conditions on the solutions. Vec operator and Kronecker product are employed in this paper, so the size of the matrix is enlarged greatly and the computation is very expensive in the process of solving solutions. Iterative algorithms have been received much attention to solve linear matrix equations in recent years. For example, by extending the well-known Jacobi and Gauss-seidel iterations for $AX = b$, Ding, Liu and Ding in [11] derived iterative solutions of matrix equations $AXB = F$ and generalized Sylvester matrix equations $AXB + CYD = F$. By absorbing the thought of the conjugate gradient method, Peng [12] presented an iterative algorithm to solve Equation (1). Peng [13], Peng, Hu and Zhang [14] put forward an iterative method for bisymmetric solution of Equation (4). These matrix-form CG methods are based on short recurrences, which keep work and storage requirement constant at each iteration. However, these iteration methods are only defined by the Galerkin condition, but lack of a minimization property, which means that the algorithm may exhibit a rather irregular convergence, and often results in a very slow convergence. Lei and Liao [15] presented that a minimal residual algorithm could remedy this problem, and this algorithm satisfies a minimization property, which ensures that this method possesses a smoothly convergence.

However, to our best knowledge, the unknown matrix with different linear constraint of linear matrix equations, such as Equations ((1)-(4)), has not been considered yet. No loss of generality, we research the following case

$$A_1 X_1 B_1 + A_2 X_2 B_2 + A_3 X_3 B_3 + A_4 X_4 B_4 = C.$$  

X. L. Zhou et al.
which has four unknown matrices and each is required to satisfy different linear constraint. We should point out that the matrices $A_i, B_i, C$ are experimentally occurring in practices, so they may not satisfy solvability conditions. Hence, we should study the least squares solutions, i.e. Problem I. Noting that it is obvious difficulties to solve this problem by conventional methods, such as matrix decomposition and ver operator, hence iterative method is considered. Absorbing the thought of the minimal residual algorithm presented by Lei and Liao [15], and combing the trait of problem, we conduct an iterative method for solving Problem I. This method can both maintain the short recurrence and satisfy a minimization property, i.e. the approximation solution minimizes the residual norm of Equation (5) over a special affine subspace, which ensures that this method converges smoothly.

The paper is organized as follows. In Section 2, we first conduct an iterative method for solving Problem I, and then describe the basic properties of this method; we also solve Problem II by using this iterative method. In Section 3, we show that the method possesses a minimization property. In Section 4, we present numerical experiments to show the efficiency of the proposed method, and use some conclusions in Section 5 to end our paper.

2. The Iterative Method for Solving Problem I and II

In this section, we firstly introduce some lemmas which are required for solving Problem I, we then conduct an iterative method to obtain the solution of Problem I. We show that, for any initial matrix group

$$[X_1^{(0)}, X_2^{(0)}, X_3^{(0)}, X_4^{(0)}] \in K,$$

the matrix group sequences $[X_1^{(i)}, X_2^{(i)}, X_3^{(i)}, X_4^{(i)}]$ generated by the iterative method converge to a solution of Problem I within finite iteration steps in the absence of roundoff errors. We also show that the unique least norm solution of Problem I can be obtained by choosing a special kind of initial matrix group.

**Lemma 1.** [16] [17]. A matrix $X \in CSR^{n \times n}$ if and only if $X = S_n X S_n^T$.

A matrix $X \in BSR^{n \times n}$ if and only if $X = X^T = S_n X S_n$.

**Lemma 2.** Suppose that a matrix $X \in R^{n \times n}$, then $X + S_n X S_n \in CSR^{n \times n}$.

Suppose that a matrix $X \in SR^{n \times n}$, then $X + S_n X S_n \in BSR^{n \times n}$.

**Proof:** Its proof is easy to obtain from Lemma 1.

**Lemma 3.** Suppose that $A \in R^{n \times n}$, $X \in SR^{n \times n}$, $Y \in CSR^{n \times n}$, $Z \in BSR^{n \times n}$ then

$$\langle A, X \rangle = \frac{1}{2} [A + A^T], X \rangle$$

$$\langle A, Y \rangle = \frac{1}{4} (A + A^T + S_n (A + A^T) S_n), Y \rangle$$

$$\langle A, Z \rangle = \frac{1}{4} (A + A^T + S_n (A + A^T) S_n), Z \rangle.$$

**Proof:** It is easy to verify from direct computation.

**Lemma 4.** (Projection Theorem) [18]. Let $X$ be a finite dimensional inner product space, $M$ be a subspace of $X$, and $M^\perp$ be the orthogonal complement subspace of $M$. For a given $x \in X$, there always exists an $m_0 \in M$ such that

$$\|x - m_0\| \leq \|x - m\|, \quad \forall m \in M,$$

where $\|\|$ is the norm associated with the inner product defined in $X$. Moreover, $m_0 \in M$ is the unique minimization vector in $M$ if and only if

$$(x - m_0) \perp M, \quad i.e., \quad (x - m_0) \in M^\perp.$$

**Lemma 5.** Suppose $\tilde{R}$ is the residual of matrix group $[\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4] \in K$ corresponding to Equation (5), i.e. $\tilde{R} = - A_1 \tilde{X}_1 B_1 - A_2 \tilde{X}_2 B_2 - A_3 \tilde{X}_3 B_3 - A_4 \tilde{X}_4 B_4$.

If the following conditions are satisfied simultaneously,

$$A_i^T \tilde{R} B_i^T = 0,$$

$$A_i^T \tilde{R} B_i^T + B_i \tilde{R}^T A_i = 0,$$

$$A_i^T \tilde{R} B_i^T + S_{a_i} A_i^T \tilde{R}^T S_{a_i} = 0,$$

$$A_i^T \tilde{R} B_i^T + B_i \tilde{R}^T A_i + S_{a_i} (A_i^T \tilde{R} B_i^T + B_i \tilde{R}^T A_i) S_{a_i} = 0,$$

$$i = 1, 2, 3, 4.$$
then the matrix group \( \begin{bmatrix} \tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4 \end{bmatrix} \) is a solution of Problem I.

**Proof:** Let

\[ L = \{ Z \mid Z = A_1X_1B_1 + A_2X_2B_2 + A_3X_3B_3 + A_4X_4B_4, [X_1, X_2, X_3, X_4] \in K \}, \]

obviously, \( Z \) is a linear subspace of \( R^{n \times q} \). For matrix group \( \begin{bmatrix} \tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4 \end{bmatrix} \in K \), denote

\[ \tilde{C} = A_1\tilde{X}_1B_1 + A_2\tilde{X}_2B_2 + A_3\tilde{X}_3B_3 + A_4\tilde{X}_4B_4, \]

then \( \tilde{C} \in L \). Applying to Lemma 4, we know that \( \begin{bmatrix} \tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4 \end{bmatrix} \) is a solution of Problem I if and only if

\[ (\tilde{C} - C) \perp L, \]

i.e. for all \( [X_1, X_2, X_3, X_4] \in K \),

\[ \langle \tilde{R}, A_1X_1B_1 + A_2X_2B_2 + A_3X_3B_3 + A_4X_4B_4 \rangle = 0. \]

By Lemma 3, it is easy to verify that if the equations of (6) are satisfied simultaneously, the expression above holds, which means \( \begin{bmatrix} \tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4 \end{bmatrix} \) is a solution of Problem I. \( \Box \)

Lemma 6. Suppose that matrix group \( \begin{bmatrix} \tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4 \end{bmatrix} \) is a solution of Problem I, then arbitrary matrix group \( [X_1, X_2, X_3, X_4] \in S_K \) can be express as \( [\tilde{X}_1 + X_1, \tilde{X}_2 + X_2, \tilde{X}_3 + X_3, \tilde{X}_4 + X_4] \) where matrix group \( \begin{bmatrix} \tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4 \end{bmatrix} \in K \) satisfies

\[ A_1\tilde{X}_1B_1 + A_2\tilde{X}_2B_2 + A_3\tilde{X}_3B_3 + A_4\tilde{X}_4B_4 = 0. \] (7)

**Proof:** Assume that matrix group \( \begin{bmatrix} \tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4 \end{bmatrix} \) is a solution of Problem I. If \( \begin{bmatrix} \tilde{X}_1 + X_1, \tilde{X}_2 + X_2, \tilde{X}_3 + X_3, \tilde{X}_4 + X_4 \end{bmatrix} \in S_K \), then by Lemma 5 and its proof process, we have

\[
\begin{align*}
&\left\| A_1\tilde{X}_1B_1 + A_2\tilde{X}_2B_2 + A_3\tilde{X}_3B_3 + A_4\tilde{X}_4B_4 - C \right\|^2 \\
= &\left\| A_1(\tilde{X}_1 + X_1)B_1 + A_2(\tilde{X}_2 + X_2)B_2 + A_3(\tilde{X}_3 + X_3)B_3 + A_4(\tilde{X}_4 + X_4)B_4 - C \right\|^2 \\
= &\left\| A_1\tilde{X}_1B_1 + A_2\tilde{X}_2B_2 + A_3\tilde{X}_3B_3 + A_4\tilde{X}_4B_4 - \tilde{R} \right\|^2 \\
= &\left\| A_1\tilde{X}_1B_1 + A_2\tilde{X}_2B_2 + A_3\tilde{X}_3B_3 + A_4\tilde{X}_4B_4 \right\|^2 + \left\| \tilde{R} \right\|^2
\end{align*}
\]

which implies matrix group \( \begin{bmatrix} \tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4 \end{bmatrix} \) satisfies (7).

Conversely, if matrix group \( \begin{bmatrix} Y_1, Y_2, Y_3, Y_4 \end{bmatrix} = \begin{bmatrix} \tilde{X}_1 + X_1, \tilde{X}_2 + X_2, \tilde{X}_3 + X_3, \tilde{X}_4 + X_4 \end{bmatrix} \in K, \)

where matrix group \( \begin{bmatrix} \tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4 \end{bmatrix} \in K \) satisfies (7), then

\[
\begin{align*}
&\left\| A_1Y_1B_1 + A_2Y_2B_2 + A_3Y_3B_3 + A_4Y_4B_4 - C \right\|^2 \\
= &\left\| A_1\tilde{X}_1B_1 + A_2\tilde{X}_2B_2 + A_3\tilde{X}_3B_3 + A_4\tilde{X}_4B_4 - \tilde{R} \right\|^2 \\
= &\left\| A_1\tilde{X}_1B_1 + A_2\tilde{X}_2B_2 + A_3\tilde{X}_3B_3 + A_4\tilde{X}_4B_4 - C \right\|^2
\end{align*}
\]

which means matrix group \( \begin{bmatrix} Y_1, Y_2, Y_3, Y_4 \end{bmatrix} \in S_K. \) \( \Box \)

Next, we develop iterative algorithm for the least-squares solutions with \( X_i \) satisfies different linear constraint of matrix equation

\[ A_1X_1B_1 + A_2X_2B_2 + A_3X_3B_3 + A_4X_4B_4 = C, \]

where \( A_i \in R^{p \times n}, B_i \in R^{n \times q}, (i = 1, 2, 3, 4) \) and \( C \) are given constant matrices, and \( [X_1, X_2, X_3, X_4] \in K \) is the unknown matrices group to be solved.
Algorithm 1. For an arbitrary initial matrix group \( \begin{bmatrix} X_1^{(0)}, X_2^{(0)}, X_3^{(0)}, X_4^{(0)} \end{bmatrix} \in K \), compute

**Step 1.** \( R_0 = C - \sum_{i=1}^{4} A_i X_i^{(0)} B_i \);

\[ P_{0,1} = A^T R_0 B_1^T; \]

\[ P_{0,2} = \frac{1}{2} \left( A^T R_0 B_2^T + B_2 R_0^T A_2 \right); \]

\[ P_{0,3} = \frac{1}{2} \left( A^T R_0 B_3^T + S_3 A^T R B_3^T S_3 \right); \]

\[ P_{0,4} = \frac{1}{4} \left( A^T R_0 B_4^T + B_4 R_0^T A_4 + S_4 \left( A^T R_0 B_4^T + B_4 R_0^T A_4 \right) S_4 \right); \]

\[ Q_{0,r} = P_{0,r}, \quad r = 1, 2, 3, 4. \]

**Step 2.** If \( \sum_{r=1}^{4} P_{r,r} = 0 \), then stop; else, \( k := k + 1 \), and compute

**Step 3.** \( M_k = \sum_{r=1}^{4} A_i Q_{r,r} B_r; \)

\[ \alpha_k = \frac{\sum_{r=1}^{4} \left\| P_{r,r} \right\|}{\| M_k \|^2}, \quad X_r^{(k+1)} = X_r^{(k)} + \alpha_k Q_{r,r}, \quad r = 1, 2, 3, 4; \]

\[ R_{k+1} = R_k - \alpha_k M_k; \]

\[ P_{k+1,1} = A^T R_{k+1} B_1^T = P_{k,1} - \alpha_k A^T M_k B_1^T; \]

\[ P_{k+1,2} = \frac{1}{2} \left( A^T R_{k+1} B_2^T + B_2 R_{k+1}^T A_2 \right) = P_{k,2} - \frac{\alpha_k}{2} \left[ A^T M_k B_2^T + B_2 M_k^T A_2 \right]; \]

\[ P_{k+1,3} = \frac{1}{4} \left( A^T R_{k+1} B_3^T + S_3 A^T R_{k+1} B_3^T S_3 \right) \]

\[ = P_{k,3} - \frac{\alpha_k}{4} \left[ A^T M_k B_3^T + S_3 A^T M_k B_3^T S_3 \right]; \]

\[ P_{k+1,4} = \frac{1}{4} \left( A^T R_{k+1} B_4^T + B_4 R_{k+1}^T A_4 + S_4 \left( A^T R_{k+1} B_4^T + B_4 R_{k+1}^T A_4 \right) S_4 \right) \]

\[ = P_{k,4} - \frac{\alpha_k}{4} \left[ A^T M_k B_4^T + B_4 M_k^T A_4 + S_4 \left( A^T M_k B_4^T + B_4 M_k^T A_4 \right) S_4 \right]; \]

\[ \beta_k = \frac{\sum_{r=1}^{4} \left\| P_{r,r} \right\|}{\sum_{r=1}^{4} \left\| P_{r,r} \right\|}, \quad Q_{k+1,r} = P_{k+1,r} + \beta_k Q_{k,r}, \quad r = 1, 2, 3, 4; \]

**Step 4.** Go to step 2.

**Remark 1.** 1) Obviously, matrices sequence \( P_r, Q_r, X_r^{(r)} \) generated by Algorithm 1 satisfies

\( P_{i,j} \in R^{m_{ij}}, \quad P_{i,j} \in SR^{m_{ij}}, \quad P_{i,j} \in CSR^{m_{ij}}, \quad P_{i,j} \in BSR^{m_{ij}} \)

\( Q_{i,j} \in R^{m_{ij}}, \quad Q_{i,j} \in SR^{m_{ij}}, \quad Q_{i,j} \in CSR^{m_{ij}}, \quad Q_{i,j} \in BSR^{m_{ij}} \)

\( X_1^{(i)} \in R^{m_{ij}}, \quad X_2^{(i)} \in SR^{m_{ij}}, \quad X_3^{(i)} \in CSR^{m_{ij}}, \quad X_4^{(i)} \in BSR^{m_{ij}} \)

2) \( R_i \) is the residual of Equation (5), when \( i = 0, 1, 2, \ldots \)

3) Algorithm 1 implies that if \( P_{r,r} = 0 \) \( (r = 1, 2, 3, 4) \), then the corresponding matrix group \( \begin{bmatrix} X_1^{(i)}, X_2^{(i)}, X_3^{(i)}, X_4^{(i)} \end{bmatrix} \) is the solution of Problem I.
In the next part, we will show the basic properties of iteration method by induction. First for convenience of discussion in the later context, we introduce the following conclusions from Algorithm 1. For all \( i, j, t \)

\[
\begin{aligned}
\left( r_i - r_j, \sum_{r=1}^{4} A_r P_r B_r \right) &= \sum_{r=1}^{4} \left( A_r^T \left( r_i - r_j \right) B_r^T, P_r \right) \\
&= \left\langle A_r^T \left( r_i - r_j \right) B_r^T, P_r \right\rangle + \frac{1}{2} \left\langle A_r^T \left( r_i - r_j \right) B_r^T + B_r \left( r_i - r_j \right)^T A_r, P_r \right\rangle \\
&\quad + \frac{1}{2} A_r^T \left( r_i - r_j \right) B_r^T + S_{\alpha_0} A_r^T \left( r_i - r_j \right) B_r^T S_{\alpha_0}, P_r \\
&\quad + \frac{1}{4} \left\langle A_r^T \left( r_i - r_j \right) B_r^T + B_r \left( r_i - r_j \right)^T A_r + S_{\alpha_0} A_r^T \left( r_i - r_j \right) B_r^T + B_r \left( r_i - r_j \right)^T A_r \right\rangle S_{\alpha_0}, P_r \rangle \\
&= \sum_{r=1}^{4} \left\langle P_r, P_r \right\rangle + \sum_{r=1}^{4} \left\langle P_r, P_r \right\rangle.
\end{aligned}
\]

**Lemma 7.** For matrices \( P_{r, \alpha}, Q_{r, \beta} (r = 1, 2, 3, 4) \) and \( M_r \) generated by Algorithm 1, if there exist a positive integer \( k \) such that \( \sum_{r=1}^{4} \| P_{r, \alpha} \| = 0 \), \( \alpha_i \neq 0 \), and \( \alpha_i \neq \infty \) for all \( i = 0, 1, 2, \ldots, k \), then we have

1) \( \sum_{r=1}^{4} \left\langle P_{r, \alpha}, P_{r, \beta} \right\rangle = 0 \), \( i, j = 0, 1, 2, \ldots, k \), \( i \neq j \);

2) \( \left\langle M_r, M_r \right\rangle = 0 \), \( i, j = 0, 1, 2, \ldots, k \), \( i \neq j \);

3) \( \sum_{r=1}^{4} \left\langle P_{r, \alpha}, Q_{r, \beta} \right\rangle = 0 \), \( 0 \leq j < i \leq k \).

**Proof:** For \( k = 1 \), it follows that

\[
\sum_{r=1}^{4} \left\langle P_{r, \alpha}, P_{r, \beta} \right\rangle = \left\langle P_{0,1} P_{0,2} - \alpha_0 A_{1,0}^T M_0 B_1^T \right\rangle + \left\langle P_{0,2} P_{0,2} - \frac{\alpha_0}{2} A_{1,0}^T M_0 B_1^T + B_2 M_0^T A_2 \right\rangle \\
+ \left\langle P_{0,3} P_{0,3} - \frac{\alpha_0}{2} A_{1,0}^T M_0 B_1^T + S_{\alpha_0} A_{1,0}^T M_0 B_1^T S_{\alpha_0} \right\rangle \\
+ \left\langle P_{0,4} P_{0,4} - \frac{\alpha_0}{4} \left\langle A_{1,0}^T M_0 B_1^T + B_2 M_0^T A_2 + S_{\alpha_0} A_{1,0}^T M_0 B_1^T + B_2 M_0^T A_2 \right\rangle S_{\alpha_0} \right\rangle \\
= \sum_{r=1}^{4} \| P_{r, \alpha} \|^2 - \alpha_0 \sum_{r=1}^{4} \left\langle A_{1,0}^T P_{r, \alpha}, M_0 \right\rangle = \sum_{r=1}^{4} \| P_{r, \alpha} \|^2 - \alpha_0 \left\langle M_0, M_0 \right\rangle = 0.
\]

\[
\left\langle M_0, M_0 \right\rangle = \left\langle M_0, \sum_{r=1}^{4} A_r \left( P_{r, \alpha} + \beta_0 Q_{r, \alpha} \right) B_r \right\rangle = \left\langle M_0, \sum_{r=1}^{4} A_r P_{r, \beta} B_r \right\rangle + \beta_0 \| M_0 \|^2 \\
= \frac{1}{\alpha_0} \left\langle R_0 - R_1, \sum_{r=1}^{4} A_r P_{r, \beta} B_r \right\rangle + \beta_0 \| M_0 \|^2 = -\frac{1}{\alpha_0} \sum_{r=1}^{4} \| P_{r, \beta} \|^2 + \beta_0 \| M_0 \|^2 = 0.
\]

Assume that the conclusions

\[
\sum_{r=1}^{4} \left\langle P_{r, \alpha}, P_{s, \beta} \right\rangle = 0; \quad \left\langle M_j, M_s \right\rangle = 0; \quad \sum_{r=1}^{4} \left\langle P_{r, \alpha}, Q_{r, \beta} \right\rangle = 0
\]

hold for all \( j \leq s - 1 (0 < s < k) \), then
\[
\sum_{r=1}^{4} \left\langle P_{j,r}, P_{s+1,r} \right\rangle = \left\langle P_{j,1} - \alpha_j A_j^T M_j B_j^T, \frac{\alpha_j}{2} \left( A_j^T M_j B_j^T + B_j M_j^T A_j \right) \right\rangle + \left\langle P_{j,2} - \alpha_j A_j^T M_j B_j^T + B_j M_j^T A_j, \frac{\alpha_j}{2} \left( A_j^T M_j B_j^T + B_j M_j^T A_j \right) \right\rangle + \left\langle P_{j,3} - \alpha_j A_j^T M_j B_j^T + B_j M_j^T A_j, \frac{\alpha_j}{2} \left( A_j^T M_j B_j^T + B_j M_j^T A_j \right) \right\rangle + \left\langle P_{j,4} - \alpha_j A_j^T M_j B_j^T + B_j M_j^T A_j, \frac{\alpha_j}{2} \left( A_j^T M_j B_j^T + B_j M_j^T A_j \right) \right\rangle \\
= -\alpha \sum_{r=1}^{4} \left\langle P_{j,r} - \alpha_j A_j^T M_j B_j^T - \alpha_j A_j^T M_j B_j^T + B_j M_j^T A_j, \right\rangle = -\alpha \sum_{r=1}^{4} \left\langle A_j^T \left( Q_{j,r} - \beta_j \left( Q_{j-1,r} \right) \right) B_j, M_j \right\rangle = -\alpha \left\langle M_j - \beta_j \left( M_{j-1} \right), M_j \right\rangle = 0.
\]

\[
\left\langle M_j, M_{s+1} \right\rangle = \left\langle M_j, \sum_{r=1}^{4} A_j \left( P_{s+1,r} + \beta_j Q_{s+1,r} \right) B_r \right\rangle = \left\langle M_j, \sum_{r=1}^{4} A_j P_{s+1,r} B_r \right\rangle = \frac{1}{\alpha_j} \left\langle R_j - R_{j+1}, \sum_{r=1}^{4} A_j P_{s+1,r} B_r \right\rangle = \frac{1}{\alpha_j} \left[ \sum_{r=1}^{4} \left\langle P_{j,r}, P_{s+1,r} \right\rangle - \sum_{r=1}^{4} \left\langle P_{j,r}, P_{s+1,r} \right\rangle \right] = -\frac{1}{\alpha_j} \sum_{r=1}^{4} \left\langle P_{j,r}, P_{s+1,r} \right\rangle.
\]

\[
\sum_{r=1}^{4} \left\langle Q_{j,r}, P_{s+1,r} \right\rangle = \left\langle Q_{j,1} - \alpha_j A_j^T M_j B_j^T, \frac{\alpha_j}{2} \left( A_j^T M_j B_j^T + B_j M_j^T A_j \right) \right\rangle + \left\langle Q_{j,2} - \alpha_j A_j^T M_j B_j^T + B_j M_j^T A_j, \frac{\alpha_j}{2} \left( A_j^T M_j B_j^T + B_j M_j^T A_j \right) \right\rangle + \left\langle Q_{j,3} - \alpha_j A_j^T M_j B_j^T + B_j M_j^T A_j, \frac{\alpha_j}{2} \left( A_j^T M_j B_j^T + B_j M_j^T A_j \right) \right\rangle + \left\langle Q_{j,4} - \alpha_j A_j^T M_j B_j^T + B_j M_j^T A_j, \frac{\alpha_j}{2} \left( A_j^T M_j B_j^T + B_j M_j^T A_j \right) \right\rangle \\
= -\alpha \sum_{r=1}^{4} \left\langle A_j Q_{j,r} B_j, M_j \right\rangle = -\alpha \left\langle M_j, M_j \right\rangle = 0.
\]

By the assumption of Equation (3), we have
\[
\sum_{r=1}^{4} \left\langle Q_{j,r}, P_{s,r} \right\rangle = \sum_{r=1}^{4} \left\langle P_{s,r} + \beta_j Q_{s+1,r}, P_{s,r} \right\rangle = \sum_{r=1}^{4} \left\| P_{s,r} \right\|^2.
\]

Then for \( j = s \),
\[
\sum_{r=1}^{4} \left\langle P_{s,r}, P_{s+1,r} \right\rangle = \left\langle P_{s,1} - \alpha_j A_j^T M_j B_j^T, \frac{\alpha_j}{2} \left( A_j^T M_j B_j^T + B_j M_j^T A_j \right) \right\rangle + \left\langle P_{s,2} - \alpha_j A_j^T M_j B_j^T + B_j M_j^T A_j, \frac{\alpha_j}{2} \left( A_j^T M_j B_j^T + B_j M_j^T A_j \right) \right\rangle + \left\langle P_{s,3} - \alpha_j A_j^T M_j B_j^T + B_j M_j^T A_j, \frac{\alpha_j}{2} \left( A_j^T M_j B_j^T + B_j M_j^T A_j \right) \right\rangle + \left\langle P_{s,4} - \alpha_j A_j^T M_j B_j^T + B_j M_j^T A_j, \frac{\alpha_j}{2} \left( A_j^T M_j B_j^T + B_j M_j^T A_j \right) \right\rangle \\
= \sum_{r=1}^{4} \left\| P_{s,r} \right\|^2 - \alpha \sum_{r=1}^{4} \left\langle A_j P_{s,r} B_j, M_j \right\rangle \\
= \sum_{r=1}^{4} \left\| P_{s,r} \right\|^2 - \alpha \left\| M_j \right\|^2 = 0.
\]

\[
\sum_{r=1}^{4} \left\langle Q_{s,r}, P_{s+1,r} \right\rangle = \left\langle Q_{s,1}, P_{s+1,1} \right\rangle + \left\langle Q_{s,2}, P_{s+1,2} \right\rangle + \left\langle Q_{s,3}, P_{s+1,3} \right\rangle + \left\langle Q_{s,4}, P_{s+1,4} \right\rangle \\
= \sum_{r=1}^{4} \left\langle A_j^T \left( Q_{s,r} - \beta_j \left( Q_{s-1,r} \right) \right) B_j, M_j \right\rangle \\
= \sum_{r=1}^{4} \left\langle A_j \left( Q_{s,r} - \beta_j \left( Q_{s-1,r} \right) \right) B_j, M_j \right\rangle = \beta \left\| M_j \right\|^2 - \frac{1}{\alpha_j} \sum_{r=1}^{4} \left\langle P_{s+1,r}, P_{s+1,r} \right\rangle = 0.
\]
Then the conclusion \( \sum_{i=1}^{4} \langle P_{i}, P_{i+1} \rangle = 0 \) and the assumption \( \sum_{i=1}^{4} \langle P_{i}, A^{i}M_{i}B^{i} \rangle = \sum_{i=1}^{4} \|P_{i}\|^{2} - \alpha_{i} \|M_{i}\|^{2} = 0 \) show that \( \langle M_{j}, M_{s+1} \rangle = 0 \) for all \( j \leq s - 1 \). By the principal of induction, we know that Eq.(3) holds for all \( 0 \leq j < i \leq k \), and Equation (1) and Equation (2) hold for all \( i, j = 0,1,2,\ldots,k, i \neq j \) due to the fact that \( \langle A, B \rangle = \langle B, A \rangle \) holds for all matrices \( A \) and \( B \) in \( R^{m \times n} \).

**Lemma 7.** shows that the matrix sequence

\[
\begin{pmatrix}
P_{0,1} & P_{0,2} & \ldots & P_{1,2} & P_{1,3} & \ldots \\
 & P_{0,3} & \ldots & P_{1,2} & P_{1,3} & \ldots \\
 & & P_{0,4} & \ldots & P_{1,3} & \ldots \\
\end{pmatrix}
\]

generated by Algorithm 1 are orthogonal each other in the finite dimension matrix space \( R^{m \times n} \). Hence the iterative method will be terminated at most \( \sum_{i=1}^{4} n_{i}^{2} \) steps in the absence of roundoff errors.

It is worth to note that the conclusions of Lemma 7 may not be true without the assumptions \( \alpha_{i} \neq 0 \) and \( \alpha_{i} \neq \infty \). Hence it is necessary to consider the case that \( \alpha_{i} = 0 \) or \( \alpha_{i} = \infty \).

If \( \alpha_{i} = 0 \), which implies \( \sum_{i=1}^{4} \|P_{i}\| = 0 \), it follows that \( P_{i,r} = 0, r = 1,2,3,4 \).

If \( \alpha_{i} = \infty \), which implies \( \|M_{i}\| = 0 \), then we have \( \sum_{i=1}^{4} A_{i}Q_{i}B_{i} = 0 \), making inner product with \( R_{i} \) by both side, yields

\[
\sum_{i=1}^{4} A_{i}Q_{i}B_{i}, R_{i} = \sum_{i=1}^{4} \langle Q_{i}, A_{i}^{2}R_{i}B_{i} \rangle = \sum_{i=1}^{4} \langle Q_{i}, P_{i,r} \rangle = \sum_{i=1}^{4} \|P_{i,r}\| = 0.
\]

So the discussions above show that if there exist a positive integer \( i \) such that the coefficient \( \alpha_{i} = 0 \) or \( \alpha_{i} = \infty \), then the corresponding matrix group \( \left[ X_{1}^{(i)}, X_{2}^{(i)}, X_{3}^{(i)}, X_{4}^{(i)} \right] \in K \) is just the solution of Problem I.

Together with Lemma 7 and the discussion about the coefficient \( \alpha_{i} \), we can conclude the following theorem.

**Theorem 1.** For an arbitrary initial matrix group \( \left[ X_{1}^{(0)}, X_{2}^{(0)}, X_{3}^{(0)}, X_{4}^{(0)} \right] \in K \), the matrix group sequence

\[
\left[ X_{1}^{(i)}, X_{2}^{(i)}, X_{3}^{(i)}, X_{4}^{(i)} \right] \in K \text{ generated by Algorithm 1 will converge to a solution of Problem I at infinite iteration steps in exact arithmetic.}
\]

By choosing a special kind of initial matrix group, we can obtain the unique least norm of Problem I. To this end, we first define a matrix set as follows

\[
S = \left\{ X_{1}, X_{2}, X_{3}, X_{4} \mid \begin{array}{l}
X_{1} = A_{1}^{2}HB_{1}^{T}, \\
X_{2} = \frac{1}{2} \left( A_{1}^{2}HB_{1}^{T} + B_{2}H^{T}A_{2} \right), \\
X_{3} = \frac{1}{2} \left( A_{2}^{2}HB_{2}^{T} + S_{3}A_{2}^{2}HB_{2}^{T}S_{3} \right), \\
X_{4} = \frac{1}{4} \left( A_{4}^{2}HB_{4}^{T} + B_{2}H^{T}A_{2} + S_{4} \left( A_{4}^{2}HB_{4}^{T} + B_{3}H^{T}A_{3} \right) \right) S_{4} \right\}
\]

where \( H \in R^{p \times q} \). Evidently, \( S \) is a linear subspace of \( K \).

**Theorem 2.** If we choose the initial matrix group \( \left[ X_{1}^{(0)}, X_{2}^{(0)}, X_{3}^{(0)}, X_{4}^{(0)} \right] \in S \), especially, let
\(X_i^{(0)} = 0 \ (i = 1, 2, 3, 4)\), we can obtain the least norm solution of Problem I.

**Proof**: By the Algorithm 1 and Theorem 1, if we choose initial matrix group \(\{X_1^{(0)}, X_2^{(0)}, X_3^{(0)}, X_4^{(0)}\} \in S\), we can obtain the solution \([\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4]\) of Problem I with finite iteration steps and there exist a matrix \(\hat{H} \in \mathbb{R}^{pq}\) such that the solution \([\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4]\) can be represented that

\[
\hat{X}_1 = A_i^T \hat{H} B_i^T,
\hat{X}_2 = \frac{1}{2} \left( A_i^T \hat{H} B_i^T + B_i \hat{H}^T A_i \right),
\hat{X}_3 = \frac{1}{2} \left( A_i^T \hat{H} B_i^T + S_{n_i} A_i^T \hat{H} B_i^T S_{n_i} \right),
\hat{X}_4 = \frac{1}{4} \left( A_i^T \hat{H} B_i^T + B_i \hat{H}^T A_i + S_{n_i} \left( A_i^T \hat{H} B_i^T + B_i \hat{H}^T A_i \right) S_{n_i} \right).
\]

By Lemma 6 we know that arbitrary solution of Problem I can be expressed as

\[
[\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4] \in S_\text{es}
\]

where matrix group \([\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4]\) satisfies (7).

Then

\[
\sum_{i=1}^{4} \langle \hat{X}_i, X_i \rangle = \langle A_i^T \hat{H} B_i^T, X_i \rangle + \frac{1}{2} \left( A_i^T \hat{H} B_i^T + B_i \hat{H}^T A_i, \hat{X}_2 \right) + \frac{1}{2} \left( A_i^T \hat{H} B_i^T + S_{n_i} A_i^T \hat{H} B_i^T S_{n_i}, \hat{X}_3 \right) + \frac{1}{4} \left( A_i^T \hat{H} B_i^T + B_i \hat{H}^T A_i + S_{n_i} \left( A_i^T \hat{H} B_i^T + B_i \hat{H}^T A_i \right) S_{n_i}, \hat{X}_4 \right)
\]

\[
= \langle \hat{H}, \sum_{i=1}^{4} A_i X_i B_i \rangle = 0.
\]

So we have

\[
\sum_{i=1}^{4} \left\| \hat{X}_i + X_i \right\| = \sum_{i=1}^{4} \left\| \hat{X}_i \right\| + \sum_{i=1}^{4} \left\| X_i \right\|
\]

which implies that matrix group \([\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4]\) is the least norm solution of Problem I. \(\square\)

**Remark 2.** Since the solution of Problem I is no empty, so the \(S_e\) is a closed convex linear subspace, hence it is certain that the least norm solution group \([\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4]\) of Problem I is unique, and \([\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4] \in S\). If matrix group \([X_1, X_2, X_3, X_4] \in S\) is a solution of Problem I, then it just be the unique least norm solution of Problem I, i.e. \(X_i = \hat{X}_i \ (i = 1, 2, 3, 4)\).

### 3. The Minimization Property of Iterative Method

In this section, the minimization property of Algorithm 1 is characterized, which ensures that the Algorithm 1 converges smoothly.

**Theorem 3.** For an arbitrary initial matrix group \([X_1^{(0)}, X_2^{(0)}, X_3^{(0)}, X_4^{(0)}] \in K\), the matrix group \([X_1^{(i)}, X_2^{(i)}, X_3^{(i)}, X_4^{(i)}] \in K\) generated by Algorithm 1 at the \(k\)th iteration step satisfies the following minimization problem

\[
\left\| A_i X_i B_i + \cdots + A_4 X_4 B_4 - C \right\| = \min_{(X_1, \ldots, X_4) \in F} \left\| A_i X_i B_i + \cdots + A_4 X_4 B_4 - C \right\|
\]

where \(F\) denote a affine subspace which has the following form
Proof: For arbitrary matrix group \([X_1, X_2, X_3, X_4] \in F\), there exist a set of real numbers \(\{t_i\}_{i=0}^{k-1}\) such that
\[
[X_1, \ldots, X_4] = \left[ X_1^{(0)}, \ldots, X_4^{(0)} \right] + \sum_{i=0}^{k-1} \left[ Q_{1,i}, \ldots, Q_{4,i} \right] B_i.
\]

Denote
\[
g(t_0, \ldots, t_{k-1}) = \left\| A_1 \left( X_1^{(0)} + \sum_{i=0}^{k-1} Q_{1,i} \right) B_1 + \cdots + A_4 \left( X_4^{(0)} + \sum_{i=0}^{k-1} Q_{4,i} \right) B_4 - C \right\|^2
\]
by the conclusion Equation (2) in Lemma 7, we have
\[
g(t_0, \ldots, t_{k-1}) = \left\| A_1 X_1^{(0)} B_1 + \cdots + A_4 X_4^{(0)} B_4 - C + \sum_{i=0}^{k-1} \left( A_i Q_{i,1} B_1 + \cdots + A_i Q_{i,4} B_4 \right) \right\|^2
\]
\[
= \left\| R_0 \right\|^2 + \sum_{i=0}^{k-1} \left\| A_i Q_{i,1} B_1 + \cdots + A_i Q_{i,4} B_4 \right\|^2 - 2 \sum_{i=0}^{k-1} \left\{ A_i Q_{i,1} B_1 + \cdots + A_i Q_{i,4} B_4, R_0 \right\}
\]
where \(R_0\) is the corresponding residual of initial matrix group \(\left[ X_1^{(0)}, X_2^{(0)}, X_3^{(0)}, X_4^{(0)} \right]\). Algorithm 1 show that the matrix \(R_0\) can be express as
\[
R_0 = R_t + \alpha_{i-1} \sum_{r=1}^{4} A_i Q_{r,i-1} B_r + \alpha_{i-2} \sum_{r=1}^{4} A_i Q_{r,i-2} B_r + \cdots + \alpha_{0} \sum_{r=1}^{4} A_i Q_{r,0} B_r.
\]

Because \(g(t_0, \ldots, t_{k-1})\) is a continuous and differentiable function with respect to the \(k\) variable \(t_i, (i = 0, 1, \cdots, k-1)\), we easily know that
\[
g(t_0, t_1, \ldots, t_{k-1}) = \min
\]
if and only if
\[
\frac{\partial g(t_0, t_1, \ldots, t_{k-1})}{\partial t_i} = 0.
\]

It follows from the conclusion in Lemma 7 that
\[
t_i = \frac{\sum_{r=1}^{4} \left\{ A_i Q_{r,i} B_r, R_0 \right\}}{\sum_{r=1}^{4} \left\| A_i Q_{r,i} B_r \right\|^2} = \frac{\sum_{r=1}^{4} \left\{ A_i Q_{r,i} B_r, R_0 \right\}}{\sum_{r=1}^{4} \left\| A_i Q_{r,i} \right\|^2} = \frac{\sum_{r=1}^{4} \left\| A_i Q_{r,i} B_r \right\|^2}{\sum_{r=1}^{4} \left\| A_i Q_{r,i} \right\|^2} = \alpha_i.
\]

By the fact that
\[
\min_{i} \left( t_0, t_1, \cdots, t_{k-1} \right) = \min_{[x_1, x_2, x_3, x_4] \in F} \left\| A_1 X_1 B_1 + \cdots + A_4 X_4 B_4 - C \right\|^2.
\]

We complete the proof.

Theorem 3 shows that the approximation solution \(\left[ X_1^{(k)}, X_2^{(k)}, X_3^{(k)}, X_4^{(k)} \right]\) minimizes the residual norm in the affine subspace \(F\) for all initial matrix group within \(K\). Furthermore, by the fact
\[
\left[ X_1^{(k-1)}, X_2^{(k-1)}, X_3^{(k-1)}, X_4^{(k-1)} \right] \in F, \text{ then we have}
\]
\[
\left\| A_1 X_1^{(k-1)} B_1 + \cdots + A_4 X_4^{(k-1)} B_4 - C \right\| \leq \left\| A_1 X_1^{(k-1)} B_1 + \cdots + A_4 X_4^{(k-1)} B_4 - C \right\|
\]
which shows that the sequence
\[
\left\| A_1 X_1^{(k)} B_1 + \cdots + A_4 X_4^{(k)} B_4 - C \right\|, \left\| A_1 X_1^{(k)} B_1 + \cdots + A_4 X_4^{(k)} B_4 - C \right\|, \ldots
\]
is monotonically decreasing. The descent property of the residual norm of Equation (5) ensures that the Algorithm 1 possesses fast and smoothly convergence.
4. Numerical Examples

In this section, we present numerical examples to illustrate the efficiency of the proposed iteration method. All the tests are performed using Matlab 7.0 which has a machine precision of around $10^{-16}$. Because of the error of calculation, the iteration will not stop within finite steps. Hence, we regard the approximation solution group $[X^{(i)}_1, X^{(i)}_2, X^{(i)}_3, X^{(i)}_4]$ as the solution of Problem I if the corresponding $P_{kr}(r = 1, 2, 3, 4)$ satisfies $\sum_{r=1}^{4} \|P_{kr}\|^2 \leq 10e-010$.

Example 1. Given matrices $A_1, B_1, A_2, B_2, A_3, B_3, A_4, B_4$ and $C$ as follows:

\[
A_1 = \begin{pmatrix}
1 & 1 & 5 & 2 & -3 & 0 \\
1 & 0 & 6 & 0 & 0 \\
3 & 12 & 7 & 6 & -12 & 3 \\
-2 & -2 & 2 & -2 & -6 & 4 \\
2 & 2 & -2 & 2 & 6 & -4 \\
0 & 0 & 0 & 0 & 0 & 0 \\
5 & 11 & 23 & 4 & -14 & 9
\end{pmatrix},
B_1 = \begin{pmatrix}
-9 & 4 & 4 & -9 & 6 \\
8 & 12 & 3 & 8 & 8 \\
5 & 7 & 0 & 5 & -12 \\
-5 & 15 & 0 & -3 & 20 \\
7 & 56 & 1 & 7 & 14 \\
-12 & 6 & 0 & -12 & -7
\end{pmatrix}
\]

\[
A_2 = \begin{pmatrix}
3 & 4 & 11 & -5 & 9 & 14 \\
3 & -2 & 6 & 5 & 4 & 12 \\
7 & 1 & 6 & 17 & -9 & 2.3 \\
12 & 2.9 & 8 & -6 & 3.8 & 44 \\
-7.9 & 3.2 & 4.3 & 9.1 & -9.2 & 8 \\
0 & -1.5 & 0 & 5 & 0 & 0 \\
12 & 2.9 & 8 & -6 & 3.8 & 44
\end{pmatrix},
B_2 = \begin{pmatrix}
5 & 0 & 0 & 5 & 2.4 \\
2 & 3 & 0 & 2 & 0 \\
23 & 6 & 0 & 23 & 0 \\
3 & 3 & 0 & 3 & 1 \\
12 & -15 & 0 & 12 & 0 \\
-10 & 9 & 0 & -10 & 0 \\
-14 & -11.4 & 0 & -14 & 8
\end{pmatrix}
\]

\[
A_3 = \begin{pmatrix}
4 & 4 & -5 & 4 & 4.2 \\
-1 & 12 & -2 & -1 & 21 \\
3 & 7 & 3 & 3 & 5 \\
-2 & 9 & 2 & -2 & -9 \\
5 & -8 & -10 & 5 & 12 \\
-9.1 & -9.4 & -12.5 & -9.1 & 7.6 \\
12 & 7 & 22 & 12 & 11 \\
-9 & 13 & 15.9 & -9 & -6.3
\end{pmatrix},
B_3 = \begin{pmatrix}
5 & 0 & 0 & 5 & 2.4 \\
2 & 3 & 0 & 2 & 0 \\
23 & 6 & 0 & 23 & 0 \\
3 & 3 & 0 & 3 & 1 \\
12 & -15 & 0 & 12 & 0 \\
-10 & 9 & 0 & -10 & 0 \\
-14 & -11.4 & 0 & -14 & 8
\end{pmatrix}
\]

\[
A_4 = \begin{pmatrix}
0 & -2 & 3 & 5 & 0 & -1 & -1 \\
-1 & -3 & 2 & 4 & -5 & 1 & -21 \\
0 & -2 & 3 & 5 & 0 & -1 & -14 \\
-1 & -3 & 2 & 4 & -5 & 1 & -21 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -3 & 2 & 1 & -5 & 1 & -2 \\
21 & -4.8 & 10.9 & 44.2 & 13.5 & 2.87 & 3.65
\end{pmatrix},
B_4 = \begin{pmatrix}
5 & 0 & 0 & 5 & 2.4 \\
2 & 3 & 0 & 2 & 0 \\
23 & 6 & 0 & 23 & 0 \\
3 & 3 & 0 & 3 & 1 \\
12 & -15 & 0 & 12 & 0 \\
-10 & 9 & 0 & -10 & 0 \\
-14 & -11.4 & 0 & -14 & 8
\end{pmatrix}
\]
Choose the initial matrices $X_i^{(0)} = 0 (i = 1, 2, 3, 4)$ where 0 denotes zero matrix in appropriate dimension. Using Algorithm 1 and iterating 74 steps, we have the unique least norm solution $\left[ X_1^{(74)}, X_2^{(74)}, X_3^{(74)}, X_4^{(74)} \right] \in K$ as follows:

$$
B_4 = \begin{pmatrix}
-2 & -2 & -4 & 0 & -2 \\
-1 & -1 & -4 & 0 & -1 \\
-2 & -6 & 0 & 0 & 0 \\
8 & 8 & -22 & 0 & 8 \\
4 & 6 & 0 & 4 & 0 \\
22 & 22 & 12 & 0 & 22 \\
21 & 21 & 21.8 & 0 & 21 \\
10.3 & 0 & 36.5 & 0 & 0
\end{pmatrix},
$$

$$
C = \begin{pmatrix}
-72 & -49 & 26 & 220 & 134 \\
342 & -67 & -50 & 251 & 39 \\
134 & 24.9 & 130 & -128 & 60 \\
190 & -28.6 & 225 & 41 & 120 \\
87 & 136 & 44 & 52 & 23 \\
-45 & 35 & -24.9 & 35.7 & -63 \\
41 & -96 & 35.9 & 64.8 & -66
\end{pmatrix}.
$$

$$X_1^{(74)} = \begin{pmatrix}
0.3488 & 0.3512 & -0.6731 & -0.8191 & 0.4135 & -0.3588 \\
1.4504 & 0.8483 & -2.2031 & -4.2978 & 1.2281 & -0.6214 \\
0.9954 & 0.5572 & -1.6690 & -1.5996 & 0.7736 & -0.4921 \\
0.6201 & 0.8376 & -1.4185 & -1.7915 & 0.6907 & -0.9980 \\
-1.5711 & -0.5939 & 2.2658 & 4.3840 & -0.7899 & 0.3719 \\
0.3831 & 0.1950 & -0.4999 & -0.8051 & 0.5968 & -0.0280
\end{pmatrix},$$

$$X_2^{(74)} = \begin{pmatrix}
0.0837 & 0.7782 & 0.1083 & -0.3906 & 0.3401 & 0.3817 & -0.1904 & 0.0314 \\
0.7782 & 0.7113 & 0.7072 & 0.1963 & -0.4179 & -0.3313 & 0.4221 & -0.5675 \\
0.1083 & 0.7072 & 0.1762 & -0.3788 & 0.2916 & 0.1869 & 0.2184 & -0.2188 \\
-0.3906 & 0.1963 & -0.3788 & -0.0473 & -0.0796 & 0.3926 & -0.2816 & 0.3870 \\
0.3401 & -0.4179 & 0.2916 & -0.0796 & 0.0976 & -0.4232 & -0.0532 & -0.2994 \\
0.3817 & -0.3313 & 0.1869 & 0.3926 & -0.4232 & 0.3104 & -0.0502 & -0.2933 \\
-0.1904 & 0.4221 & 0.2184 & -0.2816 & -0.0532 & -0.0502 & -0.1573 & 0.3146 \\
0.0314 & -0.5675 & -0.2188 & 0.3870 & -0.2994 & -0.2933 & 0.3146 & -0.8281
\end{pmatrix},$$

$$X_3^{(74)} = \begin{pmatrix}
-0.2550 & 0.0991 & 0.0210 & 0.0156 & -0.0884 & 0.0125 & -0.0180 \\
-0.2521 & -0.0117 & -0.0403 & 0.0352 & 0.2365 & -0.0112 & 0.2756 \\
0.8419 & -0.0292 & -0.0156 & 0.0419 & -0.2079 & 0.0161 & 0.1924 \\
0.3561 & 0.0560 & 0.0341 & 0.1414 & 0.0341 & 0.0560 & 0.3561 \\
0.1924 & 0.0161 & -0.2079 & 0.0419 & -0.0156 & -0.0292 & 0.8419 \\
0.2756 & -0.0112 & 0.2365 & 0.0352 & -0.0403 & -0.0117 & -0.2521 \\
-0.0180 & 0.0125 & -0.0884 & 0.0156 & 0.0210 & 0.0991 & -0.2550
\end{pmatrix},$$

$$X_4^{(74)} = \begin{pmatrix}
-0.2988 & 0.0807 & -0.0264 & 0.1130 & -0.1272 & 0.1788 & -0.0045 & 0.0002 \\
0.0807 & 0.1392 & 0.1494 & -0.1014 & 0.0934 & -0.0472 & -0.0271 & -0.0045 \\
-0.0264 & 0.1494 & 0.0308 & -0.1096 & -0.0239 & 0.0030 & -0.0472 & 0.1788 \\
0.1130 & -0.1014 & -0.1096 & -0.1113 & -0.2907 & -0.0239 & 0.0934 & -0.1272 \\
-0.1272 & 0.0934 & -0.0239 & -0.2907 & -0.1113 & -0.1096 & -0.1014 & 0.1130 \\
0.1788 & -0.0472 & 0.0030 & -0.0239 & -0.1096 & 0.0308 & 0.1494 & -0.0264 \\
-0.0045 & -0.0271 & -0.0472 & 0.0934 & -0.1014 & 0.1494 & 0.1392 & 0.0807 \\
0.0002 & -0.0045 & 0.1788 & -0.1272 & 0.1130 & -0.0264 & 0.0807 & -0.2988
\end{pmatrix}.
with
\[
\sum_{i=1}^{4} \left\| \mathbf{P}_{2,i} \right\| = 3.4778 \times 10^{-11}, \quad \text{and} \quad \sum_{i=1}^{4} \left\| \mathbf{X}^{(74)}_{2,i} \right\| = 14.0628.
\]

And
\[
\min_{[X_{1}, X_{2}, X_{3}, X_{4}] \in \mathcal{S}} \left\| A_1 X_1 B_1 + A_2 X_2 B_2 + A_3 X_3 B_3 + A_4 X_4 B_4 - C \right\| = 57.0635.
\]

If we let the initial matrix \( X^{(0)}_1 = -2I_6, X^{(0)}_2 = 5I_8, X^{(0)}_3 = I_5, X^{(0)}_4 = 3I_5 \), noting that \([-2I_6, 5I_8, I_5, 3I_5]\) within \( K \) but not within \( S \), then we have

\[
\mathbf{X}^{(82)}_1 = \begin{pmatrix}
-1.6182 & 0.5769 & -0.7270 & -0.5419 & 0.7736 & -0.5307 \\
1.6801 & -1.2937 & -2.4655 & -3.7133 & 0.3589 & -0.6108 \\
1.6186 & 0.9272 & -3.6464 & -1.7355 & 1.8356 & -0.1625 \\
0.6687 & 0.8635 & -1.6254 & -2.9306 & 0.7439 & -1.0791 \\
-2.2221 & 0.1884 & 2.4556 & 4.3019 & -1.1991 & 0.5606 \\
0.8887 & -0.0064 & -0.6313 & -1.1530 & 0.2455 & -1.5494
\end{pmatrix},
\]

\[
\mathbf{X}^{(82)}_2 = \begin{pmatrix}
4.3975 & 0.4156 & 0.0790 & -0.1835 & -0.1415 & 0.1494 & -0.2318 & 1.1825 \\
0.4156 & 4.8742 & 0.2354 & 0.7305 & -0.1579 & -1.9460 & -0.5984 & -0.4272 \\
0.0790 & 0.2354 & 4.9505 & -0.4095 & 0.6208 & 0.1762 & -0.4792 & -0.6346 \\
-0.1835 & 0.7305 & -0.4095 & 3.4749 & 1.1343 & 1.0277 & 0.7031 & -0.5118 \\
-0.1415 & -0.1579 & 0.6208 & 1.1343 & 4.2586 & -0.8962 & -0.6739 & 0.9514 \\
0.1494 & -1.9460 & 0.1762 & 1.0277 & -0.8962 & 2.2692 & 1.5568 & -0.3421 \\
-0.2318 & -0.5984 & -0.4792 & 0.7031 & -0.6739 & 1.5568 & 1.6349 & 0.1487 \\
1.1825 & -0.4272 & -0.6346 & -0.5118 & 0.9514 & -0.3421 & 0.1487 & 2.2651
\end{pmatrix},
\]

\[
\mathbf{X}^{(82)}_3 = \begin{pmatrix}
0.0400 & 0.4583 & -0.3783 & 0.2769 & 0.3650 & 0.2883 & -0.1037 \\
-0.2169 & 0.9919 & 0.0212 & 0.0438 & 0.2006 & 0.0163 & 0.1040 \\
0.4283 & 0.1148 & 0.7425 & 0.1080 & -0.0350 & 0.0991 & 0.2474 \\
0.3206 & 0.2510 & 0.0548 & 1.3436 & 0.0548 & 0.2510 & 0.3206 \\
0.2474 & 0.0991 & -0.0350 & 0.1080 & 0.7425 & 0.1148 & 0.4283 \\
0.1040 & 0.0163 & 0.2006 & 0.0438 & 0.212 & 0.9919 & -0.2169 \\
-0.1037 & 0.2883 & 0.3650 & 0.2769 & -0.3783 & 0.4583 & 0.0400
\end{pmatrix},
\]

\[
\mathbf{X}^{(82)}_4 = \begin{pmatrix}
0.6809 & -0.3258 & 0.3206 & -0.1912 & 0.7079 & 0.1287 & 0.2288 & 0.1271 \\
-0.3258 & 1.5242 & -0.7005 & -0.7838 & 0.3594 & -0.2261 & 0.1465 & 0.2288 \\
0.3206 & -0.7005 & 2.6931 & -0.4311 & -0.1002 & -0.4047 & -0.2261 & 0.1287 \\
-0.1912 & -0.7838 & -0.4311 & 2.4751 & -0.0815 & -0.1002 & 0.3594 & 0.7079 \\
0.7079 & 0.3594 & -0.1002 & -0.0815 & 2.4751 & -0.4311 & -0.7838 & -0.1912 \\
0.1287 & -0.2261 & -0.4047 & -0.1002 & -0.4311 & 2.6931 & -0.7005 & 0.3206 \\
0.2288 & 0.1465 & -0.2261 & 0.3594 & -0.7838 & -0.7005 & 1.5242 & -0.3258 \\
0.1271 & 0.2288 & 0.1287 & 0.7079 & -0.1912 & 0.3206 & -0.3258 & 0.6809
\end{pmatrix},
\]

with
\[
\sum_{i=1}^{4} \left\| \mathbf{P}_{3,i} \right\| = 5.9073 \times 10^{-11}, \quad \text{and} \quad \sum_{i=1}^{4} \left\| \mathbf{X}^{(82)}_i \right\| = 31.2518 > \sum_{i=1}^{4} \left\| \mathbf{X}^{(74)}_i \right\| = 14.0628.
\]

And
Example 2. Suppose that the matrices $A_1, B_1, A_2, B_2, A_3, B_3, A_4, B_4$ are the same as Example 1, let $C = A_i \tilde{X}_1 + A_i \tilde{X}_2 + A_i \tilde{X}_3 + A_i \tilde{X}_4$, where $\tilde{X}_1 = -2I_4, \tilde{X}_2 = 5I_4, \tilde{X}_3 = I_4, \tilde{X}_4 = 3I_4$, that is to say, Equation (5) is consistent over set $K$. Then similarly Algorithm 2.1 in Peng [14] we can conduct another iteration algorithm as follows:

**Algorithm 2.** For an arbitrary initial matrix group $[X_1^{(0)}, X_2^{(0)}, X_3^{(0)}, X_4^{(0)}] \in K$, compute

**Step 1.** $R_0 = C - \sum_{i=1}^{4} A_i X_i^{(0)} B_i$;
$$P_{0,1} = A_i^T R_0 B_i^T;$$
$$P_{0,2} = \frac{1}{2} \left( A_i^T R_0 B_i^T + B_i R_0^T A_i \right);$$
$$P_{0,3} = \frac{1}{2} \left( A_i^T R_0 B_i^T + A_i B_i^T R_0 S_{n_i} \right);$$
$$P_{0,4} = \frac{1}{4} \left( A_i^T R_0 B_i^T + B_i R_0^T A_i + S_{n_i} \left( A_i^T R_0 B_i^T + B_i R_0^T A_i \right) S_{n_i} \right);$$
$$Q_{0,r} = P_{0,r}, \quad r = 1, 2, 3, 4.$$

**Step 2.** If $R_k = 0$, then stop; else, $k := k + 1$, and compute

**Step 3.** $M_k = \sum_{i=1}^{4} A_i Q_{k,r} B_i$;
$$\alpha_k = \left\| R_k \right\|_F, \quad X_k^{(k+1)} = X_k^{(k)} + \alpha_k Q_{k,r}, \quad r = 1, 2, 3, 4;$$
$$R_{k+1} = R_k - \alpha_k M_k;$$
$$P_{k+1,1} = A_i^T R_{k+1} B_i^T = P_{k+1,1} - \alpha_k A_i^T M_k B_i^T;$$
$$P_{k+1,2} = \frac{1}{2} \left( A_i^T R_{k+1} B_i^T + B_i R_{k+1}^T A_i \right) = P_{k+2} - \frac{\alpha_k}{2} \left[ A_i^T M_k B_i^T + B_i M_k^T A_i \right];$$
$$P_{k+1,3} = \frac{1}{2} \left( A_i^T R_{k+1} B_i^T + A_i B_i^T R_{k+1} S_{n_i} \right) = P_{k+3} - \frac{\alpha_k}{2} \left[ A_i^T M_k B_i^T + A_i B_i^T M_k S_{n_i} \right];$$
$$P_{k+1,4} = \frac{1}{4} \left( A_i^T R_{k+1} B_i^T + B_i R_{k+1}^T A_i + S_{n_i} \left( A_i^T R_{k+1} B_i^T + B_i R_{k+1}^T A_i \right) S_{n_i} \right) = P_{k+4} - \frac{\alpha_k}{4} \left[ A_i^T M_k B_i^T + B_i M_k^T A_i + S_{n_i} \left( A_i^T M_k B_i^T + B_i M_k^T A_i \right) S_{n_i} \right];$$
$$\beta_k = \left\| R_{k+1} \right\|_F, \quad Q_{k+1,r} = P_{k+1,r} + \beta_k Q_{k,r}, \quad r = 1, 2, 3, 4.$$

**Step 4.** Go to step 2.

The main differences of Algorithm 1 and Algorithm 2 are: in Algorithm 1 the selection of coefficient $\alpha_k$ make $\left\| R_{k+1} \right\| = \min$, and $\beta_k$ such that $\left\{ \sum_{r=1}^{4} A_i Q_{k+1,r} B_r, \sum_{r=1}^{4} A_i Q_{k,r} B_r \right\} = 0$, but in Algorithm 2, the choosing of $\alpha_k$ such that $\left\{ R_{k+1}, R_{k} \right\} = 0$, and $\beta_k$ such that $\left\{ Q_{k+1,r}, Q_{k,r} \right\} = 0$. Noting that Algorithm 2 satisfies the Galerkin condition, but lacks of minimization property. Choosing the initial matrix $X_i^{(0)} = 0 (i = 1, 2, 3, 4)$ where 0 denotes zero matrix in appropriate dimension, by making use of Algorithm 1 and Algorithm 2, we can
obtain the same least norm solution group, and we also obtain the convergence curves of residual norm shown in Figure 1. The results in this figure show clearly that the residual norm of Algorithm 1 is monotonically decreasing, which is in accordance with the theory established in this paper, and the convergence curve is more smooth than that in Algorithm 2.

Acknowledgements

We thank the Editor and the referee for their comments. Research supported by the National Natural Science Foundation of China (11301107, 11261014, 11561015, 51268006).

References


http://dx.doi.org/10.1016/0024-3795(93)90328-L

http://dx.doi.org/10.1007/BF02253689


