Inverse Nonnegativity of Tridiagonal $M$-Matrices under Diagonal Element-Wise Perturbation

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Received 24 March 2015; accepted 6 June 2015; published 9 June 2015

Abstract

One of the most important properties of $M$-matrices is element-wise non-negative of its inverse. In this paper, we consider element-wise perturbations of tridiagonal $M$-matrices and obtain bounds on the perturbations so that the non-negative inverse persists. The largest interval is given by which the diagonal entries of the inverse of tridiagonal $M$-matrices can be perturbed without losing the property of total nonnegativity. A numerical example is given to illustrate our findings.

Keywords

Totally Positive Matrix, Totally Nonnegative Matrix, Tridiagonal Matrices, Compound Matrix, Element-Wise Perturbations

1. Introduction

In many mathematical problems, $Z$-matrices and $M$-matrices play an important role. It is often useful to know the properties of their inverses, especially when the $Z$-matrices and the $M$-matrices have a special combinatorial structure, for more details we refer the reader [1]. $M$-matrices have important applications, for instance, in iterative methods, in numerical analysis, in the analysis of dynamical systems, in economics, and in mathematical programming. One of the most important properties of some kinds of $M$-matrices is the nonnegativity of their inverses, which plays central role in many of mathematical problems.

An $n \times n$ real matrix $M = (m_{ij})$ is called $M$-matrix if $m_{ii} > 0$, $i = 1, \ldots, n$ and $m_{ij} \leq 0$, $i \neq j$, over the
years, $M$-matrices have considerable attention, in large part because they arise in many applications \[2\] \[3\]. Recently, a noticeable amount of attention has turned to the inverse of tridiagonal $M$-matrices (those matrices which happen to be inverses of special form of $M$-matrices with property $a_{ij} = 0$ whenever $|i - j| > 1$) and $M$ is generalized strictly diagonally dominant. A matrix is said to be generalized (strictly) diagonally dominant if $a_{ii} > 0$ whenever $a_{ij} = 0, i \not= j$ and $a_{ii} > 0$, and $M$ is generalized strictly diagonally dominant if $M^{-1} > 0$ where the inequality is satisfied element-wise. A rich class of $M$-matrices were introduced by Ostrowski in 1937 \[4\], with reference to the work of Minkowski \[5\] \[6\]. A condition which is easy to check is that a matrix $M$ is an $M$-matrix if and only if $a_{ij} \leq 0, i \not= j$ and $a_{ii} > 0$, and $M$ is generalized strictly diagonally dominant.

In this paper, we consider the inverse of perturbed $M$-matrix. Specifically we consider the effect of changing single elements inside the diagonal of $M^{-1}$. We are interested in the large amount by which the single diagonal element of $M^{-1}$ can be varied without losing the property of total nonnegativity.

The reminder of the paper is organized as follows. In section 2, we explain our notations and some needed important definitions are presented. In section 3, some auxiliary results and important prepositions and lemmas are stated. In section 4, we present our results.

2. Notations

In this section we introduce the notation that will be used in developing the paper. For $k, n$ we denote by $Q_{k,n}$ the set of all strictly increasing sequences of $k$ integers chosen from $\{1, 2, \ldots, n\}$. For $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\beta = (\beta_1, \ldots, \beta_k) \in Q_{k,n}$, we denote by $A[\alpha, \beta]$ the $k \times k$ submatrix of $A$ contained in the rows indexed by $\alpha_1, \ldots, \alpha_k$ and columns indexed by $\beta_1, \ldots, \beta_k$. A matrix $A$ is called totally positive (abbreviated TP henceforth) if $\det A[\alpha, \beta] > 0$ and totally nonnegative (abbreviated TN) if $\det A[\alpha, \beta] \geq 0$ for all $\alpha, \beta \in Q_{k,n}$, $k = 1, 2, \ldots, n$. A matrix $A$ is called totally positive if $\det A_{\alpha, \beta} > 0$ and totally nonnegative if $\det A_{\alpha, \beta} \geq 0$ for all $\alpha, \beta \in Q_{k,n}$, $k = 1, 2, \ldots, n$.

Throughout this paper we use the following notation for general tridiagonal $M$-matrix:

$$M = \begin{bmatrix}
a_{11} & -b_1 & 0 & \cdots & 0 \\
-c_1 & a_2 & -b_2 & \cdots & \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & -c_{n-2} & a_{n-1} - b_{n-1} \\
0 & \cdots & 0 & -c_{n-1} & a_n
\end{bmatrix}$$

where $a_i, b_i$ and $c_i > 0$, and each $a_i$ is large enough that $M$ is strictly diagonally dominant.

We let $E_{ij}$ to be the square standard basis matrix whose only nonzero entry is 1 that occurs in the $(i,j)$ position.

**Definition 2.1** Compound Matrices ([7], p. 19).

Let $A = (a_{ij})_{i,j=1}^n$ be a square matrix of order $n$. Let $\alpha = \{1, \cdots, n\}$ be the index set of cardinality $n$, defining $\alpha_i = \alpha \setminus \{n-i+1\}$, $i = \{1, \cdots, n\}$ are the index sets of cardinality $n-1$.

Construct the following table which depends on $\alpha_i$.

<table>
<thead>
<tr>
<th>$\alpha_i$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\cdots$</th>
<th>$\alpha_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1$</td>
<td>$\det A(a, \alpha)$</td>
<td>$\det A(a, \alpha)$</td>
<td>$\cdots$</td>
<td>$\det A(a, \alpha)$</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>$\det A(a, \alpha)$</td>
<td>$\det A(a, \alpha)$</td>
<td>$\cdots$</td>
<td>$\det A(a, \alpha)$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\cdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$\alpha_n$</td>
<td>$\det A(a, \alpha)$</td>
<td>$\det A(a, \alpha)$</td>
<td>$\cdots$</td>
<td>$\det A(a, \alpha)$</td>
</tr>
</tbody>
</table>
The created matrix \( C_{n-1}(A) = \begin{bmatrix} \det(A_{11}), \det(A_{12}), \cdots, \det(A_{1n}) \\ \det(A_{21}), \det(A_{22}), \cdots, \det(A_{2n}) \\ \vdots \\ \det(A_{n1}), \det(A_{n2}), \cdots, \det(A_{nn}) \end{bmatrix} \)

is called \((n-1)\)th compound matrix of \( A \).

For example, if \( A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \) with indexed sets \( \alpha_1 = \{1, 2\} \), \( \alpha_2 = \{1, 3\} \) and \( \alpha_3 = \{2, 3\} \).

Then \( C_2(A) = \begin{bmatrix} \det(A_{11}), \det(A_{12}) \\ \det(A_{21}), \det(A_{22}) \\ \det(A_{31}), \det(A_{32}) \end{bmatrix} \begin{bmatrix} \det(A_{11}), \det(A_{12}), \det(A_{1n}) \\ \det(A_{21}), \det(A_{22}), \cdots, \det(A_{2n}) \\ \det(A_{31}), \cdots, \det(A_{3n}) \end{bmatrix} = \begin{bmatrix} -3 & -6 & -3 \\ -6 & -12 & -6 \\ -3 & -6 & -3 \end{bmatrix} \).

### 3. Auxiliary Results

We start with some basic facts on tridiagonal \( M \)-matrices. We can find the determinant of any \( n \times n \) tridiagonal \( M \)-matrix \( M = (m_{ij}) \) by using the following recursion equation \[8\] \[9\].

\[
\det M = a_n \det M(2, \cdots, n) - b_{n-1} \det M(3, \cdots, n) = a_n \det M(1, \cdots, n-1) - b_{n-1} c_n \det M(1, \cdots, n-2).
\]

And we have the following proposition for finding the determinant of a \( n \times n \) tridiagonal \( M \)-matrix \( M = (m_{ij}) \) the following relation is true

\[
\det M = \det M(1, \cdots, i-1) \det M(i+1, \cdots, n) - b_i c_i \det M(1, \cdots, i-2) \det M(i+1, \cdots, n), \quad i = 2, \cdots, n.
\]

We will present now some of propositions of nonsingular totally nonnegative matrices which important for our work.

### Proposition 3.1 ([10], formula 4.1)
For any \( n \times n \) tridiagonal \( M \)-matrix \( M = (m_{ij}) \) the following relation is true

\[
\det M^{-1} = \frac{\det M^{-1}(\alpha)}{\det M(\alpha)}, \text{ when } d(\alpha) = 0.
\]

In the sequel we will make use the following lemma, see, e.g. \[12\].

### Lemma 3.4 (Sylvester Identity)
Partition square matrix \( P \) of order \( n \), \( n > 2 \), as:

\[
P = \begin{bmatrix} a & P_{12} & b \\ P_{23} & P_{22} & P_{23} \\ c & P_{32} & d \end{bmatrix},
\]

where \( P_{22} \) square matrix of order \( n-2 \) and \( a \), \( b \), \( c \) and \( d \) are scalars.

Define the submatrices

\[
A = \begin{bmatrix} a & P_{12} \\ P_{23} & P_{22} \end{bmatrix}, \quad B = \begin{bmatrix} P_{12} & b \\ P_{22} & P_{23} \end{bmatrix},
\]

\[
C = \begin{bmatrix} P_{21} & P_{22} \\ c & P_{32} \end{bmatrix}, \quad D = \begin{bmatrix} P_{22} & P_{23} \\ P_{32} & d \end{bmatrix}
\]
If $P_{22}$ is nonsingular, then
\[
\det P = \frac{\det A \det D - \det B \det C}{\det P_{22}}.
\]

**Lemma 3.5** ([11], p. 199) Let $A = (a_{ij})$ be a square matrix of order $n$, with $\det A(2, \cdots, n) \neq 0$. Then $A - xE_{11}$ is totally nonnegative $\forall x \in \left[0, \frac{\det A}{\det A(2, \cdots, n)}\right]$.

We now state an important result which links the determinant of $M$-matrix with the value of the elements of its inverse.

**Lemma 3.6** [10] Let $M = \left[m_{ij}\right]_{i,j=1}^n$ be a tridiagonal matrix of order $n$, then we can find the elements of inverse matrix $M^{-1} = \left[m_{ij}^{-1}\right]_{i,j=1}^n$ by using the following formula
\[
m_{ij}^{-1} = (-1)^{i+j} \frac{\det M\left(1, 2, \cdots, j, \cdots, n/1, 2, \cdots, i, \cdots, n\right)}{\det M}.
\]

### 4. Main Results

In this section, we present our results based on the inverse of tridiagonal $M$-matrices. Firstly we begin with the following theorem.

**Theorem 4.1**

Let
\[
M = \begin{bmatrix}
  a_1 & -b_1 & 0 & \cdots & 0 \\
  -c_1 & a_2 & -b_2 & \ddots & \\
  0 & \ddots & \ddots & \ddots & 0 \\
  \vdots & & -c_{n-2} & a_{n-1} & -b_{n-1} \\
  0 & \cdots & 0 & -c_{n-1} & a_n
\end{bmatrix}
\]

be strictly diagonally dominant $M$-matrix.

If $C_{n-1}(M)$ is the $(n-1)^{\text{st}}$ compound matrix of $M$ then the matrix $S\left(C_{n-1}(M)\right)^T S$ is totally nonnegative matrix. Moreover, $\det S\left(C_{n-1}(M)\right)^T S = (\det M)^{n-1}$ where $S = \text{diag}\left(1, -1, \cdots, (-1)^{n-1}\right)$

**Proof:** Let $M$ be strictly diagonally dominant $M$-matrix.

Then
\[
M^{-1} = \frac{S\left(C_{n-1}(M)\right)^T S}{\det M}
\]

is totally nonnegative matrix. So is $S\left(C_{n-1}(M)\right)^T S$.

You can find this formula in ([7], p. 21).

There is an explicit formula for the determinant of $\det S\left(C_{n-1}(M)\right)^T S = (\det M)^{n-1}$ given as
\[
\det S\left(C_{n-1}(M)\right)^T S = \begin{vmatrix}
  \det M(\alpha_1, \alpha_1) & -\det M(\alpha_2, \alpha_1) & \cdots & (-1)^{n+1} \det M(\alpha_n, \alpha_1) \\
  -\det M(\alpha_1, \alpha_2) & \det M(\alpha_2, \alpha_2) & \cdots & \cdots & \cdots & \cdots \\
  \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
  (-1)^{n+1} \det M(\alpha_1, \alpha_n) & \cdots & \cdots & \cdots & \cdots & \cdots \\
  1 & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  (-1)^{n+1} \det M(\alpha_1, \alpha_n) & \cdots & \cdots & \cdots & \cdots & \cdots
\end{vmatrix}
\]

\[
= \det A(\alpha_1, \alpha_1) - \det M(\alpha_1, \alpha_2) \det M(\alpha_2, \alpha_2) \cdots (-1)^{n+2} \det M(\alpha_n, \alpha_n)
\]

\[
= \det A(\alpha_1, \alpha_1) - \det M(\alpha_1, \alpha_2) \det M(\alpha_2, \alpha_2) \cdots (-1)^{n+2} \det M(\alpha_n, \alpha_n)
\]

\[
= \det A(\alpha_1, \alpha_1) - \det M(\alpha_1, \alpha_2) \det M(\alpha_2, \alpha_2) \cdots (-1)^{n+2} \det M(\alpha_n, \alpha_n)
\]
Multiply the first row by \((-1)^i \det M(\alpha_i, \alpha_i)\) and add it to the \(i\)th row to obtain

\[
\begin{vmatrix}
1 & \frac{\det M(\alpha_i, \alpha_i)}{\det M(\alpha_i, \alpha_i)} & \cdots & \frac{(-1)^{-i} \det M(\alpha_i, \alpha_i)}{\det M(\alpha_i, \alpha_i)} \\
0 & \frac{\det M(\alpha_i, \alpha_i)}{\det M(\alpha_i, \alpha_i)} & \cdots & \frac{(-1)^{-i} \det M(\alpha_i, \alpha_i)}{\det M(\alpha_i, \alpha_i)} \\
\vdots & \ddots & \ddots & \vdots \\
0 & \frac{(-1)^i \det M(\alpha_i, \alpha_i)}{\det M(\alpha_i, \alpha_i)} & \cdots & \frac{(-1)^{-i-1} \det M(\alpha_i, \alpha_i)}{\det M(\alpha_i, \alpha_i)}
\end{vmatrix}
\]

\[
= \frac{1}{(\det M(\alpha_i, \alpha_i))^2}
\begin{vmatrix}
1 & \frac{\det M(\alpha_i, \alpha_i)}{\det M(\alpha_i, \alpha_i)} & \cdots & \frac{(-1)^{i-1} \det M(\alpha_i, \alpha_i)}{\det M(\alpha_i, \alpha_i)} \\
0 & \frac{\det M(\alpha_i, \alpha_i)}{\det M(\alpha_i, \alpha_i)} & \cdots & \frac{(-1)^{i-1} \det M(\alpha_i, \alpha_i)}{\det M(\alpha_i, \alpha_i)} \\
\vdots & \ddots & \ddots & \vdots \\
0 & \frac{(-1)^{-i+1} \det M(\alpha_i, \alpha_i)}{\det M(\alpha_i, \alpha_i)} & \cdots & \frac{(-1)^{-i-1} \det M(\alpha_i, \alpha_i)}{\det M(\alpha_i, \alpha_i)}
\end{vmatrix}
\]

where \((\alpha_i)_i = \alpha_i \setminus \{n-i\} = \left(\frac{\det M(\alpha_i, \alpha_i)}{\det M(\alpha_i, \alpha_i)}\right)^{i-1} \frac{\det M(\alpha_i, \alpha_i)}{\det M(\alpha_i, \alpha_i)}

And now apply an induction argument to get the result.

**Numerical Example:** Let \(M = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 5 & -1 \\ 0 & -2 & 3 \end{bmatrix}\) be strictly diagonally dominant M-matrix, then

\[
C_2(M) = \begin{bmatrix} 13 & -3 & 1 \\ -6 & 9 & -3 \\ 4 & -6 & 13 \end{bmatrix}
\]

and \(S(C_2(M))^T S = \begin{bmatrix} 13 & 6 & 4 \\ 3 & 9 & 6 \\ 1 & 3 & 13 \end{bmatrix}\) is totally nonnegative.

Note that \(\det(M) = 33\) and \(\det(S(C_2(M))^T S) = 33^2 = 1089\)

Numerically we can conclude the following fact.

**Fact:** For any \(n \times n\) tridiagonal M-matrix \(M = (m_{ij})\) the following formula is true.

\[
\det M^{-1}(1, \cdots, n-1/2, \cdots, n) = \det M^{-1}(2, \cdots, n/1, \cdots, n-1) = 0 \quad \text{for} \quad n \geq 3
\]

Moreover, \(\det M^{-1} = \frac{1}{\det M}\)

To prove this result we use **Theorem 4.1**.

Suppose \(M\) is nonsingular then \(M^{-1} = \frac{S(C_{n-1}(M))^T S}{\det M}\), so

\[
\det M^{-1} = \det \left(\frac{S(C_{n-1}(M))^T S}{\det M}\right) = \frac{\det \left(\frac{S(C_{n-1}(M))^T S}{\det M}\right)}{(\det M)^n} = \frac{(\det M)^{n-1}}{(\det M)^n} = \frac{1}{\det M}
\]

For example, when \(n = 3\), the M-matrix of our form
$M = \begin{bmatrix} a_1 & -b_1 & 0 \\ -c_1 & a_2 & -b_2 \\ 0 & -c_2 & a_3 \end{bmatrix}$ has an inverse given as

$$M^{-1} = \begin{bmatrix} m_{11}^{-1} & m_{12}^{-1} & m_{13}^{-1} \\ m_{21}^{-1} & m_{22}^{-1} & m_{23}^{-1} \\ m_{31}^{-1} & m_{32}^{-1} & m_{33}^{-1} \end{bmatrix} = \begin{bmatrix} \frac{\text{det}(2,3/2,3)}{\text{det}M} & -\frac{\text{det}(1,3/2,3)}{\text{det}M} & \frac{\text{det}(1,2/2,3)}{\text{det}M} \\ -\frac{\text{det}(2,3/1,3)}{\text{det}M} & \frac{\text{det}(2,3/1,2)}{\text{det}M} & -\frac{\text{det}(1,2/1,3)}{\text{det}M} \\ -\frac{\text{det}(2,3/1,2)}{\text{det}M} & -\frac{\text{det}(1,3/1,2)}{\text{det}M} & \frac{\text{det}(1,2/1,2)}{\text{det}M} \end{bmatrix}$$

\begin{equation*}
\text{det}M^{-1}(1,2/2,3) = \left( -\frac{\text{det}(1,3/2,3)}{\text{det}M} \right) - \left( -\frac{\text{det}(1,2/1,3)}{\text{det}M} \right) - \left( -\frac{\text{det}(1,2/2,3)}{\text{det}M} \right) (\text{det}(1,3/1,3))
\end{equation*}

= $(b,a_3)(a_2) - (b,a_3)(a_3)$ = 0.

Similarly we can find $\text{det}M^{-1}(2,3/1,2) = 0$.

**Illustrative Example:** Let $M = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 4 & -2 \\ 0 & -3 & 5 \end{bmatrix}$ be a tridiagonal $M$-matrix and

$$M^{-1} = \begin{bmatrix} 0.3784 & 0.1351 & 0.0541 \\ 0.1351 & 0.4054 & 0.1622 \\ 0.0811 & 0.2432 & 0.2973 \end{bmatrix}$$

Note that $\text{det}M^{-1}(1,2/2,3) = \text{det}M^{-1}(2,3/1,2) \equiv 0$

Observe that the error came from the rounded to the nearest part of 10,000.

**Theorem 4.2** Let $M$ be a strictly diagonally dominant $M$-matrix, if $\alpha = \{1, \cdots, n-1\}$, $\beta = \{2, \cdots, n\}$, $\gamma = \alpha \cap \beta$ then

$$\text{det}M^{-1} = \frac{\text{det}M^{-1}[\alpha][\alpha] \times \text{det}M^{-1}[\beta][\beta]}{\text{det}M^{-1}(\gamma)}$$

**Proof:**

Assume $M^{-1} = \begin{bmatrix} a & m_{12}^{-1} & b \\ m_{21}^{-1} & M_{22}^{-1} & m_{23}^{-1} \\ c & m_{32}^{-1} & d \end{bmatrix}$ and

$$M^{-1}[\alpha][\alpha] = \begin{bmatrix} a & m_{12}^{-1} \\ m_{21}^{-1} & M_{22}^{-1} \\ c & m_{32}^{-1} \end{bmatrix}, \quad M^{-1}[\beta][\beta] = \begin{bmatrix} m_{12}^{-1} & b \\ m_{21}^{-1} & M_{22}^{-1} \\ m_{32}^{-1} & d \end{bmatrix}, \quad M^{-1}[\beta][\alpha] = \begin{bmatrix} m_{12}^{-1} & M_{22}^{-1} \\ c & m_{32}^{-1} \end{bmatrix}, \quad M^{-1}[\alpha][\beta] = \begin{bmatrix} M_{22}^{-1} \\ m_{32}^{-1} \end{bmatrix}.$$

Note that $M^{-1}[\beta][\beta] = M^{-1}[\alpha][\alpha] = 0$ by previous fact, and by using Sylvester's identity, we have

$$\det M^{-1} = \frac{\text{det}M^{-1}[\alpha][\alpha] \times \text{det}M^{-1}[\beta][\beta] - \text{det}M^{-1}[\alpha][\beta] \times \text{det}M^{-1}[\beta][\alpha]}{\text{det}M^{-1}(\gamma)}.$$

Moreover we conclude the following theorem.

**Theorem 4.3** Let $M$ be the $M$-matrix defined above then
\[
\det \left( M^{-1} - xE_{ij} \right) = \frac{\det M^{-1} - x \cdot a_i}{\det M}
\]

For example

\[
M = \begin{bmatrix}
2 & -1 & 0 & 0 \\
-1 & 3 & -1 & 0 \\
0 & -2 & 4 & -1 \\
0 & 0 & -3 & 5
\end{bmatrix}, \quad M^{-1} = \begin{bmatrix}
0.6308 & 0.2615 & 0.0769 & 0.0154 \\
0.2615 & 0.5231 & 0.1538 & 0.0308 \\
0.1538 & 0.3077 & 0.3846 & 0.0769 \\
0.0923 & 0.1846 & 0.2308 & 0.2462
\end{bmatrix}
\]

Let \( x = 0.2 \) then \( M^{-1} + 0.2E_{11} = \)

\[
\begin{bmatrix}
0.4308 & 0.2615 & 0.0769 & 0.0154 \\
0.2615 & 0.5231 & 0.1538 & 0.0308 \\
0.1538 & 0.3077 & 0.3846 & 0.0769 \\
0.0923 & 0.1846 & 0.2308 & 0.2462
\end{bmatrix}
\]

\[
\det \left( M^{-1} + 0.2E_{11} \right) = 0.0092 \quad \text{and} \quad \det M^{-1} - \frac{x \cdot a_i}{\det M} = 0.0154 - \frac{0.2 \times 2}{65} = 0.0092
\]

Now, we will perturb elements inside the diagonal band of the inverse of \( M \)-matrix without losing the nonnegativity property. We begin with the \((1,1)\) element then generalize to other elements.

**Theorem 4.4** Let \( M \) be a strictly diagonally dominant tridiagonal \( M \)-matrix. Then the matrix \( M^{-1} - xE_{11} \) is totally nonnegative for all \( x \in \left[ 0, \frac{1}{a_1} \right] \).

**Proof:**

Let \( M = [a_i, -b_i, 0, \ldots, 0]
\]

Be a nonsingular strictly diagonally dominant tridiagonal \( M \)-matrix then \( M^{-1} \) is totally nonnegative.

By Lemma 3.5 and Proposition 3.2, we have

\( M^{-1} - xE_{11} \) is totally nonnegative \( \forall x \in \left[ 0, \frac{\det M^{-1}}{\det M^{-1} \{2, \ldots, n\}} \right] \).

By using the formula in Proposition 3.3

\[
\frac{\det M^{-1}}{\det M^{-1} \{2, \ldots, n\}} = \frac{(\det M^{-1})^{1-i} \cdot a_i}{\det M^{-1} \{2, \ldots, n\}^{1-i}} = \frac{1}{a_i}
\]

Note that a similar result holds for decreasing the element \( m_{nn} \) by considering the matrix \( \tilde{S}M^{-1}\tilde{S} \), which reverses the matrix \( M^{-1} \) as the relation \( m_{ij}^{-1} = m_{n-i+1,n-j+1}^{-1} \).

We can generalize this result for the other elements of diagonal.

**Theorem 4.5** Assume \( M \) is a strictly diagonally dominant tridiagonal \( M \)-matrix. Then the matrix \( M^{-1} - xE_{ij} \) is totally nonnegative for all \( x \in \left[ 0, \frac{1}{a_i} \right] \).

**Proof:** Suppose that \( X = M^{-1} - xE_{ij} \) is not totally nonnegative for all \( x \in \left[ 0, \frac{1}{a_i} \right] \), then there exist \( \alpha, \beta \in Q_{+} \) both contain \( i \) such that \( \det X = \det \left( \left( M^{-1} - xE_{ij} \right)(\alpha, \beta) \right) < 0 \).

To compute \( \det X \) expand the determinant along the \( i^{th} \) row of \( X \) then
\[ \det(1) = (-1)^{\alpha_1 + \beta_1} \det(X(\{\alpha_1, \beta_1\})) + \det(M^{-1}(\alpha', \beta')) \text{ where } \alpha' \subset \alpha, \ \beta' \subset \beta, \text{ and } \det(M^{-1}(\alpha', \beta')) \text{ is some minor of } M^{-1}. \]

Take the case when \( \alpha_i + \beta_i \) odd. Thus, \( \det(X) \) is a positive linear combination of minors of \( M^{-1} \) and hence is positive, which contradicts the assumption.

Now suppose \( \exists x \not\in \left( 0, \frac{1}{a_i} \right) \) such that \( M^{-1} - xE_{ii} \) is totally nonnegative matrix.

Suppose that \( x > \frac{1}{a_i} \), then by Theorem 4.3.

\[
\det(M^{-1} + xE_{ii}) = \det(M^{-1}) - \frac{x \times a_i}{\det(M)} < \det(M^{-1}) - \frac{1}{\det(M)} = 0, \text{ since } \det(M^{-1}) = \frac{1}{\det(M)}
\]

which contradicts the nonnegativity of \( M^{-1} - xE_{ii} \).

**Numerical Example:** Let \( M = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -2 & 4 & -1 \\ 0 & 0 & -3 & 5 \end{bmatrix} \) is strictly diagonally dominant tridiagonal \( M \)-matrix

\[
M^{-1} = \begin{bmatrix} 0.6308 & 0.2615 & 0.0769 & 0.0154 \\ 0.2615 & 0.5231 & 0.1538 & 0.0308 \\ 0.1538 & 0.3077 & 0.3846 & 0.0769 \\ 0.0923 & 0.1846 & 0.2308 & 0.2462 \end{bmatrix}
\]

The matrices

\[
M^{-1} + xE_{11} = \begin{bmatrix} 0.1308, 0.6308, 0.2615, 0.0769, 0.0154 \\ 0.2615 & 0.5231 & 0.1538 & 0.0308 \\ 0.1538 & 0.3077 & 0.3846 & 0.0769 \\ 0.0923 & 0.1846 & 0.2308 & 0.2462 \end{bmatrix}, \ x \in [0, 0.5],
\]

\[
M^{-1} + xE_{22} = \begin{bmatrix} 0.6308 & 0.2615 & 0.0769 & 0.0154 \\ 0.2615 & 0.1898, 0.5231, 0.1538 & 0.3077 & 0.3846 & 0.0769 \\ 0.1538 & 0.3077 & 0.3846 & 0.0769 \\ 0.0923 & 0.1846 & 0.2308 & 0.2462 \end{bmatrix}, \ x \in [0, 0.3333],
\]

\[
M^{-1} + xE_{33} = \begin{bmatrix} 0.6308 & 0.2615 & 0.0769 & 0.0154 \\ 0.2615 & 0.5231 & 0.1538 & 0.0308 \\ 0.1538 & 0.3077 & 0.1346, 0.3846 & 0.0769 \\ 0.0923 & 0.1846 & 0.2308 & 0.2462 \end{bmatrix}, \ x \in [0, 0.25], \text{ and}
\]

\[
M^{-1} = \begin{bmatrix} 0.6308 & 0.2615 & 0.0769 & 0.0154 \\ 0.2615 & 0.5231 & 0.1538 & 0.0308 \\ 0.1538 & 0.3077 & 0.3846 & 0.0769 \\ 0.0923 & 0.1846 & 0.2308 & 0.2462 \end{bmatrix}, \ x \in [0, 0.2].
\]

are TNN matrices.

Note that \( \det(M^{-1}(\{1, 2, 3\} \setminus \{2, 3, 4\})) = \det(M^{-1}(\{2, 3, 4\} \setminus \{1, 2, 3\})) = 0 \).

**References**


