Tight Monomials in Quantum Group for Type $A_5$ with $t \leq 6$

Yuwang Hu*, Guiwei Li, Jun Wang

College of Mathematics and Information Science, Xinyang Normal University, Xinyang, China
Email: *hywzrn@163.com

Received 5 June 2015; accepted 23 August 2015; published 26 August 2015

Copyright © 2015 by authors and Scientific Research Publishing Inc.
This work is licensed under the Creative Commons Attribution International License (CC BY).
http://creativecommons.org/licenses/by/4.0/

Abstract
All tight monomials in quantum group for type $A_5$ with $t \leq 6$ are determined in this paper.

Keywords
Quantum Group, Canonical Basis, Tight Monomial

1. Introduction
The term “quantum groups” was popularized by V. G. Drinfel’d in his address to the International Congress of Mathematicians (ICM) in Berkeley (1986). However, quantum groups are actually not groups; they are nontrivial deformations of the universal enveloping algebras of semisimple Lie algebras, also called quantized enveloping algebras. These algebras were introduced independently by Drinfel’d [1] (in his definition, these algebras were infinitesimal, i.e., they were Hopf algebras over the field of formal power series) and Jimbo [2] (in his definition, these algebras were Hopf algebras over the field of rational functions in one variable) in 1985 in their study of exactly solvable models in the statistical mechanics. Quantum groups play an important role in the study of Lie groups, Lie algebras, algebraic groups, Hopf algebras, etc.; they are also closely linked with conformal field theory, quiver theory and knot theory.

The positive part of a quantum group has a kind of important basis, i.e., canonical basis introduced by Lusztig [3], which plays an important role in the theory of quantum groups and their representations. However, it is difficult to determine the elements in canonical basis, which is interested in seeking the simplest elements in canonical basis, i.e., monomial basis elements. Some efforts have been done for monomial basis elements in quantum group of type $A_n$. Lusztig firstly introduced algebraic definition of canonical basis of quantum groups for the simply laced case (i.e., $A_n$, $D_n$, $E_6$), and gave explicitly the longest monomials for type $A_1$, $A_2$, which were all of canonical basis elements (see [3]). Then, Lusztig [4] associated a quadratic form to every monomial. He showed

How to cite this paper: Hu, Y.W., Li, G.W. and Wang, J. (2015) Tight Monomials in Quantum Group for Type $A_5$ with $t \leq 6$. Advances in Linear Algebra & Matrix Theory, 5, 63-75. http://dx.doi.org/10.4236/alamt.2015.53007
that, given certain linear conditions, the monomial was tight, i.e., it belonged to canonical basis (respectively, semitight, i.e., it was a linear combination of elements in canonical basis with constant coefficients in \( \mathbb{N} \)) provided that the quadratic form satisfied a certain positivity condition (respectively, nonnegativity condition). He showed that the positivity condition (for tightness) always held in type \( A_2 \) and computed 8 longest tight monomials of type \( A_2 \). He also asked when we had (semitightness in type \( A_n \). Based on Lusztig’s work, Xi [5] found explicitly all 14 canonical basis elements of type \( A_3 \) (consisting of 8 longest monomials and 6 polynomials with one-dimensional support). For type \( A_4 \), Hu, Ye and Yue [6] determined all 62 longest monomials in canonical basis, and Hu and Ye [7] gave all 144 polynomials with one-dimensional support in canonical basis. Li and Hu [8] got 112 polynomials with two-dimensional support. For type \( A_5 \), (n ≥ 5), Marsh [9] carried out thorough investigation. He presented a semitight longest monomial for type \( A_5 \). However, he proved that a class of special longest monomials did not satisfy sufficient condition of tightness or semitightness for type \( A_n \) (n ≥ 6) although it might turn out that the corresponding monomials were still tight. Reineke [10] associated a new quadratic form to every monomial, and gave a sufficient and necessary condition for the monomial to be tight if it is a linear combination of elements in canonical basis with constant coefficients in \( \mathbb{N} \). By use of this criterion, Wang [11] listed all tight monomials for type \( A_3 \), in which 8 longest tight monomials were the same as Lusztig and Xi’s results.

Based on Reineke’s criterion and some other results, all tight monomials for type \( A_5 \) with \( t \leq 6 \) are determined in this paper.

2. Preliminaries

Let \( C = \{c_{ij}\}_{i,j\in\Gamma_0} \) be a Cartan matrix of finite type, \( D = \text{diag}(d_i)_{i\in\Gamma_0} \) be a diagonal matrix with integer entries making the matrix \( DC \) symmetric. Let \( g = g(C) \) be the complex semisimple Lie algebra associated with \( C \), and let \( U = U_v^+(g) \) (here \( v \) is an indeterminate) be the corresponding quantized enveloping algebra, whose positive part \( U^+ \) is the \( \mathbb{Q}(v) \)-subalgebra of \( U \) generated by \( E_i, i\in\Gamma_0 \), subject to the relations

\[
\sum_{r,s\in\Gamma_0 \setminus \{i,j\}} (-1)^{r+s} E_i^{(r)} E_j^{(s)} = 0, \forall i,j \in \Gamma_0, \nonumber
\]

where \( E_i^{(r)} = E_i^r/s^r, s = [1],[2], \ldots, [s], a^{(r)} = (v^{a_d} - v^{-a_d})/(v^a - v^{-a}) \). Let \( A = \mathbb{Z}[v, v^{-1}] \); \( U^+ \) be the \( A \)-subalgebra of \( U^+ \) generated by \( E_i^{(r)}, \forall i \in \Gamma_0, \forall s \in \mathbb{N} \). Corresponding to every reduced expression \( i \) of the longest element of the Weyl group of \( g \), one constructs a PBW basis \( B_i \) of \( U^+ \). Lusztig proved that the \( \mathbb{Z}[v^{-1}] \)-lattice \( L_i \) spanned by \( B_i \) is independent of the choice of \( i \), write \( L \); and the image of \( B_i \) in the \( \mathbb{Z} \)-module \( L/v^{-1}L \) is a basis \( B \) of \( L/v^{-1}L \) independent of \( i \). Let \( L \) be the image of \( L \) under the bar map of \( U^+ \) defined by \( E_i \mapsto \bar{E}_i, i \in \Gamma_0 \) and \( v \mapsto v^{-1} \). Canonical basis \( \mathbf{B} \) is the preimage of \( B \) under \( \mathbb{Z} \)-module isomorphism \( L(L) \cong L/v^{-1}L \).

A monomial in \( U^+ \) is an element of the form

\[
E_{i_1}^{r_1}E_{i_2}^{r_2}\cdots E_{i_q}^{r_q} \tag{*}
\]

where \( i_1, i_2, \ldots, i_q \in \Gamma_0, a_1, a_2, \ldots, a_q \in \mathbb{N} \). When \( t = v, s, s_1 \cdots s_q = w_0 \) is the longest element of Weyl group, we say that \((*)\) is tight if it belongs to \( \mathbf{B} \); we say that \((*)\) is semitight if it is a linear combination of elements in \( \mathbf{B} \) with constant coefficients.

Let \( Q = (Q_0, Q_1) \) be a finite quiver with vertex set \( Q_0 \) and arrow set \( Q_1 \). Write \( \rho \in Q_1 \) as \( t_{\rho} \rightarrow h_{\rho} \), where \( h_{\rho} \) and \( t_{\rho} \) denote the head and the tail of \( \rho \) respectively. An automorphism \( \sigma \) of \( Q \) is a permutation on the vertices of \( Q \) and on the arrows of \( Q \) such that \( \sigma(h_{\rho}) = h_{\sigma(\rho)} \) and \( \sigma(t_{\rho}) = t_{\sigma(\rho)} \) for any \( \rho \in Q_1 \). Denote the quiver with automorphism \( \sigma \) as \( (Q, \sigma) \). Attach to the pair \( (Q, \sigma) \) a valued quiver \( \Gamma = \Gamma(Q, \sigma) = (\Gamma_0, \Gamma_1) \) as follows. Its vertex set \( \Gamma_0 \) and arrow set \( \Gamma_1 \) are simply the sets of \( \sigma \)-orbits in \( Q_0 \) and \( Q_1 \), respectively. The valuation of \( \Gamma \) is given by \( d_\rho = |\text{vertices in the } \sigma \text{-orbit of } \rho| \), \( \forall \rho \in \Gamma_1 \); \( m_\rho = |\text{arrows in the } \sigma \text{-orbit of } \rho| \), \( \forall \rho \in \Gamma_1 \). The Euler form of \( \Gamma \) is defined to be the bilinear form \( \langle \cdot, \cdot \rangle : \mathbb{Z}[\Gamma_0] \times \mathbb{Z}[\Gamma_0] \rightarrow \mathbb{Z} \) given by

\[
\langle X, Y \rangle = \sum_{i \in \Gamma_0} d_i x_i y_i - \sum_{\rho \in \Gamma_1} m_\rho x_{t_{\rho}} y_{h_{\rho}}, \nonumber
\]
where \( X = \sum_{i \in \Gamma_0} x_i, Y = \sum_{i \in \Gamma_0} y_i \in \mathbb{Z}[\Gamma_0] \), so \( X \cdot Y = \langle X,Y \rangle + \langle Y,X \rangle \) is the symmetric Euler form. The valued quiver \( \Gamma \) defines a Cartan matrix \( C_i = C_{i,a} = (c_{g})_{i,j \in \Gamma_0} \), where

\[
c_{ij} = \begin{cases} 
2 - 2 \sum_{\rho \in \Gamma_1} \frac{m_{\rho}}{d_i}, & i = j; \\
- \sum_{\rho \in \Gamma_1} \frac{m_{\rho}}{d_i}, & i \neq j. 
\end{cases}
\]

Let \( t \) be a non-negative integer. Let \( \mathbf{i} = (i_1, i_2, \cdots, i_t) \in \Gamma_0^t \) and \( \mathbf{a} = (a_1, a_2, \cdots, a_t) \in \mathbb{N}^t \). We write

\[ E_{\mathbf{i}}^{(\mathbf{a})} = E_{i_1}^{(a_1)} E_{i_2}^{(a_2)} \cdots E_{i_t}^{(a_t)} \in U^+. \]

Define

\[ \mathcal{M}_t = \{ A = (a_{im})_{i,m} \mid a_{im} \in \mathbb{N}, \text{ro}(A) = \text{co}(A) = \mathbf{a}, a_{im} = 0, \forall i, \neq i_m \}, \]

where

\[ \text{ro}(A) = \left( \sum_{m=1}^t a_{i_1 m}, \cdots, \sum_{m=1}^t a_{i_t m} \right), \text{co}(A) = \left( \sum_{m=1}^t a_{1 m}, \cdots, \sum_{m=1}^t a_{t m} \right). \]

Obviously, \( D_\mathbf{i} = \text{diag}(a_1, a_2, \cdots, a_t) \in \mathcal{M}_t \).

The following results are very useful in the determination of tight monomials.

**Theorem 2.1** [4] (Lusztig, 1993). Let \( U \) be the quantum group of type \( A_n, D_n, E_6, i \in \Gamma_0^t, \mathbf{a} \in \mathbb{N}^t \) as before. If the following quadratic form takes only values < 0 on \( \mathcal{M}_t \setminus \{ D_\mathbf{i} \} \), then monomial \( E_{\mathbf{i}}^{(\mathbf{a})} \) is tight.

\[ Q_{\mathbf{i}}^{(\mathbf{a})} (A) = \sum_{1 \leq m, i \leq t} a_{i m} a_{i m} - \sum_{1 \leq p < r \leq t} a_{i p} a_{i r}. \]

**Theorem 2.2** [10] (Reineke, 2001). Let \( U \) be the quantum group of type \( A_n, D_n, E_6, i \in \Gamma_0^t, \mathbf{a} \in \mathbb{N}^t \) as before, the monomial \( E_{\mathbf{i}}^{(\mathbf{a})} \) is tight if and only if the following quadratic form takes only values < 0 on \( \mathcal{M}_t \setminus \{ D_\mathbf{i} \} \)

\[ Q_{\mathbf{i}}^{t} (A) = \sum_{1 \leq m, i \leq t} a_{i m} a_{i m} + \sum_{1 \leq p < r \leq t} (i_i \cdot i_m) a_{i p} a_{i r} + \sum_{1 \leq p < r \leq t} a_{i p} a_{i r}. \]

If \( i_1, i_2, \ldots, i_t \) are mutually different, then \( \mathcal{M}_t = \{ D_\mathbf{i} \} \), by Theorem 2.2, we have the following Corollaries.

**Corollary 2.3.** When \( i_1, i_2, \ldots, i_t \) are mutually different, monomial \( E_{i_1}^{(a_1)} E_{i_2}^{(a_2)} \cdots E_{i_t}^{(a_t)} \) is tight.

**Corollary 2.4.** If \( E_{i_1}^{(a_{i_1})} E_{i_2}^{(a_{i_2})} \cdots E_{i_p}^{(a_{i_p})} \) is tight, then for any mutually different \( j_1, j_2, \ldots, j_p \) and any mutually different \( i_1, i_2, \ldots, i_p \) and \( p + t + q \leq l(w_0) \),

\[ E_{i_1}^{(a_{i_1})} E_{j_1}^{(a_{j_1})} E_{i_2}^{(a_{i_2})} \cdots E_{j_p}^{(a_{j_p})} E_{i_1}^{(a_{i_1})} E_{j_2}^{(a_{j_2})} \cdots E_{j_p}^{(a_{j_p})} \]

is also tight.

**Theorem 2.5** [12] (Deng-Du, 2010). Let \( \mathbf{i} = (i_1, i_2, \cdots, i_t) \in \Gamma_0^t \) and \( \mathbf{a} = (a_1, a_2, \cdots, a_t) \in \mathbb{N}^t \). If \( E_{\mathbf{i}}^{(\mathbf{a})} \) is tight, then

(a) For \( \forall 1 \leq r \leq s \leq t \), monomial \( E_{i_r}^{(a_r)} E_{i_{r+1}}^{(a_{r+1})} \cdots E_{i_s}^{(a_s)} \) is also tight;

(b) For \( \forall 1 \leq r < t \), \( i_r \neq i_{r+1} \).

**Theorem 2.6** [4] (Lusztig, 1993). Let \( \Phi \) be the non-trivial automorphism of \( U^+ \) induced by Dynkin diagram automorphism of \( g \), and \( \Psi: U^+ \rightarrow \left(U^+\right)^{opp} \) be the unique \( Q(v) \)-algebra isomorphism such that \( E_j \rightarrow E_j \).

If \( E_{\mathbf{i}}^{(\mathbf{a})} \) is tight, then \( \Phi \left( E_{\mathbf{i}}^{(\mathbf{a})} \right) \) and \( \Psi \left( E_{\mathbf{i}}^{(\mathbf{a})} \right) \) are all tight.
3. Main Results

Let \( \mathbf{i} = (i_1, i_2, \ldots, i_t) \in \Gamma_t^r \), \( \mathbf{a} = (a_1, a_2, \ldots, a_t) \in \mathbb{N}^t \). For convenience, we abbreviate a monomial \( E_{i_1}^{(a_1)}E_{i_2}^{(a_2)}\cdots E_{i_t}^{(a_t)} \) as a word \( i_1i_2\cdots i_t \) (1 as 0), an inequality \( a_{i_1} + \cdots + a_{i_t} \leq a_i + \cdots + a_q \) as \( j_1 + \cdots + j_p \leq l_1 + \cdots + l_q \). For example, a monomial \( E_{i_1}^{(a_1)}E_{i_2}^{(a_2)}E_{i_3}^{(a_3)}(a_1 + a_2 \leq a_3) \) is abbreviated to 121(1+3≤2), a monomial \( E_{i_1}^{(a_1)}E_{i_2}^{(a_2)}E_{i_3}^{(a_3)}E_{i_4}^{(a_4)} \) to 1234, etc.

By Theorem 2.5(b), we only consider those words \( i_1i_2\cdots i_t \) with \( i_1 \neq i_{t+1}, \forall 1 \leq r < t \) in determining tight monomials, in this case, we call \( i_1i_2\cdots i_t \) the word with \( t \)-value, \( E_{i_1}^{(a_1)}E_{i_2}^{(a_2)}\cdots E_{i_t}^{(a_t)} \) the monomial with \( t \)-value.

If \( i_1i_2\cdots i_t = 0 \) for some \( 1 \leq r < t \), we identify the word \( i_1i_2\cdots i_{t-1}i_{t+1} \) with the word \( i_1i_2\cdots i_{t-1}i_{t+1} \). Let us present the so called word-procedure for making the words with \( (t+1) \)-value from the words with \( t \)-value. Let \( i_1i_2\cdots i_t \) be a word with \( t \)-value, we firstly add a number \( 1 \leq i_{t+1} < 6 \) different from \( i_1 \) (or \( i_t \)) in the front (or behind) of \( i_1 \) (or \( i_t \)), secondly delete the words with \( t \)-value, lastly apply the automorphism \( \Phi \) and isomorphism \( \Psi \). After the above procedure put into practice for all the words with \( t \)-value, we get all words with \( (t+1) \)-value by deleting repeated words. For example, by applying the above word-procedure to the word 13 with 2-value, we get the words with 3-value as follows: 132, 134, 135, 143, 213, 235, 325, 354, 435.

Theorem 3.1. Let \( M_t \) be the set of all tight monomials with \( t \)-value in quantum group for type \( A_5 \), we have the following results.

(1) \( t = 0 \), \( M_0 = \{0\} \), tight monomial has only one;

(2) \( t = 1 \), if \( S_2 = \{1, 2, 3\} \), then \( M_1 = \Phi(S_2) \), tight monomials have 5 families;

(3) \( t = 2 \), if \( S_4 = \{12, 13, 14, 15, 23\} \), then \( M_2 = \Psi\Phi(S_4) \), tight monomials have 14 families;

(4) \( t = 3 \), if \( S_6 = S_1^1 \cup S_1^2 \), where

\[
S_1^1 = \{123, 124, 125, 132, 134, 135, 234, 243\}, \quad S_1^2 = \{121, 212, 232, 323(1 + 3 \leq 2)\},
\]

then \( M_3 = \Psi\Phi(S_6) \), tight monomials have 33 families;

(5) \( t = 4 \), if \( S_8 = \bigcup_{i=0}^6 S_4^i \), where

\[
S_4^0 = \{1234, 1235, 1243, 1245, 1254, 1324, 1325, 1432\}, \quad S_4^1 = \{1213, 1214, 1215, 2123, 2124, 2125, 2321, 2324, 2325, 3231, 3234, 3235(1 + 3 \leq 2)\}, \quad S_4^2 = \{2132, 3243(1 + 4 \leq 2 + 3)\},
\]

then \( M_4 = \Psi\Phi(S_8) \), tight monomials have 67 families;

(6) \( t = 5 \), if \( S_8 = \bigcup_{i=0}^7 S_4^i \), where

\[
S_4^0 = \{12345, 12354, 12435, 12543, 13254, 14325\}, \quad S_4^1 = \{12134, 12135, 12143, 12145, 21234, 21243, 21245, 21241, 23245, 32341, 32345(1 + 3 \leq 2)\}, \quad S_4^2 = \{12324, 12325, 13234, 13235, 42325, 43235(2 + 4 \leq 3)\}, \quad S_4^3 = \{21324, 21325, 32431(1 + 4 \leq 2 + 3)\}, \quad S_4^4 = \{12321, 23432, 32123(1 + 5 \leq 2 + 4, 2 + 4 \leq 3)\}, \quad S_4^5 = \{12132, 23243, 32312(1 + 3 \leq 2, 2 + 5 \leq 3 + 4)\}, \quad S_4^6 = \{21322, 32343(1 + 3 \leq 2, 3 + 5 \leq 4)\}, \quad S_4^7 = \{31231, 42342(2 + 5 \leq 3, 1 + 4 \leq 3)\},
\]
then \( M_t = \Psi \Phi(S_t) \), tight monomials have 125 families;
if \( t = 6 \), \( S_6 = \bigcup_{i=5}^{17} S_i \), where
\[
S_6^0 = \{123245,123254,132354,532341,521234,132345,523241,321245(2 + 4 \leq 3)\},
\]
\[
S_6^1 = \{121345,121354,121435,121543,212345,212435,232451,323145(1 + 3 \leq 2)\},
\]
\[
S_6^2 = \{213245,324351(1 + 4 \leq 2 + 3)\},
\]
\[
S_6^3 = \{521324,132435(2 + 5 \leq 3 + 4)\},
\]
\[
S_6^4 = \{121434,121344,121454,121544,212433,212543,212453,212543(1 + 3 \leq 2, 4 + 6 \leq 5)\},
\]
\[
S_6^5 = \{132343,532343,523213,421323,423123(2 + 4 \leq 3, 4 + 6 \leq 5)\},
\]
\[
S_6^6 = \{123214,123215,234321,234325,321243,321243(1 + 5 \leq 2 + 4, 2 + 4 \leq 3)\},
\]
\[
S_6^7 = \{312314,312315,423421(1 + 4 \leq 3, 2 + 5 \leq 3)\},
\]
\[
S_6^8 = \{123243,532312,523243,412132(2 + 4 \leq 3, 3 + 6 \leq 4 + 5)\},
\]
\[
S_6^9 = \{121324,121325,232431,232435,323124,323124(1 + 3 \leq 2, 2 + 5 \leq 3 + 4)\},
\]
\[
S_6^{10} = \{132431,421324(1 + 6 \leq 3, 2 + 5 \leq 3 + 4)\},
\]
\[
S_6^{11} = \{213243(1 + 4 \leq 2 + 3, 6 + 4 \leq 5)\},
\]
\[
S_6^{12} = \{213234(1 + 4 \leq 2 + 3, 3 + 4 \leq 5)\},
\]
\[
S_6^{13} = \{321234,432354(1 + 6 \leq 2 + 4, 5 + 2 \leq 3)\},
\]
\[
S_6^{14} = \{123212,432343,321232(2 + 4 \leq 3, 4 + 6 \leq 5, 1 + 5 \leq 2 + 4)\},
\]
\[
S_6^{15} = \{121321,232432,323123(1 + 3 \leq 2, 3 + 6 \leq 5, 2 + 5 \leq 3 + 4)\},
\]
\[
S_6^{16} = \{121321,232432,323123(1 + 4 \leq 3, 2 + 5 \leq 3 + 6 \leq 4 + 5)\},
\]

then \( M_6 = \Psi \Phi(S_6) \), tight monomials have 222 families;

4. Proof of Theorem 3.1

Consider the quiver \( Q = (Q_0, Q_\sigma) \) of type \( A_5 \), where \( Q_1 = \{1 \to 2, 2 \to 3, 3 \to 4, 4 \to 5\} \), \( Q_0 = \{1, 2, 3, 4, 5\} \). Let \( \sigma = \text{id} \) be the identity automorphism of \( Q \), then valued quiver of \( (Q, \sigma) \) is \( \Gamma = (\Gamma_0, \Gamma_1) = (Q_0, Q_\sigma) = Q \). The valuation is given by \( d_i = d_2 = d_3 = d_4 = d_5 = 1 \), \( m_{\rho_1} = m_{\rho_2} = m_{\rho_3} = m_{\rho_4} = 1 \). Euler form \( \langle \cdot, \cdot \rangle \) on \( Q = \Gamma \) is
\[
\langle X, Y \rangle = \sum_{i=1}^{5} d_i x_i y_i - \sum_{i=1}^{4} m_{\rho_i} x_i y_{\rho_i} = \sum x_i y_i - \sum x_i y_{i+1},
\]
Symmetric Euler form \( \cdot \cdot \cdot \) on \( Q = \Gamma \) is
\[
X \cdot Y = \langle X, Y \rangle + \langle Y, X \rangle = \sum_{i=1}^{5} d_i x_i y_i - \sum_{i=1}^{4} m_{\rho_i} x_i y_{\rho_i} - \sum x_i y_{i+1},
\]
where \( X = \sum_{i=1}^{m} x_i, Y = \sum_{i=1}^{n} y_i \in \mathbb{Z}[\Gamma] \).

By simple computation, we have
\[
\langle i, i \rangle = 1, i \cdot i = 2 \ (i = 1, 2, 3, 4, 5), \quad i \cdot (i + 1) = 1 (i = 1, 2, 3, 4), \text{ and } 1 \cdot 3 = 4 = 1 \cdot 5 = 2 \cdot 4 = 2 \cdot 5 = 3 \cdot 5 = 0. \]

Let us prove Theorem 3.1.

Case 1. \( t \leq 2 \). By Corollary 2.3, monomials with \( t \leq 2 \) are all tight.

Case 2. \( t = 3 \). Applying the word-procedure on \( S_2 \), we get 33 words with 3-value. By considering \( \Phi \) and \( \Psi \), we get \( S_3 \). By Corollary 2.3, monomials in \( S_3 \) are all tight. For \( S_3 \), it suffices to consider \( i = (1,2,1) \). For any \( a = (a_1, a_2, a_3) \in \mathbb{N}^3 \), we have \( \mathcal{M}_{a} = \{ A | 0 \leq x \leq \min \{a_1, a_2, a_3 \} \} \), where

\[
A_x = \begin{pmatrix}
    a_1 - x & 0 & x \\
    0 & a_2 & 0 \\
    x & 0 & a_3 - x
\end{pmatrix}
\]

and
\[
q(A_x) = \sum_{1 \leq m \leq c \leq 3} a_{pm}a_{rm} + \sum_{1 \leq l \leq c \leq 3} (i \cdot i_m)a_{pm}a_{rl} + \sum_{1 \leq m \leq c \leq 3} a_{rm}a_{rl}
\]
\[
= a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} + a_{21}a_{31} + a_{33}a_{33} + (i_1 \cdot i_2)a_{22}a_{31} + (i_2 \cdot i_3)a_{13}a_{32} + (i_3 \cdot i_1)a_{13}a_{31}
\]
\[
= 2x(a_1 - x) + 2x(a_2 - x) - 2a_3x + 2x^2
\]
\[
= -2x^2 + 2(a_1 + a_3 - a_2)x.
\]

Obviously, \( q(A_x) < 0 \) if and only if \( a_1 + a_3 < a_2 \). So monomial \( E_1^{(a_1)} E_2^{(a_2)} E_3^{(a_3)} (a_1 + a_3 \leq a_2) \) is tight by Theorem 2.2.

Case 3. \( t = 4 \). Applying the word-procedure on \( S_3 \), we get 75 words with 4-value. By considering \( \Phi \) and \( \Psi \), we get \( S_3 \cup \{12, 23, 23\} \). When \( i \in \{(1,2,1,2),(2,3,2,3)\} \), for any \( a = (a_1, a_2, a_3, a_4) \in \mathbb{N}^4 \), we have \( \mathcal{M}_{a} = \{ A | 0 \leq x \leq \min \{a_1, a_2, a_3, a_4 \} \} \), where

\[
A_{x,y} = \begin{pmatrix}
    a_1 - x & 0 & x & 0 \\
    0 & a_2 - y & 0 & y \\
    x & 0 & a_3 - x & 0 \\
    0 & y & 0 & a_4 - y
\end{pmatrix}
\]

and
\[
q(A_{x,y}) = \sum_{1 \leq m \leq c \leq 4} a_{pm}a_{rm} + \sum_{1 \leq l \leq c \leq 4} (i \cdot i_m)a_{pm}a_{rl} + \sum_{1 \leq m \leq c \leq 4} a_{rm}a_{rl}
\]
\[
= a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} + a_{14}a_{34} + (i_1 \cdot i_2)a_{22}a_{31} + (i_2 \cdot i_3)a_{13}a_{32} + (i_3 \cdot i_4)a_{13}a_{31}
\]
\[
+ (i_1 \cdot i_3)a_{22}a_{31} + a_{11}a_{21} + a_{22}a_{32} + a_{33}a_{33} + a_{44}a_{42}
\]
\[
= 2x(a_1 - x) + 2y(a_2 - y) + 2x(a_3 - x) + 2y(a_4 - y) - 2a_1x + 2x^2 - 2a_3y + 2xy + 2y^2
\]
\[
= 2(a_1 + a_3 - a_2)x + 2(a_2 + a_4 - a_3)y - (x - y)^2 - x^2 - y^2.
\]

Obviously, \( q(A_{x,y}) < 0 \) if and only if \( a_1 + a_3 < a_2, a_2 + a_4 < a_3 \), this is a contradiction. Applying \( \Phi, \Psi \), one gets that the monomials corresponding to
\( i \in \{(1,2,1,2),(2,1,2,1),(2,3,2,3),(3,2,3,2),(3,4,3,4),(4,3,4,3),(4,5,4,5),(5,4,5,4)\} \).
are all not tight for any \( \mathbf{a} = (a_1, a_2, a_3, a_4) \in \mathbb{N}^4 \).

Monomials in \( S_1^3 \) are all tight by Corollary 2.3. By \( S_2^3 \) and Corollary 2.4, monomials in \( S_1^4 \) are all tight. For \( S_1^5 \), it suffices to consider \( i = (2, 1, 3, 2) \). For any \( \mathbf{a} = (a_1, a_2, a_3, a_4) \in \mathbb{N}^4 \), we have
\[
\mathcal{M}_{\mathbf{a}} = \left\{ A_{i}\mid 0 \leq x \leq \min \{a_1, a_3\} \right\},
\]
where
\[
A_x = \begin{pmatrix}
a_1 - x & 0 & 0 & x \\
0 & a_2 & 0 & 0 \\
0 & 0 & a_3 & 0 \\
x & 0 & 0 & a_4 - x
\end{pmatrix}
\]
and
\[
g(A_x) = \sum_{i \in 0, 1} a_{pm}a_{m} + \sum_{i \in 0, 1} (i_1 \cdot i_2)a_{pm}a_{m} + \sum_{i \in 0, 1} a_{pm}a_{m}
\]
\[
= a_1a_2a_4 + a_2a_4a_5 + a_3a_4a_5 + (i_1i_2)a_2a_4a_5 + (i_1i_2)\mathbb{a}_{a_1a_3a_5}
\]
\[
+ (i_1i_2)\mathbb{a}_{a_1a_3a_5} + a_2a_4a_5 + (i_1i_2)a_2a_4a_5 + (i_1i_2)a_2a_4a_5
\]
\[
= 2x(a_1 - x) + 2x(a_2 - y) + 2x(a_3 - x) - a_4x + 2x^2 - a_4y - a_4x
\]
\[
= 2(a_1 + a_3 - a_4 - a_5)x + 2(a_2 + a_3 - a_4)y - 2x^2 - 2y^2.
\]

Case 4. \( t = 5 \). Applying the word-procedure on \( S_1^5 \), and deleting words including subwords \( 1212, 2121, 2323, 3232, 3434, 4334, 4545 \) and \( 5454 \) (considering Theorem 2.5(a)), we get 125 words with 5-value. By considering \( \Phi \) and \( \Psi \), we get \( S_2^5 \). By Corollary 2.3, monomials in \( S_1^5 \) are all tight. Monomials in \( S_2^5 \) and Corollary 2.4. Monomials in \( S_3^5 \) are all tight by \( S_2^5 \) and Corollary 2.4.

For \( S_4^5 \), it suffices to consider \( i = (1, 2, 3, 2, 1) \). For any \( \mathbf{a} = (a_1, a_2, a_3, a_4, a_5) \in \mathbb{N}^5 \), we have
\[
\mathcal{M}_{\mathbf{a}} = \left\{ A_{i,y}\mid 0 \leq x \leq \min \{a_1, a_3\}, 0 \leq y \leq \min \{a_2, a_4\} \right\},
\]
where
\[
A_{x,y} = \begin{pmatrix}
a_1 - x & 0 & 0 & 0 & x \\
0 & a_2 - y & 0 & y & 0 \\
0 & 0 & a_3 & 0 & 0 \\
y & 0 & 0 & a_5 - y & 0 \\
x & 0 & 0 & 0 & a_5 - x
\end{pmatrix}
\]
and
\[
g(A_{x,y}) = \sum_{i \in 0, 1} a_{pm}a_{m} + \sum_{i \in 0, 1} (i_1 \cdot i_2)a_{pm}a_{m} + \sum_{i \in 0, 1} a_{pm}a_{m}
\]
\[
= a_1a_2a_3a_5 + a_2a_3a_5 + a_3a_5 + a_2a_3a_5 + a_2a_3a_5 + a_2a_3a_5 + a_2a_3a_5 + a_2a_3a_5 + a_2a_3a_5
\]
\[
+ (i_1i_2)\mathbb{a}_{a_1a_2a_3a_5} + (i_1i_2)\mathbb{a}_{a_1a_2a_3a_5} + (i_1i_2)\mathbb{a}_{a_1a_2a_3a_5} + (i_1i_2)\mathbb{a}_{a_1a_2a_3a_5}
\]
\[
+ (i_1i_2)\mathbb{a}_{a_1a_2a_3a_5} + (i_1i_2)\mathbb{a}_{a_1a_2a_3a_5} + (i_1i_2)\mathbb{a}_{a_1a_2a_3a_5} + (i_1i_2)\mathbb{a}_{a_1a_2a_3a_5}
\]
\[
= 2x(a_1 - x) + 2y(a_2 - y) + 2x(a_3 - x) - a_4x + 2x^2 - a_4y - a_4x
\]
\[
= 2(a_1 + a_3 - a_4 - a_5)x + 2(a_2 + a_3 - a_4)y - 2x^2 - 2y^2.
\]
\[ q(A_{x,y}) < 0 \] if and only if \[ a_1 + a_5 \leq a_2 + a_4, a_2 + a_4 \leq a_5. \] So
\[ E_1^{(a_1)}E_2^{(a_2)}E_3^{(a_3)}E_4^{(a_4)}E_5^{(a_5)} \left( a_1 + a_5 \leq a_2 + a_4, a_2 + a_4 \leq a_5 \right) \]
is tight by Theorem 2.2.

For \( S_5^6 \), it suffices to consider \( \mathbf{i} = (1, 2, 1, 3, 2) \). For any \( \mathbf{a} = (a_1, a_2, a_3, a_4, a_5) \in \mathbb{N}^5 \), we have
\[ M_{k,a} = \left\{ A_{x,y} \left| 0 \leq x \leq \min\{a_1, a_5\}, 0 \leq y \leq \min\{a_2, a_4\} \right\} \],
where

\[
A_{x,y} = \begin{pmatrix}
a_1 - x & 0 & x & 0 & 0 \\
0 & a_2 - y & 0 & 0 & y \\
x & 0 & a_3 - x & 0 & 0 \\
0 & 0 & 0 & a_4 & 0 \\
0 & y & 0 & 0 & a_5 - y
\end{pmatrix}
\]

and

\[
q(A_{x,y}) = \sum_{1 \leq p < r \leq 5} a_{pm}a_{rn} + \sum_{1 \leq p < r < s \leq 5} (i_r \cdot i_p) a_{pm}a_{rs} + \sum_{1 \leq r < s < t \leq 5} a_{rw}a_{st} = a_1a_3 + a_2a_5 + a_1a_3 + a_2a_5 + a_3a_5 + a_4a_2 + a_1a_3 + a_2a_5 + a_3a_5 + a_4a_2 + (i_1 \cdot i_2)a_{23}a_{31} + (i_1 \cdot i_3)a_{24}a_{43} + (i_2 \cdot i_3)a_{25}a_{52} + (i_1 \cdot i_2)a_{23}a_{31} + (i_1 \cdot i_3)a_{24}a_{43} + (i_2 \cdot i_3)a_{25}a_{52} + (i_1 \cdot i_2)a_{23}a_{31} + (i_1 \cdot i_3)a_{24}a_{43} + (i_2 \cdot i_3)a_{25}a_{52} \]
\[
= 2x(a_1 - x) + 2y(a_2 - y) + 2x(a_1 - x) + 2y(a_2 - y) - (a_2 - y)x + 2x^2 - xy - a_2x - (a_3 - x)y - a_4y + 2y^2 - (a_1 - x)y - a_4y + 2(a_1 + a_3 - a_2)x + 2(a_2 + a_5 - a_3 - a_4)y - (x - y)^2 - x^2 - y^2.
\]

\[ q(A_{x,y}) < 0 \] if and only if \[ a_1 + a_5 \leq a_2 + a_4, a_2 + a_4 \leq a_5. \] So
\[ E_1^{(a_1)}E_2^{(a_2)}E_3^{(a_3)}E_4^{(a_4)}E_5^{(a_5)} \left( a_1 + a_5 \leq a_2 + a_4, a_2 + a_4 \leq a_5 \right) \]
is tight by Theorem 2.2.

For \( S_7^7 \), it suffices to consider \( \mathbf{i} = (2, 1, 2, 3, 2) \). For any \( \mathbf{a} = (a_1, a_2, a_3, a_4, a_5) \in \mathbb{N}^5 \), we have
\[ M_{k,a} = \left\{ A = A_{x,y,z} \left| \text{entries in matrix are all non-negative integer} \right\} \],
where
\[
A = \begin{pmatrix}
a_1 - x - x_1 & 0 & x & 0 & x_1 \\
0 & a_2 & 0 & 0 & 0 \\
x_2 & 0 & a_3 - x_1 & 0 & x_3 \\
0 & 0 & 0 & a_4 & 0 \\
x + x_1 - x_2 & 0 & x_2 + x_3 - x & 0 & a_5 - x_1 - x_3
\end{pmatrix}
\]

and
\[ q(A) = \sum_{1 \leq m < 5} a_{pm} a_{mp} + \sum_{1 \leq p < m \leq 5} (i_j \cdot i_m) a_{pm} a_{mp} + \sum_{1 \leq m \leq 5} a_{pm} a_{mp} \]

\[ = a_{11} a_{31} + a_{12} a_{31} + a_{13} a_{31} + a_{14} a_{31} + a_{15} a_{31} + a_{23} a_{31} + a_{33} a_{31} + a_{43} a_{31} + a_{53} a_{31} + a_{11} a_{32} + a_{12} a_{32} + a_{13} a_{32} + a_{14} a_{32} + a_{15} a_{32} + a_{23} a_{32} + a_{33} a_{32} + a_{43} a_{32} + a_{53} a_{32} + (i_1 \cdot i_2) (a_{22} a_{31} + a_{24} a_{31}) + (i_1 \cdot i_3) (a_{32} a_{31} + a_{33} a_{31}) + (i_1 \cdot i_4) a_{42} a_{31} + (i_1 \cdot i_5) a_{52} a_{31} + (i_2 \cdot i_3) a_{33} a_{22} + (i_2 \cdot i_4) a_{43} a_{22} + (i_2 \cdot i_5) a_{53} a_{22} + (i_3 \cdot i_4) a_{44} a_{32} + (i_3 \cdot i_5) a_{54} a_{32} + (i_4 \cdot i_5) a_{55} a_{32} + (i_4 \cdot i_3) a_{45} a_{32} + (i_5 \cdot i_3) a_{36} a_{22} + (i_5 \cdot i_4) a_{46} a_{22} + (i_5 \cdot i_2) a_{26} a_{22} + (i_5 \cdot i_1) a_{16} a_{22} + (i_1 \cdot i_4) a_{44} a_{41} + (i_2 \cdot i_4) a_{44} a_{41} + (i_3 \cdot i_4) a_{44} a_{41} + (i_4 \cdot i_4) a_{44} a_{41} + (i_5 \cdot i_4) a_{44} a_{41} + (i_1 \cdot i_3) a_{33} a_{41} + (i_2 \cdot i_3) a_{33} a_{41} + (i_3 \cdot i_3) a_{33} a_{41} + (i_4 \cdot i_3) a_{33} a_{41} + (i_5 \cdot i_3) a_{33} a_{41} + (i_1 \cdot i_2) a_{22} a_{41} + (i_2 \cdot i_2) a_{22} a_{41} + (i_3 \cdot i_2) a_{22} a_{41} + (i_4 \cdot i_2) a_{22} a_{41} + (i_5 \cdot i_2) a_{22} a_{41} \]

\[ = 2(a_1 + a_3 - a_2) + 2(a_1 + a_3 + a_5 - a_4) x_3 + 2(a_1 + a_3 - a_4) x_3 - x^2 - (x - x_1)^2 - x_2^2 - 2x_3^2 - 2x_4 x_3 - 2x_5 x_3 - 2x_6 x_3 - 2x_7 x_3 - 2x_8 x_3 - 2x_9 x_3. \]

\[ q(A) < 0 \quad \text{if and only if} \quad a_1 + a_3 \leq a_2, a_3 \leq a_4. \]

is tight by Theorem 2.2.

For \( S_5 \), it suffices to consider \( i = (3, 1, 2, 3, 1) \). For any \( a = (a_1, a_2, a_3, a_4, a_5) \in \mathbb{N}^5 \), we have

\[ M_{h,a} = \left\{ A_{i,y} \middle| 0 \leq x \leq \min \{a_1, a_4\}, 0 \leq y \leq \min \{a_2, a_5\} \right\}, \]

where

\[
A_{i,y} = \begin{pmatrix}
  a_i - x & 0 & 0 & x & 0 \\
  0 & a_2 - y & 0 & 0 & y \\
  0 & 0 & a_3 & 0 & 0 \\
  x & 0 & 0 & a_4 - x & 0 \\
  0 & y & 0 & 0 & a_5 - y
\end{pmatrix}
\]

and

\[ q(A_{i,y}) = \sum_{1 \leq m < 5} a_{pm} a_{mp} + \sum_{1 \leq p < m \leq 5} (i_j \cdot i_m) a_{pm} a_{mp} + \sum_{1 \leq m \leq 5} a_{pm} a_{mp} \]

\[ = a_{11} a_{41} + a_{12} a_{41} + a_{13} a_{41} + a_{14} a_{41} + a_{15} a_{41} + a_{22} a_{41} + a_{33} a_{41} + a_{44} a_{41} + a_{55} a_{41} + (i_1 \cdot i_2) a_{22} a_{41} + (i_1 \cdot i_3) a_{33} a_{41} + (i_1 \cdot i_4) a_{44} a_{41} + (i_1 \cdot i_5) a_{55} a_{41} + (i_2 \cdot i_3) a_{33} a_{22} + (i_2 \cdot i_4) a_{44} a_{22} + (i_2 \cdot i_5) a_{55} a_{22} + (i_3 \cdot i_4) a_{44} a_{41} + (i_3 \cdot i_5) a_{55} a_{41} + (i_4 \cdot i_5) a_{55} a_{41} + (i_4 \cdot i_3) a_{44} a_{41} + (i_5 \cdot i_3) a_{33} a_{22} + (i_5 \cdot i_4) a_{44} a_{22} + (i_5 \cdot i_2) a_{22} a_{41} + (i_5 \cdot i_1) a_{11} a_{41} + (i_1 \cdot i_4) a_{44} a_{41} + (i_2 \cdot i_4) a_{44} a_{41} + (i_3 \cdot i_4) a_{44} a_{41} + (i_4 \cdot i_4) a_{44} a_{41} + (i_5 \cdot i_4) a_{44} a_{41} + (i_1 \cdot i_3) a_{33} a_{22} + (i_2 \cdot i_3) a_{33} a_{22} + (i_3 \cdot i_3) a_{33} a_{22} + (i_4 \cdot i_3) a_{33} a_{22} + (i_5 \cdot i_3) a_{33} a_{22} + (i_1 \cdot i_2) a_{22} a_{41} + (i_2 \cdot i_2) a_{22} a_{41} + (i_3 \cdot i_2) a_{22} a_{41} + (i_4 \cdot i_2) a_{22} a_{41} + (i_5 \cdot i_2) a_{22} a_{41} \]

\[ = 2(a_1 + a_4 - a_2) x + 2(a_1 + a_4 - a_3) y - 2x^2 - 2y^2 \]

\[ q(A_{i,y}) < 0 \quad \text{if and only if} \quad a_1 + a_4 \leq a_2, a_4 \leq a_3. \]

is tight by Theorem 2.2.

Case 5. \( t = 6 \). Applying the word-procedure on \( S_6 \), and deleting words including subwords 1212, 2121, 2323, 3232, 3434, 4343, 4545 and 5454(considering Theorem 2.5(a)), we get 228 words with 6-value. By considering \( \Phi \) and \( \Psi \), we get \( S_6 \bigcup \{121323, 232434\} \). When \( i \in \{(1,2,1,3,2,3), (2,3,2,4,3,4)\} \), for any \( a = (a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{N}^6 \), we have

\[ M_{h,a} = \left\{ A_{i,y,z} \middle| 0 \leq x \leq \min \{a_1, a_4\}, 0 \leq y \leq \min \{a_2, a_5\}, 0 \leq z \leq \min \{a_4, a_6\} \right\}, \]

where
\[
A_{x,y,z} = \begin{pmatrix}
a_t - x & 0 & x & 0 & 0 & 0 \\
0 & a_z - y & 0 & y & 0 & 0 \\
x & 0 & a_3 - x & 0 & 0 & 0 \\
0 & 0 & 0 & a_z - z & 0 & z \\
0 & 0 & 0 & 0 & a_z - y & 0 \\
0 & 0 & 0 & 0 & 0 & a_b - z
\end{pmatrix}
\]

and

\[
q(A_{x,y,z}) = \sum_{1 \leq p < r < m \leq 6} a_{pr}a_{rm} + \sum_{1 \leq p \leq 6 \leq m \leq 6} (i \cdot j) a_{pm}a_{rl} + \sum_{1 \leq p \leq m \leq 6} a_{pm}a_{rl}
\]

\[
= a_{11}a_{31} + a_{22}a_{52} + a_{33}a_{33} + a_{44}a_{64} + a_{55}a_{55} + a_{66}a_{66} + a_{11}a_{13} + a_{22}a_{25} + a_{33}a_{33} + a_{44}a_{46} + a_{55}a_{58} + a_{66}a_{68} + (i_{1} \cdot i_{2})a_{22}a_{31} + (i_{1} \cdot i_{3})a_{33}a_{31} + (i_{1} \cdot i_{2})a_{22}a_{31} + (i_{1} \cdot i_{3})a_{33}a_{31} + (i_{2} \cdot i_{3})a_{22}a_{31} + (i_{2} \cdot i_{4})a_{44}a_{52} + (i_{1} \cdot i_{5})a_{22}a_{52} + (i_{2} \cdot i_{6})a_{44}a_{56} + (i_{3} \cdot i_{6})a_{56}a_{52} + (i_{4} \cdot i_{6})a_{46}a_{52} + (i_{3} \cdot i_{4})a_{56}a_{52} + (i_{3} \cdot i_{5})a_{56}a_{52} + (i_{3} \cdot i_{6})a_{56}a_{52}
\]

\[
= 2x(a_t - x) + 2y(a_z - y) + 2x(a_3 - x) + 2z(a_z - z) + 2y(a_z - y) + 2z(a_z - z) - (a_t - y)x + 2x^2 - xy - a_t x - a_z y + xy - (a_t - y) y
\]

\[
+ 2y^2 - yz - (a_3 - x)y - a_t x - a_z y + yz + 2z^2 - (a_z - y) z
\]

\[
= 2(a_t + a_3 - a_z) x + 2(a_t + a_3 - a_z) y + 2(a_t + a_3 - a_z) z
\]

\[
- (x - y)^2 - (y - z)^2 - x^2 - z^2
\]

\[
q(A_{x,y,z}) < 0 \text{ if and only if } a_t + a_3 \leq a_2, a_4 + a_6 \leq a_3, a_5 + a_6 \leq a_5 \text{. This is a contradiction. Applying } \Phi, \Psi \text{, one gets that the monomials corresponding to } i \in \{1, 2, 3, 4, 5, 6\} \}
\]

are all not tight for any \( a = (a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{N}^6 \).

By Corollary 2.4, we have \( S_1^2 \Rightarrow S_0^6, S_6^2 \Rightarrow S_0^6, S_1^3 \Rightarrow S_0^6, S_6^2 \Rightarrow S_0^6, S_1^4 \Rightarrow S_0^6, S_6^2 \Rightarrow S_0^6, S_6^5 \Rightarrow S_0^5, S_6^2 \Rightarrow S_0^6, S_6^6 \Rightarrow S_0^6, S_6^7 \Rightarrow S_0^6, S_6^8 \Rightarrow S_0^6, S_6^9 \Rightarrow S_0^6, S_6^{10} \Rightarrow S_6^{10}, S_6^{11} \Rightarrow S_6^{11}, \) \( S_6^{12} \Rightarrow S_6^{12} \).

For \( S_6^{14} \), it suffices to consider \( i = (3, 2, 1, 2, 4, 3) \). For any \( a = (a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{N}^6 \), we have

\[
M_{a,x} = \{ a_{x,y} | 0 \leq x \leq \min \{a_t, a_b \}, 0 \leq y \leq \min \{a_2, a_4 \} \}
\]

where

\[
A_{x,y} = \begin{pmatrix}
a_t - x & 0 & 0 & 0 & 0 & x \\
0 & a_2 - y & 0 & y & 0 & 0 \\
0 & 0 & a_3 & 0 & 0 & 0 \\
0 & y & 0 & a_4 - y & 0 & 0 \\
0 & 0 & 0 & a_5 & 0 & 0 \\
x & 0 & 0 & 0 & a_b - x & 0
\end{pmatrix}
\]

and
\[ q(A_{s_5}) = \sum_{1 \leq m \leq 6} a_{pm} a_{rm} + \sum_{1 \leq p < m \leq 6} \left( i_i \cdot i_m \right) a_{pm} a_{ri} + \sum_{1 \leq m \leq 6} a_{rm} a_{il} \]

\[ = a_1 a_{d1} + a_2 a_{d2} + a_3 a_{d3} + a_4 a_{d4} + a_5 a_{d5} + a_6 a_{d6} + (i_1 \cdot i_2) (a_{d1} a_{d2} + a_{d3} a_{d4}) + (i_2 \cdot i_3) (a_{d2} a_{d3} + a_{d4} a_{d5}) + (i_3 \cdot i_4) (a_{d3} a_{d4} + a_{d5} a_{d6}) + (i_4 \cdot i_5) (a_{d4} a_{d5} + a_{d6} a_{d1}) + (i_5 \cdot i_6) (a_{d5} a_{d6} + a_{d1} a_{d2}) + (i_6 \cdot i_1) (a_{d6} a_{d1} + a_{d2} a_{d3}) + (i_1 \cdot i_3) a_{d1} a_{d3} + (i_3 \cdot i_4) a_{d3} a_{d4} + (i_1 \cdot i_5) a_{d1} a_{d5} + (i_5 \cdot i_6) a_{d5} a_{d6} + (i_6 \cdot i_2) a_{d6} a_{d2} + (i_2 \cdot i_4) a_{d2} a_{d4} + (i_4 \cdot i_1) a_{d4} a_{d1} \]

\[ = 2x(a_1 - x) + 2y(a_2 - y) + 2y(a_3 - y) + 2x(a_6 - x) - a_2 x - a_4 x - a_5 y - a_7 y - a_8 x - a_8 y \]

\[ = 2(a_4 - 2x + 2y - x) x + 2(a_4 + 2x - a_4) x - 2x^2 - 2y^2. \]

\[ q(A_{s_5}) < 0 \text{ if and only if } a_2 + a_4 \leq a_5, a_1 + a_6 \leq a_3 + a_4 + a_5. \]

So

\[ E_{s_5}^{(a)} E_{s_5}^{(a)} E_{s_5}^{(a)} E_{s_5}^{(a)} E_{s_5}^{(a)} E_{s_5}^{(a)} \text{ is tight by Theorem 2.2.} \]

For \( S_{15} \), it suffices to consider \( i = (1, 2, 3, 2, 1, 2) \). For any \( a = (a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{N}_6 \), we have

\[ \mathcal{M}_a = \{ A = A_{s_5, y_1, y_2, y_3} \text{ entries in matrix are all non-negative integer} \} \]

where

\[ A = \begin{pmatrix} a_1 - x & 0 & 0 & 0 & x & 0 \\ 0 & a_2 - y - y_1 & 0 & y & 0 & y_1 \\ 0 & 0 & a_3 & 0 & 0 & 0 \\ 0 & y_2 & 0 & a_4 - y_2^2 - y_3 & 0 & y_3 \\ x & 0 & 0 & 0 & a_6 - x & 0 \\ 0 & y + y_1 - y_2 & 0 & y_2 + y_3 - y & 0 & a_6 - y_1 - y_3 \end{pmatrix} \]

and

\[ q(A) = \sum_{1 \leq m \leq 6} a_{pm} a_{rm} + \sum_{1 \leq p < m \leq 6} \left( i_i \cdot i_m \right) a_{pm} a_{ri} + \sum_{1 \leq m \leq 6} a_{rm} a_{il} \]

\[ = a_1 a_{d1} + a_2 a_{d2} + a_3 a_{d3} + a_4 a_{d4} + a_5 a_{d5} + a_6 a_{d6} + (i_1 \cdot i_2) (a_{d1} a_{d2} + a_{d3} a_{d4}) + (i_2 \cdot i_3) (a_{d2} a_{d3} + a_{d4} a_{d5}) + (i_3 \cdot i_4) (a_{d3} a_{d4} + a_{d5} a_{d6}) + (i_4 \cdot i_5) (a_{d4} a_{d5} + a_{d6} a_{d1}) + (i_5 \cdot i_6) (a_{d5} a_{d6} + a_{d1} a_{d2}) + (i_6 \cdot i_1) (a_{d6} a_{d1} + a_{d2} a_{d3}) + (i_1 \cdot i_3) a_{d1} a_{d3} + (i_3 \cdot i_4) a_{d3} a_{d4} + (i_1 \cdot i_5) a_{d1} a_{d5} + (i_5 \cdot i_6) a_{d5} a_{d6} + (i_6 \cdot i_2) a_{d6} a_{d2} + (i_2 \cdot i_4) a_{d2} a_{d4} + (i_4 \cdot i_1) a_{d4} a_{d1} \]

\[ = 2(a_4 + a_5 + a_6 - x + 2(a_2 - a_4) y + 2(a_6 - a_4) y + 2(a_4 + a_5 + a_6 - a_3) y + (a_4 + a_6 - a_3) y_3 - (y - y_2)^2 - (x - y_1)^2 - y^2 - y_2^2 - 2y_1y - 2y_2y_3 - 2y_1y_3) y. \]

\[ q(A) < 0 \text{ if and only if } a_2 + a_4 \leq a_5, a_3 + a_4 + a_6 \leq a_3 + a_4 + a_5 \leq a_2 + a_4 \leq a_2 + a_4 + a_6. \]

So

\[ E_{s_5}^{(a)} E_{s_5}^{(a)} E_{s_5}^{(a)} E_{s_5}^{(a)} E_{s_5}^{(a)} E_{s_5}^{(a)} \text{ is tight by Theorem 2.2.} \]
is tight by Theorem 2.2. For \( S_0^6 \), it suffices to consider \( i = (1, 2, 1, 3, 2, 1) \). For any \( a = (a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{N}^6 \), we have
\[
\mathcal{M}_a = \{ A = A_{x,y,z} \mid \text{entries in matrix are all non-negative integer} \}
\]
where
\[
A = \begin{pmatrix}
    a_i - x - x_1 & 0 & x & 0 & 0 & x_1 \\
    0 & a_2 - y & 0 & 0 & y & 0 \\
    x_2 & 0 & a_3 - x - x_3 & 0 & 0 & x_3 \\
    0 & 0 & 0 & a_4 & 0 & 0 \\
    0 & y & 0 & 0 & a_5 - y & 0 \\
    x + x_1 - x_2 & 0 & x_2 + x_3 - x & 0 & 0 & a_6 - x_1 - x_3 \\
\end{pmatrix}
\]
and
\[
q(A) = \sum_{1 \leq a \leq 6} a_{pm}a_{rn} + \sum_{1 \leq p < r < 6} (i_i \cdot i_p) a_{pm}a_{rt} + \sum_{1 \leq r < m < 6} a_{rm}a_{rt}
\]
\[
= a_{11}a_{31} + a_{12}a_{61} + a_{13}a_{61} + a_{22}a_{62} + a_{23}a_{63} + a_{33}a_{63} + a_{a5}a_{65} + a_{a5}a_{65} + a_{a5}a_{66} + a_{56}a_{66} + (i_1 \cdot i_4) (a_{a5}a_{65} + a_{a5}a_{65}) + (i_1 \cdot i_5) (a_{a5}a_{65} + a_{a5}a_{65}) + (i_1 \cdot i_6) (a_{a5}a_{65} + a_{a5}a_{65})
\]
\[
+ (i_2 \cdot i_4) (a_{a5}a_{65} + a_{a5}a_{65}) + (i_2 \cdot i_5) (a_{a5}a_{65} + a_{a5}a_{65}) + (i_2 \cdot i_6) (a_{a5}a_{65} + a_{a5}a_{65}) + (i_3 \cdot i_4) (a_{a5}a_{65} + a_{a5}a_{65}) + (i_3 \cdot i_5) (a_{a5}a_{65} + a_{a5}a_{65}) + (i_3 \cdot i_6) (a_{a5}a_{65} + a_{a5}a_{65})
\]
\[
+ (i_4 \cdot i_5) (a_{a5}a_{65} + a_{a5}a_{65}) + (i_4 \cdot i_6) (a_{a5}a_{65} + a_{a5}a_{65}) + (i_5 \cdot i_6) (a_{a5}a_{65} + a_{a5}a_{65})
\]
\[
= 2(a_1 + a_2 - a_3) x + 2(a_2 + a_3 - a_4) y + 2(a_1 + 2a_3 + a_6 - a_3) x_3 + 2(a_3 + a_6 - a_3) x_3
\]
\[
- (x - x_1)^2 - (x_1 - y)^2 - (x - y)^2 - x^2 - 2x_1 x - 2y_1 y - x_1 y - 2x_1 x_3 - 2y_1 y_3 - x_1 y_3.
\]
\[
q(A) < 0 \quad \text{if and only if} \quad a_1 + a_2 \leq a_3, a_1 + a_6 \leq a_5, a_2 + a_5 \leq a_3 + a_4.
\]
So
\[
E_{11}^{(a_1)} E_{12}^{(a_2)} E_{31}^{(a_3)} E_{32}^{(a_4)} E_{33}^{(a_5)} E_{36}^{(a_6)} (a_1 + a_2 \leq a_3, a_2 + a_5 \leq a_3 + a_4, a_3 \leq a_5 + a_6),
\]
is tight by Theorem 2.2. For \( S_0^{17} \), it suffices to consider \( i = (3, 1, 2, 3, 1, 2) \). For any \( a = (a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{N}^6 \), we have
\[
\mathcal{M}_a = \{ A = A_{x,y,z} \mid 0 \leq x \leq \min \{a_1, a_4\}, 0 \leq y \leq \min \{a_2, a_5\}, 0 \leq z \leq \min \{a_3, a_6\} \}
\]
where
\[
A = \begin{pmatrix}
    a_i - x & 0 & 0 & x & 0 & 0 \\
    0 & a_2 - y & 0 & 0 & y & 0 \\
    0 & 0 & a_3 - z & 0 & 0 & z \\
    x & 0 & 0 & a_4 - x & 0 & 0 \\
    0 & y & 0 & 0 & a_5 - y & 0 \\
    0 & 0 & z & 0 & 0 & a_6 - z \\
\end{pmatrix}
\]
\[ q(A) = \sum_{1 \leq m < n \leq \infty} a_{mn}a_{nm} + \sum_{1 \leq i, j \leq \infty} (i \cdot j) a_{mn}a_{ni} + \sum_{1 \leq i, j \leq \infty} a_{mn}a_{ij} \]

\[ = a_{11}a_{41} + a_{22}a_{52} + a_{33}a_{63} + a_{44}a_{74} + a_{55}a_{85} + a_{66}a_{96} + a_{11}a_{14} + a_{22}a_{25} + a_{33}a_{36} + a_{44}a_{44} + a_{55}a_{55} + a_{66}a_{66} \]

\[ + (i \cdot j_1) a_{22}a_{41} + (i \cdot j_2) a_{33}a_{41} + (i \cdot j_3) a_{44}a_{41} + (i \cdot j_4) a_{55}a_{41} + (i \cdot j_5) a_{66}a_{41} + (i \cdot j_6) a_{22}a_{52} + (i \cdot j_7) a_{33}a_{52} + (i \cdot j_8) a_{44}a_{52} + (i \cdot j_9) a_{55}a_{52} + (i \cdot j_{10}) a_{66}a_{52} + (i \cdot j_{11}) a_{22}a_{63} + (i \cdot j_{12}) a_{33}a_{63} + (i \cdot j_{13}) a_{44}a_{63} + (i \cdot j_{14}) a_{55}a_{63} + (i \cdot j_{15}) a_{66}a_{63} \]

\[ + (i \cdot j_1) a_{22}a_{33} + a_{33}a_{44} + a_{44}a_{55} + (i \cdot j_3) a_{33}a_{44} + (i \cdot j_5) a_{44}a_{55} + (i \cdot j_7) a_{55}a_{44} + (i \cdot j_9) a_{66}a_{44} + (i \cdot j_{12}) a_{66}a_{55} \]

\[ = 2(a_1 + a_4 - a_2) x + 2(a_2 + a_5 - a_3) y + 2(a_3 + a_6 - a_4 - a_5) z - (x - z)^2 - (y - z)^2 - x^2 - y^2. \]

\[ q(A) < 0 \quad \text{if and only if} \quad a_2 + a_5 \leq a_3, a_4 \leq a_2 + a_5, a_4 \leq a_5, a_2 + a_5 \leq a_4 + a_5. \]

So
\[ E^{(n)}_1 E^{(n)}_2 E^{(n)}_3 E^{(n)}_4 E^{(n)}_5 \quad (a_1 + a_4 - a_2, a_3, a_4, a_5, a_6, a_7, a_8) \]

is tight by Theorem 2.2.

**Funding**

This paper is supported by the NSF of China (No. 11471333) and Basic and advanced technology research project of Henan Province (142300410449).

**References**


