Low-Rank Positive Approximants of Symmetric Matrices

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Abstract
Given a symmetric matrix $X$, we consider the problem of finding a low-rank positive approximant of $X$. That is, a symmetric positive semidefinite matrix, $S$, whose rank is smaller than a given positive integer, $\ell$, which is nearest to $X$ in a certain matrix norm. The problem is first solved with regard to four common norms: The Frobenius norm, the Schatten $p$-norm, the trace norm, and the spectral norm. Then the solution is extended to any unitarily invariant matrix norm. The proof is based on a subtle combination of Ky Fan dominance theorem, a modified pinching principle, and Mirsky minimum-norm theorem.

Keywords
Low-Rank Positive Approximants, Unitarily Invariant Matrix Norms

1. Introduction
Let $X$ be a given real symmetric $n \times n$ matrix. In this paper we consider the problem of finding a low-rank symmetric positive semidefinite matrix which is nearest to $X$ with regard to a certain matrix norm. Let $\|\cdot\|$ be a given unitarily invariant matrix norm on $\mathbb{R}^{n \times n}$. (The basic features of such norms are explained in the next section.) Let $\ell$ be a given positive integer such that $1 \leq \ell \leq n - 1$, and define

$$S_{n,\ell}^+ = \{S \mid S \in \mathbb{R}^{n \times n}, S \succeq 0, \text{ and } \text{rank}(S) \leq \ell\},$$

where the notation $S \succeq 0$ means that $S$ is symmetric and positive semidefinite. Then the problem to solve has the form

$$\begin{align*}
\text{minimize} & \quad F(S) = \|X - S\| \\
\text{subject to} & \quad S \in S_{n,\ell}^+.
\end{align*}$$

(1.1)
The need for solving such problems arises in certain matrix completion methods that consider Euclidean distance matrices, see [1] or [2]. Since $X$ is assumed to be a symmetric matrix, it has a spectral decomposition

$$X = Q \Lambda Q^T,$$

(1.2)

where $Q \in \mathbb{R}^{n \times n}, Q^T Q = I$, is an orthonormal matrix

$$\Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$$

is a diagonal matrix, and

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

are the eigenvalues of $X$ in decreasing order. If $X \geq 0$, then $\lambda_n \geq 0$ and the spectral decomposition (1.2) coincides with the SVD of $X$. In this case the solution of (1.1) is given by the Eckart-Young-Mirsky theorem. See the next section.

The rest of the paper assumes, therefore, that $\lambda_n < 0$. In this case the solution of (1.1) is related to that of the problem

$$\text{minimize } F(S) = \|X - S\|$$

subject to $S \in S_n^+$,

(1.3)

where

$$S_n^+ = \{S | S \in \mathbb{R}^{n \times n} \text{ and } S \geq 0\}.$$ 

Let $q$ be a non-negative integer that counts the number of positive eigenvalues. That is

$$\lambda_j > 0 \text{ for } j = 1, \ldots, q, \text{ and } \lambda_j \leq 0 \text{ for } j = q+1, \ldots, n.$$ 

(1.4)

Let the diagonal matrix $\Lambda_q$ denotes the positive part of $\Lambda$,

$$\Lambda_q = \text{diag}\{\lambda_1, \ldots, \lambda_q, 0, \ldots, 0\}.$$ 

(1.5)

(If $\lambda_1 \leq 0$ then $q = 0$, and $\Lambda_n$ is the null matrix.) Then, as we shall see in Section 3, the matrix

$$X_q = Q \Lambda_q Q^T \in S_n^+$$

solves (1.3) in any unitarily invariant norm. If $q \leq \ell$ then, clearly, $X_q$ is also a solution of (1.1). Hence in the rest of the paper we assume that $1 \leq \ell < q$. This assumption implies that the diagonal matrix

$$\Lambda_{\ell} = \text{diag}\{\lambda_1, \ldots, \lambda_{\ell}, 0, \ldots, 0\}$$

(1.6)

belongs to $S_{n,\ell}^+$. The aim of the paper is to show that the matrix

$$X_{\ell} = Q \Lambda_{\ell} Q^T$$

solves (1.1) for any unitarily invariant norm.

Let $A \in \mathbb{R}^{n \times n}$ be a given real $n \times n$ matrix. Then another related problem is

$$\text{minimize } F(S) = \|A - S\|$$

subject to $S \in S_n^+$,

(1.7)

The relation between (1.7) and (1.3) is seen when using the Frobenius matrix norm. Let $S \in \mathbb{R}^{n \times n}$ be a symmetric matrix and let $T \in \mathbb{R}^{n \times n}$ be a skew-symmetric matrix. Then, clearly,

$$\|S + T\|^2_F = \|S\|^2_F + \|T\|^2_F.$$ 

(1.8)

Recall also that any matrix $A \in \mathbb{R}^{n \times n}$ has a unique presentation as the sum $A = X + Y$ where $X = (A^T + A)/2$ is symmetric and $Y = (A^T - A)/2$ is skew-symmetric. Consequently, for any symmetric
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matrix, \( S \) say,

\[
\|A - S\|_F^2 = \|X - S + Y\|_F^2 = \|X - S\|_F^2 + \|Y\|_F^2.
\]  

Therefore, when using the Frobenius norm, a solution of (1.3) provides a solution of (1.7). This observation is due to Higham [3]. A matrix that solves (1.7) or (1.3) is called “positive approximant”. Similarly, the term “low-rank positive approximant” refers to a matrix that solves (1.1).

The current interest in positive approximants was initiated in Halmos’ paper [4], which considers the solution of (1.7) in the spectral norm. Rogers and Ward [5] considered the solution of (1.7) in the Schatten-\( p \)-norm, Ando [6] considered this problem in the trace norm, and Higham [3] considered the Frobenius norm. Halmos [4] has considered the positive approximant problem in a more general context of linear operators on a Hilbert space. Other positive approximants problems (in the operators context) are considered in [7]-[11]. The problems (1.1), (1.3) and (1.7) fall into the category of “matrix nearness problems”. Further examples of matrix (or operator) nearness problems are discussed in [12]-[18]. A review of this topic is given in Higham [19].

The plan of the paper is as follows. In the next section we introduce notations and tools which are needed for the coming discussions. In Section 3 we show that \( X_q \) solves (1.3). Section 4 considers the solution of (1.1) in Frobenius norm. This involves the Eckart-Young theorem. In the next sections Mirsky theorem extends the results to Schatten-\( p \)-norms, the trace norm, and the spectral norm. Then it is proved that \( X_q \) solves (1.1) in any unitarily invariant norm. The proof of this claim requires a subtle combination of Ky Fan dominance theorem, a modified pinching principle, and Mirsky theorem.

### 2. Notations and Tools

In this section we introduce notations and facts which are needed for coming discussions. Here \( A \) denotes a real \( m \times n \) matrix with \( m \geq n \). Let

\[
A = USV^T
\]  

be an SVD of \( A \), where \( U = [u_1, \ldots, u_m] \) is an \( m \times m \) orthogonal matrix, \( V = [v_1, \ldots, v_n] \) is an \( n \times n \) orthogonal matrix, and \( S = \text{diag} \{\sigma_1, \ldots, \sigma_n\} \) is an \( m \times n \) diagonal matrix. The singular values of \( A \) are assumed to be nonnegative and sorted to satisfy

\[
\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0.
\]  

The columns of \( U \) and \( V \) are called left singular vectors and right singular vectors, respectively. These vectors are related by the equalities

\[
A v_j = \sigma_j u_j \quad \text{and} \quad A^T u_j = \sigma_j v_j, \quad j = 1, \ldots, n.
\]  

A further consequence of (2.1) is the equality

\[
A = \sum_{j=1}^r \sigma_j u_j v_j^T.
\]  

Moreover, let \( r \) denotes the rank of \( A \). Then, clearly,

\[
\sigma_1 \geq \cdots \geq \sigma_r > 0 \quad \text{and} \quad \sigma_j = 0 \quad \text{for} \quad j = r+1, \ldots, n.
\]  

So (2.4) can be rewritten as

\[
A = \sum_{j=1}^r \sigma_j u_j v_j^T.
\]  

Let the matrices

\[
U_k = [u_1, \ldots, u_k] \in \mathbb{R}^{m \times k} \quad \text{and} \quad V_k = [v_1, \ldots, v_k] \in \mathbb{R}^{n \times k}
\]  

be constructed from the first \( k \) columns of \( U \) and \( V \), respectively. Let \( S_k = \text{diag} \{\sigma_1, \ldots, \sigma_k\} \) be a \( k \times k \) diagonal matrix. Then the matrix

\[
T_k = U_k S_k V_k^T = \sum_{j=1}^k \sigma_j u_j v_j^T
\]  

be constructed from the first \( k \) columns of \( U \) and \( V \), respectively. Let \( S_k = \text{diag} \{\sigma_1, \ldots, \sigma_k\} \) be a \( k \times k \) diagonal matrix. Then the matrix

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\]  

be constructed from the first \( k \) columns of \( U \) and \( V \), respectively. Let \( S_k = \text{diag} \{\sigma_1, \ldots, \sigma_k\} \) be a \( k \times k \) diagonal matrix. Then the matrix
is called a rank-$k$ truncated SVD of $A$. (If $\sigma_k = \sigma_{k+1}$ then this matrix is not unique.)

Let $a_{ij}, u_{ij}, v_{ij}$ denote the $(i,j)$ entries of the matrices $A, U, V$, respectively. Then (2.4) indicates that

$$a_{ij} = \sum_{j=1}^{n} \sigma_j u_{ij} v_{ij} \quad \text{for } i = 1, \ldots, n$$

(2.9)

and

$$\sum_{i=1}^{n} |a_{ij}| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |\sigma_j| |u_{ij}| |v_{ij}| = \sum_{j=1}^{n} \sigma_j \sum_{i=1}^{n} |u_{ij}| |v_{ij}| \leq \sum_{j=1}^{n} \sigma_j,$$

(2.10)

where the last inequality follows from the Cauchy-Schwarz inequality and the fact that the columns of $U$ and $V$ have unit length.

Another useful property regards the concepts of majorization and unitarily invariant norms. Recall that a matrix norm $\| \cdot \|$ on $\mathbb{R}^{m \times n}$ is called unitarily invariant if the equalities

$$\| A \| = \| X^T A \| = \| A Y \| = \| X^T A Y \|$$

(2.11)

are satisfied for any matrix $A \in \mathbb{R}^{m \times n}$, and any pair of unitary matrices $X \in \mathbb{R}^{m \times m}$ and $Y \in \mathbb{R}^{n \times n}$. Let $B$ and $C$ be a given pair of $m \times n$ matrices with singular values

$$\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n \geq 0 \quad \text{and} \quad \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n \geq 0,$$

respectively. Let $\beta = (\beta_1, \ldots, \beta_n)^T$ and $\gamma = (\gamma_1, \ldots, \gamma_n)^T$ denote the corresponding $n$-vectors of singular values. Then the weak majorization relation $\beta \prec \gamma$ means that these vectors satisfy the inequality

$$\sum_{j=1}^{k} \beta_j \leq \sum_{j=1}^{k} \gamma_j \quad \text{for } k = 1, \ldots, n.$$

(2.12)

In this case we say that $\beta$ is weakly majorized by $\gamma$, or that the singular values of $B$ are weakly majorized by those of $C$. The dominance theorem of Fan [20] [21] relates these two concepts. It says that if the singular values of $B$ are weakly majorized by those of $C$ then the inequality

$$\| B \| \leq \| C \|$$

(2.13)

holds for any unitarily invariant norm. For detailed proof of this fact see, for example, [8], [20]-[23]. The most popular example of an unitarily invariant norm is, perhaps, the Frobenius matrix norm

$$\| A \|_F = \left( \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \right)^{\frac{1}{2}},$$

(2.14)

which satisfies

$$\| A \|_F = \text{trace} (A^T A) = \text{trace} (AA^T) = \sum_{j=1}^{n} \sigma_j^2.$$

(2.15)

Other examples are the Schatten $p$-norms,

$$\| A \|_p = \left( \sum_{j=1}^{n} \sigma_j^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty$$

(2.16)

and Ky Fan $k$-norms,

$$\| A \|_k = \sum_{j=1}^{k} \sigma_j, \quad k = 1, \ldots, n.$$  

(2.17)

The trace norm,

$$\| A \|_1 = \sum_{j=1}^{n} \sigma_j$$

(2.18)

is obtained for $k = n$ and $p = 1$, while the spectral norm
\[ \|A\|_p = \sigma_1 = \max_j \sigma_j \]  

(2.19)

corresponds to \( k = 1 \) and \( p = \infty \). The use of Ky Fan \( k \)-norms enables us to state the dominance principle in the following way.

**Theorem 1 (Ky Fan dominance theorem)** The Inequality (2.13) holds for any unitarily invariant norm if and only if

\[ \|B\|_{(k)} \leq \|C\|_{(k)} \quad \text{for } k = 1, \ldots, n. \]  

(2.20)

Another useful tool is the following “rectangular” version of Cauchy interlacing theorem. For a proof of this result see ([24], p. 229) or ([25], p. 1250).

**Theorem 2 (A rectangular Cauchy interlace theorem)** Let the \( m \times n \) matrix \( A \) have the singular values (2.2). Let the \( \tilde{m} \times \tilde{n} \) matrix \( \tilde{A} \) be a submatrix of \( A \) which is obtained by deleting \( m' \) rows and \( n' \) columns of \( A \). That is, \( \tilde{m} + m' = m \) and \( \tilde{n} + n' = n \). Define \( \tilde{k} = \min\{\tilde{m}, \tilde{n}\} \) and let

\[ \tilde{\sigma}_1 \geq \tilde{\sigma}_2 \geq \cdots \geq \tilde{\sigma}_{\tilde{k}} \geq 0 \]

denote the singular values of \( \tilde{A} \). Then

\[ \sigma_j \geq \tilde{\sigma}_j \geq \sigma_{m',n',j} \quad \text{for } j = 1, \ldots, \tilde{k}. \]  

(2.21)

To ease the coming discussions we return to square matrices. In the next assertions \( W = (w_{ij}) \in \mathbb{R}^{n \times n} \) is an arbitrary real \( n \times n \) matrix. Combining the interlace theorem with the dominance theorem leads to the following corollary.

**Theorem 3** Let the \( n \times n \) matrix \( B_k \) be obtained from \( W \) by setting to zero all the entries in the last \( n - k \) rows and columns of \( W \). Then the inequality

\[ \|B_k\| \leq \|W\| \]  

(2.22)

holds for any unitarily invariant norm.

**Theorem 4** Let the \( n \times n \) diagonal matrix

\[ \text{diag}(W) = \text{diag}\{w_{11}, \ldots, w_{nn}\}, \]

be obtained from the diagonal entries of \( W \). Then

\[ \|\text{diag}(W)\| \leq \|W\| \]  

(2.23)

in any unitarily invariant norm.

**Proof.** There is no loss of generality in assuming that the diagonal entries of \( W \) are ordered such that

\[ |w_{11}| \geq |w_{22}| \geq \cdots \geq |w_{nn}|. \]

Let the matrix \( B_k \) be defined as in Theorem 3. Then from (2.10) and (2.22) we conclude that

\[ \|\text{diag}(W)\|_{(k)} \leq \|B_k\|_{(k)} \leq \|W\|_{(k)} \quad \text{for } k = 1, \ldots, n, \]

which proves (2.23). \( \square \)

**Corollary 5** The diagonal matrix

\[ \text{diag}(B_k) = \text{diag}\{w_{11}, \ldots, w_{kk}, 0, 0, \ldots, 0\} \]

satisfies

\[ \|\text{diag}(B_k)\| \leq \|\text{diag}(W)\| \leq \|W\| \]  

(2.24)

in any unitarily invariant norm.

**Lemma 6** Let \( X \) and \( Y \) be a pair of real symmetric \( n \times n \) matrices that satisfy

\[ 0 \leq X \leq Y. \]

Then

\[ \|X\| \leq \|Y\| \]
in any unitarily invariant norm.

Proof. Using the spectral decomposition of \( X \) it is possible to assume that \( X \) is a diagonal matrix:

\[
X = \text{diag} \{ \lambda_1, \ldots, \lambda_n \}.
\]

The matrix \( Y - X \) is positive semidefinite and, therefore, has non-negative diagonal entries. This observation implies the inequalities

\[
y_{jj} \geq \lambda_j \quad \text{for } j = 1, \ldots, n,
\]

and

\[
\| X \| = \| \text{diag} \{ \lambda_1, \ldots, \lambda_n \} \| \leq \| \text{diag} \{ y_{11}, \ldots, y_{nn} \} \| = \| \text{diag} (Y) \|,
\]

while (2.23) gives

\[
\| \text{diag} (Y) \| \leq \| Y \|.
\]

\[
\text{Theorem 7 (The pinching principle)} \quad \text{Let the matrix } W \in \mathbb{R}^{n \times n} \text{ be partitioned in the form}
\]

\[
\begin{pmatrix}
W_{11} & W_{12} \\
W_{21} & W_{22}
\end{pmatrix}
\]

where \( W_{11} \in \mathbb{R}^{p \times q} \) and \( W_{22} \in \mathbb{R}^{(n-q) \times (n-q)} \). Let the \( n \times n \) matrix

\[
\hat{W} = \begin{pmatrix}
W_{11} & 0 \\
0 & W_{22}
\end{pmatrix}
\]

(2.26)

denote the “pinched” version of \( W \). Then the inequality

\[
\| \hat{W} \| \leq \| W \|
\]

(2.27)

holds in any unitarily invariant norm.

Proof. Using the SVD of \( W_{11} \) we obtain a pair of \( q \times q \) orthonormal matrices, \( U_{11} \) and \( V_{11} \), such that \( U_{11}^T W_{11} V_{11} \) is a diagonal matrix that contains the singular values of \( W_{11} \). Similarly there exists a pair of \( (n-q) \times (n-q) \) orthonormal matrices, \( U_{22} \) and \( V_{22} \), such that \( U_{22}^T W_{22} V_{22} \) is a diagonal matrix that contains the singular values of \( W_{22} \). The related \( n \times n \) matrices

\[
U = \begin{pmatrix}
U_{11} & 0 \\
0 & U_{22}
\end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix}
V_{11} & 0 \\
0 & V_{22}
\end{pmatrix}
\]

are orthonormal matrices, and

\[
U^T \hat{W} V = \begin{pmatrix}
U_{11}^T W_{11} V_{11} \\
U_{22}^T W_{22} V_{22}
\end{pmatrix}
\]

(2.28)

is a diagonal matrix. Moreover, comparing \( U^T W V \) with \( U^T \hat{W} V \) shows that

\[
U^T \hat{W} V = \text{diag} \{ U^T W V \}.
\]

(2.29)

Hence a further use of (2.23) gives

\[
\| \hat{W} \| = \| U^T \hat{W} V \| = \| \text{diag} (U^T W V) \| \leq \| U^T W V \| = \| W \|.
\]

Equality (2.28) relates the singular values of \( \hat{W} \) with those of the matrices \( W_{11} \) and \( W_{22} \). Each singular value of \( W_{11} \) is a singular value of \( \hat{W} \). Similarly, each singular value of \( W_{22} \) is a singular value of \( \hat{W} \). Conversely, each singular value of \( \hat{W} \) is a singular value of \( W_{11} \) or a singular value of \( W_{22} \). The last observation enables us to sharpen the results in certain cases. This is illustrated in Lemmas 8-11 below, which seem to be new. We will use these lemmas in the proofs of Theorems 18-21.
Lemma 8 (Pinching in Schatten \( p \)-norms)

\[
\|W\|_p \geq \|\hat{W}\|_p = \|W_{1}\|_p + \|W_{2}\|_p.
\]  

(2.30)

Lemma 9 (Pinching in the trace norm)

\[
\|W\|_\text{tr} \geq \|\hat{W}\|_\text{tr} = \|W_{1}\|_\text{tr} + \|W_{2}\|_\text{tr}.
\]  

(2.31)

Lemma 10 (Pinching in the spectral norm)

\[
\|W\|_\text{sp} \geq \|\hat{W}\|_\text{sp} = \max \{\|W_{1}\|_\text{sp}, \|W_{2}\|_\text{sp}\}.
\]  

(2.32)

Lemma 11 (Pinching in Ky Fan \( k \)-norms) Let \( k_1 \) and \( k_2 \) be a pair of positive integers that satisfy

\[
1 \leq k_1 \leq q \quad \text{and} \quad 1 \leq k_2 \leq n - q.
\]

Then

\[
\|W\|_{(k_1+k_2)} \geq \|\hat{W}\|_{(k_1+k_2)} \geq \|W_{1}\|_{(k_1)} + \|W_{2}\|_{(k_2)}.
\]  

(2.33)

Proof. The sum \( \|W_{1}\|_{(k_1)} + \|W_{2}\|_{(k_2)} \) is formed from \( k_1 + k_2 \) singular values of \( \hat{W} \), while the sum defined by \( \|\hat{W}\|_{(k_1+k_2)} \) is composed from the \( k_1 + k_2 \) largest singular values of \( \hat{W} \). \( \square \)

The next tools consider the problem of approximating one matrix by another matrix of lower rank. Let \( A \in \mathbb{R}^{m \times n} \) by a given matrix with SVD that satisfies (2.1)-(2.8). Let \( 1 \leq k < n \) be a given integer, and let

\[
\mathbb{B}_k = \{B | B \in \mathbb{R}^{m \times n} \text{ and } \text{rank}(B) \leq k\}
\]

denote the related set of low-rank matrices. Then here we seek a matrix \( B \in \mathbb{B}_k \) that is nearest to \( A \) in a certain matrix norm. The difficulty stems from the fact that \( \mathbb{B}_k \) is not a convex set. Let \( T_k \) denote a rank-\( k \) truncated SVD of \( A \) as defined in (2.8). Then the Eckart-Young theorem [26] says that \( T_k \) solves this problem in the Frobenius norm. The extension of this result to any unitarily invariant norm is due to Mirsky [27]. (Recall that \( T_k \) is not always unique. In such cases the nearest matrix is not unique.) A detailed statement of these assertions is given below. For recent discussions and proofs see [25].

Theorem 12 (Eckart-Young) The inequality

\[
\|A - B\|_F^2 \geq \sum_{j=k+1}^{n} \sigma_j^2
\]

holds for any matrix \( B \in \mathbb{B}_k \). Moreover, the matrix \( T_k \) solves the problem

\[
\begin{align*}
\text{minimize} & \quad F(B) = \|A - B\|_F^2 \\
\text{subject to} & \quad B \in \mathbb{B}_k,
\end{align*}
\]

giving the optimal value of

\[
\|A - T_k\|_F^2 = \left\| \sum_{j=k+1}^{n} \sigma_j u_j v_j^T \right\|_F^2 = \sum_{j=k+1}^{n} \sigma_j^2.
\]

Theorem 13 (Mirsky) Let \( \| \cdot \| \) be any unitarily invariant norm on \( \mathbb{R}^{m \times n} \). Then the inequality

\[
\|A - B\| \geq \|A - T_k\|
\]

holds for any matrix \( B \in \mathbb{B}_k \). In other words, the matrix \( T_k \) solves the problem

\[
\begin{align*}
\text{minimize} & \quad \mu(B) = \|A - B\| \\
\text{subject to} & \quad B \in \mathbb{B}_k,
\end{align*}
\]


3. Positive Approximants of Symmetric Matrices

In this section we consider the solution of problem (1.3). Since $\|\cdot\|$ is a unitarily invariant norm, the spectral decomposition (1.2) enables us to convert (1.3) into the simpler form

$$\begin{align*}
\text{minimize} & \quad F(S) = \|\Lambda - S\|_F \\
\text{subject to} & \quad S \in \mathbb{S}_n^+,
\end{align*}$$

(3.1)

whose solution provides a solution of (1.3).

**Theorem 14** Let the matrix $\Lambda_q$ be defined as in (1.5). Then $\Lambda_q$ solves (3.1) in any unitarily invariant norm.

*Proof.* Let the diagonal matrix $D_q$ be defined by the equality

$$\Lambda + D_q = \Lambda_q.$$ 

That is

$$D_q = \text{diag} \left[ 0, \cdots, 0, \lambda_{q+1}, \cdots, \lambda_1 \right].$$

(3.2)

Let $S = \left( s_{ij} \right)$ be some matrix in $\mathbb{S}_n^+$ and let the matrix $W = \left( w_{ij} \right) \in \mathbb{R}^{n \times n}$ be defined by the equality

$$\Lambda + W = S.$$ 

(3.3)

Then the proof is concluded by showing that

$$\|W\| \geq \|D_q\|.$$ 

(3.4)

Let the diagonal matrix

$$W_q = \text{diag} \left[ 0, \cdots, 0, w_{q+1}, \cdots, w_n \right]$$

be obtained from the last $n - q$ diagonal entries of $W$. Then Corollary 5 implies that

$$\|W\| \geq \|W_q\|.$$ 

(3.5)

(3.6)

On the other hand, since $S \succeq 0$, the diagonal entries of $S$ are non-negative, which implies the inequalities

$$w_{jj} \geq |\lambda_j| \quad \text{for} \quad j = q+1, \cdots, n,$$ 

(3.7)

and

$$\|W_q\| \geq \|D_q\|.$$ 

(3.8)

Now combining (3.6) and (3.8) gives (3.4)

Theorem 14 is not new, e.g. ([8], p. 277) and [9]. However, the current proof is simple and short. In the next sections we extend these arguments to derive low-rank approximants.

4. Low-Rank Positive Approximants in the Frobenius Norm

In this section we consider the solution of problem (1.1) in the Frobenius norm. As before, the spectral decomposition (1.2) can be used to “diagonalize” the problem and the actual problem to solve has the form

$$\begin{align*}
\text{minimize} & \quad F(S) = \|\Lambda - S\|_F \\
\text{subject to} & \quad S \in \mathbb{S}_n^+.
\end{align*}$$

(4.1)

**Theorem 15** Let the matrix $\Lambda_\ell$ be defined as in (1.6). Then this matrix solves (4.1)

*Proof.* Let the diagonal matrix $D_\ell$ be defined by the equality

$$\Lambda + D_\ell = \Lambda_\ell.$$ 

That is,

$$D_\ell = \text{diag} \left[ 0, \cdots, 0, -\lambda_{q+1}, \cdots, -\lambda_1, |\lambda_{q+1}|, \cdots, |\lambda_1| \right],$$

(4.2)
and

\[ |D|_F^2 = \sum_{j=1}^{q} \lambda_j^2 + \sum_{j=1}^{n} \lambda_j^2. \]  (4.3)

Let \( S = (s_{ij}) \) be some matrix in \( S_{n,\ell}^* \) and let the matrix \( W = (w_{ij}) \in \mathbb{R}^{n \times n} \) be defined by the equality \( \Lambda + W = S \). \hfill (4.4)

Then the proof is concluded by showing that

\[ \|W\|_F^2 \geq |D|_F^2. \]  (4.5)

This aim is achieved by considering a partition of \( W \) and \( S \) in the form

\[
W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}
\]  (4.6)

where \( W_{11} \) and \( S_{11} \) are \( q \times q \) matrices, while \( W_{22} \) and \( S_{22} \) are \( (n-q) \times (n-q) \) matrices. Then, clearly,

\[
\|W\|_F^2 \geq \|W_{11}\|_F^2 + \|W_{22}\|_F^2.
\]  (4.7)

Also, as before, since \( S \) is a positive semidefinite matrix it has non-negative diagonal entries, which implies the inequalities

\[
w_{jj} \geq \lambda_j \quad \text{for} \quad j = q + 1, \ldots, n\]  (4.8)

and

\[
\|W_{22}\|_F^2 \geq \sum_{j=q+1}^{n} w_{jj}^2 \geq \sum_{j=q+1}^{n} \lambda_j^2.
\]  (4.9)

It is left, therefore, to show that

\[
\|W_{11}\|_F^2 \geq \sum_{j=1}^{q} \lambda_j^2.
\]  (4.10)

Observe that the matrices \( W_{11} \) and \( S_{11} \) are related by the equality

\[
\Lambda_{11} + W_{11} = S_{11}
\]  (4.11)

where

\[
\Lambda_{11} = \text{diag}\{\lambda_1, \ldots, \lambda_q\} \in \mathbb{R}^{q \times q}
\]  (4.12)

and

\[
\lambda_1 \geq \cdots \geq \lambda_q > 0.
\]  (4.13)

Moreover, since \( S_{11} \) is a principal submatrix of \( S \),

\[
\text{rank}(S_{11}) \leq \text{rank}(S) \leq \ell.
\]  (4.14)

Hence from the Eckart-Young theorem we obtain that

\[
\|W_{11}\|_F^2 = \|\Lambda_{11} - S_{11}\|_F^2 \geq \sum_{j=1}^{q} \lambda_j^2. \quad \Box
\]  (4.15)

**Corollary 16** Let \( X \) be a given real symmetric \( n \times n \) matrix with the spectral decomposition \((1.2)\). Then the matrix \( X_\ell = Q \Lambda \ell Q^T \) solves the problem

\[
\text{minimize} \quad F(S) = \|X - S\|_F^2 \\
\text{subject to} \quad S \in S_{n,\ell}^*.
\]  (4.16)
Corollary 17 Let $A \in \mathbb{R}^{m \times n}$ be a given matrix, let the matrix $X = \left( A + A^T \right)/2$ have the spectral decomposition (1.2), and let the matrix $X'$ be defined in (4.16). Then $X'$ solves the problem

$$
\begin{align*}
\text{minimize} & \quad G(S) = \|A - S\|^p_p \\
\text{subject to} & \quad S \in \mathbb{S}_{n,d}^+.
\end{align*}
$$

(4.18)

5. Low-Rank Positive Approximants in the Schatten $p$-Norm

Let the diagonal matrix $\Lambda$ be obtained from the spectral decomposition (1.2). In this section we consider the problem

$$
\begin{align*}
\text{minimize} & \quad F(S) = \|A - S\|^p_p \\
\text{subject to} & \quad S \in \mathbb{S}_{n,d}^+.
\end{align*}
$$

(5.1)

**Theorem 18** Let the matrix $\Lambda$ be defined in (1.6). Then this matrix solves (5.1)

*Proof.* Let the matrices $D, W$, and $S$ be defined as in the proof of Theorem 15. Then here it is necessary to prove that

$$
\|W\|^p_p \geq \|D\|^p_p,
$$

(5.2)

where

$$
\|D\|^p_p = \sum_{j=1}^{d} \lambda_j^p + \sum_{j=q+1}^{n} |\lambda_j|^p.
$$

(5.3)

Let $W$ and $S$ be partitioned as in (4.6). Then from Lemma 8 we have

$$
\|W\|^p_p \geq \|W_{11}\|^p_p + \|W_{22}\|^p_p.
$$

(5.4)

Now Theorem 4 and (4.8) imply

$$
\|W_{22}\|^p_p \geq \|\operatorname{diag}(W_{22})\|^p_p = \sum_{j=q+1}^{n} |w_{jj}|^p \geq \sum_{j=q+1}^{n} |\lambda_j|^p,
$$

(5.5)

while applying Mirsky theorem on (4.11)-(4.14) gives

$$
\|W_{11}\|^p_p \geq \|\Lambda_{11} - S_{11}\|^p_p \geq \sum_{j=1}^{q} \lambda_j^p.
$$

(5.6)

Finally substituting (5.5) and (5.6) into (5.4) gives (5.2).

6. Low-Rank Positive Approximants in the Trace Norm

Using the former notations, here we consider the problem

$$
\begin{align*}
\text{minimize} & \quad F(S) = \|A - S\|_\text{tr} \\
\text{subject to} & \quad S \in \mathbb{S}_{n,d}^+.
\end{align*}
$$

(6.1)

**Theorem 19** The matrix $\Lambda$ solves (6.1).

*Proof.* It is needed to show that

$$
\|W\|_\text{tr} \geq \|D\|_\text{tr},
$$

(6.2)

where

$$
\|D\|_\text{tr} = \sum_{j=1}^{d} |\lambda_j| + \sum_{j=q+1}^{n} |\lambda_j|.
$$

(6.3)

The use of Lemma 9 yields

$$
\|W\|_\text{tr} \geq \|W_{11}\|_\text{tr} + \|W_{22}\|_\text{tr}.
$$

(6.4)
Here Theorem 4 and (4.8) imply the inequalities
\[ \|W_{22}\|_p \geq \|\text{diag}(W_{22})\|_p = \sum_{j=q+1}^n |w_j| \geq \sum_{j=q+1}^n |\lambda_j|, \]  
and Mirsky theorem gives
\[ \|W_{11}\|_p = \|\Lambda_{11} - S_{11}\|_p \geq \sum_{j=1}^n |\lambda_j|, \]  
which completes the proof.

7. Low-Rank Positive Approximants in the Spectral Norm

In this case we consider the problem
\[ \begin{aligned} & \text{minimize} & & F(S) = \|\Lambda - S\|_p \\ & \text{subject to} & & S \in S^+_{n,t}. \end{aligned} \]  
\[ (7.1) \]

**Theorem 20** The matrix \( \Lambda \) solves (7.1).

**Proof.** Following the former notations and arguments, here it is needed to show that
\[ \|W\|_p \geq \|D\|_p. \]
Define
\[ \alpha = \max_{j=q+1,\ldots,n} |\lambda_j| \quad \text{and} \quad \beta = \max_{j=q+1,\ldots,n} |\lambda_j|, \]
Then, clearly,
\[ \|D\|_p = \max_{j=q+1,\ldots,n} |\lambda_j| = \max \{\alpha, \beta\}. \]
Using Lemma 10 we see that
\[ \|W\|_p \geq \max \left\{ \|W_{11}\|_p, \|W_{22}\|_p \right\}. \]
Now Theorem 4 and (4.8) imply
\[ \|W_{22}\|_p \geq \|\text{diag}(W_{22})\|_p = \max_{j=q+1,\ldots,n} |w_j| \geq \max_{j=q+1,\ldots,n} |\lambda_j| = \beta, \]
while Mirsky theorem gives
\[ \|W_{11}\|_p = \|\Lambda - S_{11}\|_p \geq \max_{j=1,\ldots,q} |\lambda_j| = \alpha. \]  
\[ \square \]

8. Unitarily Invariant Norms

Let the diagonal matrices \( \Lambda \) and \( \Lambda_{\ell} \) be defined as in Section 1, and let \( \|\cdot\| \) denote any unitarily invariant norm on \( \mathbb{R}^{n \times p} \). Below we will show that \( \Lambda_{\ell} \) solves the problem
\[ \begin{aligned} & \text{minimize} & & F(S) = \|\Lambda - S\| \\ & \text{subject to} & & S \in S^+_{n,t}. \end{aligned} \]  
\[ (8.1) \]
The derivation of this result is based on the following assertion, which considers Ky Fan \( k \)-norms.

**Theorem 21** The matrix \( \Lambda_{\ell} \) solves the problem
\[ \begin{aligned} & \text{minimize} & & F(S) = \|\Lambda - S\|_{k} \\ & \text{subject to} & & S \in S^+_{n,t}. \end{aligned} \]  
\[ (8.2) \]
for \( k = 1, \ldots, n. \)
Proof. We have already proved that \( \Lambda \) solves (8.2) for the spectral norm \( (k = 1) \) and the trace norm \( (k = n) \). Hence it is left to consider the case when \( 2 \leq k \leq n - 1 \). As before, the diagonal matrix \( D_i \) is defined in (4.2), and the matrices \( S \) and \( W \) satisfy (4.4) as well as the partition (4.6). With these notations at hand it is needed to show that

\[
\|W\|_{(k)} \geq \|D_i\|_{(k)} ,
\]

(8.3)

Let \( D_i \) be partitioned in a similar way:

\[
D_i = \begin{pmatrix} D_{11} & 0 \\ 0 & D_{22} \end{pmatrix},
\]

(8.4)

where

\[
D_{11} = \text{diag}\{0, \cdots, 0, -\lambda_{q+1}, \cdots, -\lambda_q\} \in \mathbb{R}^{q \times q}
\]

(8.5)

and

\[
D_{22} = \text{diag}\{\lambda_{q+1}, \cdots, \lambda_q\} \in \mathbb{R}^{(n-q) \times (n-q)}.
\]

(8.6)

Then there are three different cases to consider.

**The first case** occurs when

\[
\|D_i\|_{(k)} = \|D_{11}\|_{(k)}.
\]

(8.7)

Here Theorem 3 implies the inequalities

\[
\|W\|_{(k)} \geq \|W_{1i}\|_{(k)} ,
\]

(8.8)

while from (4.11)-(4.14) and Mirsky theorem we obtain

\[
\|W_{1i}\|_{(k)} \geq \|D_{11}\|_{(k)} = \|D_i\|_{(k)} ,
\]

(8.9)

which proves (8.3).

**The second case** occurs when

\[
\|D_i\|_{(k)} = \|D_{22}\|_{(k)}.
\]

(8.10)

Here Theorem 3 implies

\[
\|W\|_{(k)} \geq \|W_{2i}\|_{(k)} ,
\]

(8.11)

while Theorem 4 and the inequalities (4.8) give

\[
\|W_{2i}\|_{(k)} \geq \|\text{diag}(W_{2i})\|_{(k)} \geq \|D_{22}\|_{(k)} = \|D_i\|_{(k)} ,
\]

(8.12)

which proves (8.3).

**The third case** occurs when neither (8.7) nor (8.10) hold. In this case there exist two positive integers, \( k_1 \) and \( k_2 \), such that

\[
k_1 + k_2 = k
\]

(8.13)

and

\[
\|D_i\|_{(k)} = \|D_{11}\|_{(k_1)} + \|D_{22}\|_{(k_2)}.
\]

(8.14)

Now Lemma 11 shows that

\[
\|W\|_{(k)} \geq \|W_{1i}\|_{(k_1)} + \|W_{2i}\|_{(k_2)}.
\]

(8.15)

A further use of (4.11)-(4.14) and Mirsky theorem give

\[
\|W_{1i}\|_{(k)} \geq \|D_{11}\|_{(k_1)} ,
\]

(8.16)
and from Theorem 4 and (4.8) we obtain
\[ \left\| W_{22} \right\|_{\ell_2^2} \geq \left\| \text{diag}(W_{22}) \right\|_{\ell_2^2} \geq \left\| D_{22} \right\|_{\ell_2^2}. \]  
(8.17)

Hence by substituting (8.16) and (8.17) into (8.15) we get (8.3).

The fact that (8.3) holds for \( k = 1, \cdots, n \) means that the inequality
\[ \left\| W \right\| \geq \left\| D \right\| \]  
(8.18)
holds for any unitarily invariant norm. This observation is a direct consequence of Ky Fan dominance theorem. The last inequality proves our final results.

**Theorem 22** The matrix \( \Lambda \) solves (8.1) in any unitarily invariant norm.

**Theorem 23** Using the notations of Section 1, the matrix
\[ X = QAQ^T \]
solves (1.1) in any unitarily invariant norm.

### 9. Concluding Remarks

In view of Theorem 14 and Mirsky theorem, the observation that \( \Lambda \) solves (8.1) is not surprising. However, as we have seen, the proof of this assertion is not straightforward. A key argument in the proof is the inequality (8.15), which is based on Lemma 11.

Once Theorem 22 is proved, it is possible to use this result to derive Theorems 15-18. Yet the direct proofs that we give clearly illustrate why these theorems work. In fact, the proof of Theorem 15 paves the way for the other proofs. Moreover, as Corollary 17 shows, when using the Frobenius norm we get stronger results: In this case we are able to compute a low-rank positive approximant of any matrix \( A \in \mathbb{R}^{n \times n} \).

### References


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