On the Spectral Characterization of $H$-Shape Trees

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Abstract
A graph $G$ is said to be determined by its spectrum if any graph having the same spectrum as $G$ is isomorphic to $G$. An $H$-shape is a tree with exactly two of its vertices having maximal degree 3. In this paper, a formula of counting the number of closed 6-walks is given on a graph, and some necessary conditions of a graph $\Gamma$ cospectral to an $H$-shape are given.

Keywords
Spectra of Graphs, Cospectral Graphs, Spectra Radius, $H$-Shape Trees, Determined by Its Spectrum

1. Introduction
Let $G = (V, E)$ be a simple undirected graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E$. Let $A(G)$ be the adjacency matrix of $G$. Since $A(G)$ is a real symmetric matrix, its eigenvalues must be real, and may be ordered as $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. The sequence of $n$ eigenvalues is called the spectrum of $G$, the largest eigenvalue $\lambda_1(G)$ is often called the spectral radius of $G$. The characteristic polynomial of $A(G)$ is called the characteristic polynomial of the graph $G$ and is denoted by $\phi(G, \lambda)$.

Two graphs are cospectral if they share the same spectrum. A graph $G$ is said to be determined by its spectrum (DS for short) if for any graph $H$, $\phi(H, \lambda) = \phi(G, \lambda)$ implies that $H$ is isomorphic to $G$.

Determining what kinds of graphs are DS is an old problem, yet far from resolved, in the theory of graph spectra. Numerous examples of cospectral but non-isomorphic graphs are reported in literature [1]. However, there are few results known about DS graphs. For the background and some recent surveys of the known results about this problem and related topics, we refer the reader to [2]-[6] and references therein.

Because the kind of problems above are generally very hard to deal with, some more modest ones suggested by van Dam and Haemers [2], say, “Which trees are DS?”, this problem is also very hard to deal with, because we know a famous result of Schwenk [7], which says that almost all trees have non-isomorphic cospectral

A T-shape $T(l_1, l_2, l_3)$ is a tree with exactly one of its vertices having maximal degree 3 such that $T(l_1, l_2, l_3) - v = p_h \cup p_i \cup p_i$, where $p_i$ is the path on $l_i (i = 1, 2, 3)$ vertices, and $v$ is the vertex of degree 3.

More recently, Wang proved that T-shape tree $T(l_1, l_2, l_3)$ is DS; Wang and Xu [6] proved that T-shape tree $T(l_1, l_2, l_3), l_1 \leq l_2 \leq l_3 \leq 1$ with degree 3.

In this paper, we give a formula of counting the number of closed 6-walks on a graph, and give some necessary conditions of a graph $\Gamma$ cospectral to an $H$-shape.

2. Some Lemmas

In the section, we will present some lemmas which are required in the proof of the main result.

**Lemma 2.1** [8] The characteristic polynomial of a graph satisfies the following identities:

1) $\varphi(G, \lambda) = \varphi(G_1 \cup G_2, \lambda) \varphi(G_2, \lambda)$,

2) $\varphi(G, \lambda) = \varphi(G - e, \lambda) - \varphi(G - v_1v_2, \lambda)$ if $e = v_1v_2$ is a cut-edge of $G$.

where $G - e$ denotes the graph obtained from $G$ by deleting the edge $e$ and $G - v_1v_2$ denotes the graph obtained from $G$ by deleting the vertices $v_1, v_2$ and the edges incident to it.

**Lemma 2.2** [1] Let $C_n, P_n$ denote the cycle and the path on $n$ vertices respectively. Then

$$\varphi(C_n, \lambda) = \prod_{j=1}^{n} \left( \lambda - 2 \cos \frac{2\pi j}{n} \right) = 2 \cos \left( n \arccos \frac{\lambda}{2} \right) - 2$$

$$\varphi(P_n, \lambda) = \prod_{j=1}^{n} \left( \lambda - 2 \cos \frac{\pi j}{n+1} \right) = \frac{\sin \left( (n+1) \arccos \frac{\lambda}{2} \right)}{\sin \left( \arccos \frac{\lambda}{2} \right)}$$

Let $\lambda = 2 \cos \theta$, set $t^{1/2} = e^{i\theta}$, we get $\lambda = t^{1/2} + t^{-1/2}$, it is can be write the characteristic polynomial of $C_n, P_n$ in the following form [6]:

$$\varphi(C_n, t^{1/2} + t^{-1/2}) = t^{n/2} + t^{-n/2} - 2 = t^{n/2} \left( t^{1/2} - 1 \right)^2$$

$$\varphi(P_n, t^{1/2} + t^{-1/2}) = t^{n/2} \left( t^{1/2} - 1 \right)^2$$

**Lemma 2.3** [4] [9] Let $\varphi(G, x) = \sum a_i x^{n-i}$ be the characteristic polynomial of graph $G$ with $n$ vertices, then the coefficient of $x^{n-i}$ is

$$a_i = \sum_{\gamma} (-1)^{\text{comp}(\gamma)} 2^{\text{cyc}(\gamma)}$$

where $a_0 = 1$ and the sum is over all subgraphs $\gamma$ of $G$ consisting of disjoint edges and cycles, and having $i$ vertices. If $\gamma$ is such a subgraph then $\text{comp}(\gamma)$ is the number of components in it and $\text{cyc}(\gamma)$ is the number of cycles.

**Lemma 2.4** [2] [10] Let $G$ be a graph. For the adjacency matrix, the following can be obtained from the spectrum.

1) The number of vertices.
2) The number of edges.
3) Whether $G$ is regular.
4) Whether $G$ is regular with any fixed girth.
5) The number of closed walk of any length.
6) Whether $G$ is bipartite.
3. Main Results

The total number of closed $k$-walks in a graph $G$, denoted by $w_k(G)$. 

**Lemma 3.1** ([6] p. 657) Let $G$ be a graph with $e$ edges, $x_i$ vertices of degree $i$, and $y$ 4-cycles. Then 

$$
[w_k(G)] = 2e + 4\sum_{i=2}^{\infty} \binom{i}{2} x_i + 8y
$$

(4)

**Lemma 3.2** Let $\Gamma$ be a graph with $n$ vertices. If $\Gamma$ cospectral to an $H$-shape and $\Gamma \neq W_n$, then 

1) $\Gamma$ have the same degree sequences as the $H$-shape tree or $\Gamma$ have the degree sequences $(3, 2, 2, \cdots, 2, 1, 0)$. 

2) $\Gamma$ contains no 4-cycles. 

**Proof.** Let $\Gamma$ be a graph with $e$ edges, $x_i$ vertices of degree $i$, and $y$ 4-cycles. By lemma 2.4 we known that cospectral graphs have the same number of edges and closed 4-walks, respectively. Since $\Gamma$ is cospectral to an $H$-shape tree, hence by (4) we have 

$$
2e + 4\sum_{i=2}^{\infty} \binom{i}{2} x_i + 8y = 6n - 2
$$

namely 

$$
\sum_{i=2}^{\infty} \binom{i}{2} x_i + 2y = n = \sum_{i=2}^{\infty} x_i
$$

(5)

Since 

$$
\sum_{i=2}^{n} (i-1) x_i = \sum_{i=2}^{\infty} ix_i - \sum_{i=2}^{\infty} x_i = (2e - x_i) - \left(n - x_0 - x_i\right) = 2e - n + x_0 = n - 2 + x_0,
$$

(6)

from (5), we have 

$$
\sum_{i=2}^{\infty} \binom{i-1}{2} x_i + 2y = n - \sum_{i=2}^{\infty} (i-1) = 2 - x_0
$$

(7)

the (7) imply to $y = 1$ or 0. 

**Case 1.** $y = 1$. By (7) we get $x_0 = 0$ and $x_1 = x_2 = \cdots = 0$, by (5) we get $x_2 = n - 2$ and $x_4 = 2$, then $\Gamma = C_4 \cup P_{n-2}$. 

We known that “the spectrum of graph $W_n$ is the union of the spectra of the circuit $C_4$ and the path $P_{n-4}$” [1], that is 

$$
\varphi(W_n, \lambda) = \varphi(C_4 \cup C_1 \cup P_{n-4}, \lambda)
$$

**Case 2.** $y = 0$. By (7) we have $x_0 \leq 2$. 

If $x_0 = 0$, then $x_2 = 2, x_4 = x_5 = \cdots = 0$, by (5) we get $x_2 = n - 6$ and $x_4 = 4$. Thus $\Gamma$ have the same degree sequences as the $H$-shape tree. 

If $x_0 = 1$, then $x_2 = 1, x_3 = x_4 = \cdots = 0, x_2 = n - 3$ and $x_4 = 1$. The degree sequences of $\Gamma$ is $(3, 2, 2, \cdots, 2, 1, 0)$. 

If $x_0 = 2$, then $x_3 = x_4 = \cdots = 0, x_2 = n, |\Gamma| \geq n + 2$, a contradiction. □

**Lemma 3.3** Let $G$ be a graph with $e$ edges, $x_i$ vertices of degree $i$, and $z$ 6-cycles. Then 

$$
[w_k(G)] = 2e + 12\sum_{i=2}^{\infty} \binom{i}{2} x_i + 6p_4 + 12k_{1,3} + 12z
$$

(8)

where $p_4$ is the number of induced paths of length three and $k_{1,3}$ is the number of induced star $K_{1,3}$. 

**Proof.** A close walk of length 6 can be produced from in the following five classes graphs, they are $P_3, P_4, P_5, K_{1,3}$ and $C_6$. For an edge and a 6-cycle, it is easy to see that the number of close 6-walks equals 2 and 12, respectively. For a $P_3$, the number of close 6-walks of a 1-degree vertex is 3 and the number of close 6-walks of the 2-degree vertex is 6, since the number of induced paths of length two is $\sum_{i=2}^{\infty} \binom{i}{2} x_i$, hence for all induced paths $P_3$, the number of close 6-walks is $12\sum_{i=2}^{\infty} \binom{i}{2} x_i$. For a $P_4$, since the number of close 6-walks of a 1-degree ver-
tex is 1 and the number of close 6-walks of a 2-degree vertex is 2, hence for all induced paths \( P_4 \), the number of close 6-walks of a 2-degree vertex is 6, thus for all induced stars \( K_{1,3} \), the number of close 6-walks is 12 \( k_{1,3} \).

**Corollary 3.4** Let \( H = H (l_1, l_2, l_3, l_4, l_5) \), then

\[
[w_k (H)] = \begin{cases} 
20n + 28 - 6k & \text{if } l_i \geq 1 \text{ and have } k \text{ elements are 1 in } \{l_2, l_3, l_4, l_5\} \\
20n + 34 - 6k & \text{if } l_i = 0 \text{ and have } k \text{ elements are 1 in } \{l_2, l_3, l_4, l_5\}
\end{cases}
\]

(9)

where \( 0 \leq k \leq 4 \).

**Proof.**

**Case 1.** \( l_1 \geq 1 \).

1) If \( k = 0 \), that is \( l_i \geq 2 (i = 2, 3, 4, 5) \), then

\[
w_k (H) = 2(n - 1) + 12[(n - 6) + 3 \times 2] + 6[(l_2 + l_3 - 2) + (l_4 + l_5 - 2) + (l_1 - 1) + 8] + 12 \times 2 = 20n + 28
\]

where \((l_2 + l_3 - 2), (l_4 + l_5 - 2)\) and \((l_1 - 1)\) are the number of induced paths \( P_4 \) in \( P_{l_2+l_3+1}, P_{l_4+l_5+1} \) and \( P_{l_1+2} \), respectively. The \( 8(=4+4) \) is the number of induced paths of through a 3-degree vertex \( u \) (or \( v \)). If \( P_4 \) is such a induced path, then \( u \) is an internal vertex in the \( P_4 \) and have at least a vertex in the \( l_2 \) (or \( l_3 \)).

2) If \( k \neq 0 \), then

\[
w_k (H) = 2(n - 1) + 12[(n - 6) + 3 \times 2] + 6[(l_2 + l_3 - 2) + (l_4 + l_5 - 2) + (l_1 - 1) + (8 - k)] + 12 \times 2 = 20n + 28 - 6k
\]

**Case 2.** \( l_1 = 0 \).

1) If \( k \neq 0 \), then

\[
w_k (H) = 20n + 34 - 6k
\]

2) If \( k = 0 \), similarly, we have \( w_0 (H) = 20n + 34 - 6k \).

**Example 1.** Let \( H_1 = H (0, 1, 1, 1, 1) \), by (9) we have \( w_6 (H_1) = 20 \times 6 + 34 - 6 = 130 \), if we give to a suitable label for the \( H_1 \), by a simple calculation we can get the diagonal matrix of \( A^6 (H_1) \), that is

\[
\text{diag} (A^6 (H_1)) = [11, 11, 43, 43, 11, 11]
\]

clearly, the sum of the elements in the diagonal matrix equals \( 4 \times 11 + 2 \times 43 = 130 \).

**Example 2.** Let \( H_2 = H (2, 2, 2, 2, 2) \), by (9) we have \( w_6 (H_2) = 20 \times 12 + 28 = 268 \), similarly, if we give to a suitable label for the \( H_2 \), then we can get the diagonal matrix of \( A^6 (H_2) \), that is

\[
\text{diag} (A^6 (H_2)) = [6, 6, 6, 6, 22, 22, 22, 22, 29, 29, 49, 49]
\]

clearly, the sum of the elements in the diagonal matrix equals \( 4 \times 6 + 4 \times 22 + 2 \times 29 + 2 \times 49 = 268 \).

**Lemma 3.5** Let \( \Gamma \) be a graph with \( n \) vertices, \( e \) edges, \( x_i \) vertices of degree \( i \), and \( z \) 6-cycles. If \( \Gamma \) cospectral to \( H (l_1, l_2, l_3, l_4, l_5) \) and \( \Gamma \neq W_n \), then

\[
2 \sum_{i=2}^{5} \left( \binom{i-1}{2} x_i + p_4 + 2k_{1,3} + 2z = \begin{cases} 
n + 9 - k - 2x_6 & \text{if } l_i \geq 1 \text{ and have } k \text{ elements are 1 in } \{l_2, l_3, l_4, l_5\} \\
n + 10 - 2x_6 & \text{if } l_i = 0 \text{ and have } k \text{ elements are 1 in } \{l_2, l_3, l_4, l_5\}
\end{cases}
\]

(10)

where \( k (0 \leq k \leq 4) \) is the number of elements of equals 1 in \( \{l_2, l_3, l_4, l_5\} \) and \( p_4 \) is the number of induced paths of length three and \( k_{1,3} \) is the number of induced star \( K_{1,3} \) in \( \Gamma \).

**Proof.** If \( l_i \geq 1 \), by Lemma 3.3 we have

\[
2e + 12 \sum_{i=1}^{5} \binom{i}{2} x_i + 6p_4 + 12k_{1,3} + 12z = 20n + 28 - 6k,
\]

\[
2 \sum_{i=2}^{5} \left( \binom{i}{2} x_i + p_4 + 2k_{1,3} + 2z = 3n + 5 - k \right),
\]

\[
2 \sum_{i=2}^{5} \left( \binom{i-1}{2} x_i + p_4 + 2k_{1,3} + 2z = 3n + 5 - 2 \sum_{i=2}^{5} (i-1)x_i = 3n + 5 - 2(n-2 + x_6) = n + 9 - k - 2x_6.
\]
Similarly, when \( l_i = 0 \) the (10) hold. □

**Definition 1.** Let \( U \) be a graph obtained from a cycle \( C_6 \) (g is even and \( 6 \leq g \leq n_1 - 2 \)) and a path \( P_{n_{1-1}} \), such that identifying an end vertex in the path and any one vertex in the cycle, and uniting an isolated vertex \( K_1 \).

If a graph have the degree sequences \( (3, 2, 2, \cdots, 2, 1, 0) \), then the graph is \( H \)-shaping some cycle.

**Lemma 3.6** Let \( U' \) be a graph with degree sequences \( (3, 2, 2, \cdots, 2, 1, 0) \). If \( U' \) cosppectral to an \( H \)-shape, then \( U' \) and \( H \) satisfying one of the following conditions.

1) There are one 6-cycle in \( U' \) and \( l_i \geq 1, l_2, l_3, l_4, l_5 \geq 2 \).
2) There are one 6-cycle in \( U' \) and \( l_i = 0 \), have an element is 1 in \( \{ l_2, l_3, l_4, l_5 \} \).
3) No 6-cycle in \( U' \) and \( l_i \geq 1 \), have two elements are 1 in \( \{ l_2, l_3, l_4, l_5 \} \).
4) No 6-cycle in \( U' \) and \( l_i = 0 \), have three elements are 1 in \( \{ l_2, l_3, l_4, l_5 \} \).

**Proof.** Without loss of generality, Let \( U' = U \cup C_{n_2} \), where \( n_2 \geq 6 \) is even and \( n_1 + n_2 = n \). Let \( U' \) have \( e \) edges, \( x_i \) vertices of degree \( i \), and \( z \) 6-cycles.

**Case 1.** \( l_i \geq 1 \). By Lemma 3.5 we have \( 2 \times 1 + [ g + (n_i - g - 3) + 4 + n_2 ] + 2 \times 1 + 2z = n + 9 - k - 2, 2z = 2 - k \), get \( k = 0, z = 1 \) or \( k = 2, z = 0 \).

**Case 2.** \( l_i = 0 \), we have \( 2 \times 1 + [ g + (n_i - g - 3) + 4 + n_2 ] + 2 \times 1 + 2z = n + 10 - k - 2, 2z = 3 - k \), get \( k = 1, z = 1 \) or \( k = 3, z = 0 \). □

**Lemma 3.7** Let \( \lambda = t^{\lfloor z/2 \rfloor} + t^{\lfloor z/2 \rfloor} \), then

\[
\phi \left( H \left( l_1, l_2, l_3, l_4, l_5 \right), \lambda \right) = \frac{t^{\lfloor z/2 \rfloor}}{(t - 1)^3} \left[ (t - 1)^2 \left( t^{l_{k+1}} - 1 \right) \left( t^{l_{k+2}} - 1 \right) \left( t^{l_{k+3}} - 1 \right) \left( t^{l_{k+4}} - 1 \right) - t(t - 1) \left( t^{l_{k+1}} - 1 \right) \left( t^{l_{k+2}} - 1 \right) \left( t^{l_{k+3}} - 1 \right) \left( t^{l_{k+4}} - 1 \right) \right] \]

(11)

**Proof.** By Lemma 2.1 (b) and Lemma 2.2 we have

\[
\phi \left( H \left( l_1, l_2, l_3, l_4, l_5 \right), \lambda \right) = \phi \left( P_{l_{k+1}}, \lambda \right) \phi \left( T \left( l_1, l_2, l_3 \right), \lambda \right) - \phi \left( P_{l_{k+1}}, \lambda \right) \phi \left( P_{l_{k+1}}, \lambda \right) \phi \left( T \left( l_1, l_2, l_3 \right), \lambda \right)
\]

(12)

If a graph has the same degree sequences as the \( H \)-shape, then \( \Gamma' \) is one of the following graphs \( G_1, G_2, G_3, G_4, G_5 \) in figure or it is an \( H \)-shape.

![Graphs G1, G2, G3, G4, G5](image)

**Lemma 3.8** If \( \Gamma \) is cospectrally to an \( H \)-shape tree, then \( \Gamma \) contains no \( P_{n_1} \cup P_{n_2} \) \((n_1, n_2 < n)\) as two connected component.

**Proof.** Assume that \( \Gamma \) contains a \( P_{n} \) as a connected component, by (11) some \( l_i \) is equal, without loss of generality, let \( l_i = l_2 = l_4 = n_1 \), then
If $\Gamma$ contains a $P_n$ as a connected component, then $l_3 = l_4$ and $l_1 + l_2 + 1 = l_3$, a contradiction. 

Thus, if a graph $\Gamma (\Gamma \neq W_n)$ is one of the following graphs $G_3, G_4, G_5$ (Fig.) uniting some even cycle, respectively, or it is an $H$-shape.

**Lemma 3.9** If $H_i = H(m_1, m_2, m_3, m_4, m_5)$ and $H = H(l_1, l_2, l_3, l_4, l_5)$ are cospectral, then $H(m_1, m_2, m_3, m_4, m_5) \cong H(l_1, l_2, l_3, l_4, l_5)$.

**Proof.** By (11) we have

\[
\phi(H(l_1, l_2, l_3, l_4, l_5), \lambda) = \phi(H(m_1, m_2, m_3, m_4, m_5), \lambda) - \phi(H(l_1, l_2, l_3, l_4, l_5), \lambda) + \phi(H(l_1, l_2, l_3, l_4, l_5), \lambda) - \phi(H(l_1, l_2, l_3, l_4, l_5), \lambda)
\]

which completes the proof.

Thus, if $\Gamma$ contains a $P_n$ as a connected component, then $l_3 = l_4$ and $l_1 + l_2 + 1 = l_3$, a contradiction. 

**Lemma 3.9** If $H_i = H(m_1, m_2, m_3, m_4, m_5)$ and $H = H(l_1, l_2, l_3, l_4, l_5)$ are cospectral, then $H(m_1, m_2, m_3, m_4, m_5) \cong H(l_1, l_2, l_3, l_4, l_5)$.

**Proof.** By (11) we have

\[
\phi(H(l_1, l_2, l_3, l_4, l_5), t^{1/2} + t^{-1/2}) t^{n/2} (t-1)^3
\]

which completes the proof. 

**Theorem 3.10** Let $G_5$ be a graph in Figure, then $G_5$ and $H$-shape are not cospectral.

**Proof.** Let $G_5 - u - v = P_{m_1} \cup P_{m_2} \cup P_{m_3} \cup P_{m_4} \cup P_{m_5-1} = 1$ (where $m_1, m_2, m_3, m_4, m_5 \geq 4$), that is, $m_1 + m_2 + m_3 + m_4 + m_5 + 1 = n$. Denote the first component by $G_{S_1}$ and the second component by $G_{S_2}$. By Lemma 2.1 and Lemma 2.3 we have
\[
\varphi(G_{5,1}, \lambda) = \varphi(C_m, \lambda) \varphi(P_{m_4}, \lambda) - \varphi(P_{m_4-1}, \lambda) \varphi(P_{m_4-1}, \lambda)
\]

\[
\varphi(G_{5,2}, t^{l/2} + t^{-l/2})
\]

\[
= \frac{t^{-m_2/2} (r_{m_2} - 1)^2}{(t-1)^2} \left[ \left( \frac{t^{-m_2/2} (r_{m_2} + m_2 + 1)}{t-1} \right)^2 - \left( \frac{t^{-m_2/2} (r_{m_2} - 1)}{t-1} \right)^2 \right]
\]

By Lemma 2.1 (a) we have

\[
\varphi(G_{5,2}, t^{l/2} + t^{-l/2}) t^{m_2/2} (t-1)^5
\]

\[
= \frac{t^{-m_2/2} (r_{m_2} - 1)^2}{(t-1)^2} \left[ \left( \frac{t^{-m_2/2} (r_{m_2} + m_2 + 1)}{t-1} \right)^2 - \left( \frac{t^{-m_2/2} (r_{m_2} - 1)}{t-1} \right)^2 \right]
\]

\[
= \psi_{G_5}(t)
\]

Comparing (14) and (15), since \( \psi_{G_5}(0) = -1 \) for any \( l_i (i = 1, 2, \cdots, 5) \) and \( \psi_{G_5}(0) = 1 \) for any \( m_i (i = 1, 2, \cdots, 5) \), hence \( \psi_{G_5}(t) \neq \psi_{G_5}(t) \). \( G_5 \) and \( H \)-shape are not cospectral.

**Remark.** If \( G_5 \) uniting some \( C_{n_i} \), without loss of generality, let \( G_{5,i} = G_5 \cup C_{n_{n_i}} \), where \( m_1 + m_2 + m_3 + m_4 + m_5 + 1 = n_1 \). Since \( \varphi(C_{n-n_1} t^{l/2} + t^{-l/2}) = t^{(n-n_1)/2} \left( \frac{l^{(n-n_1)/2} - 1}{t-1} \right)^2 \), we have \( \psi_{G_5}(t) = \psi_{G_5}(t) \left( \frac{l^{(n-n_1)/2} - 1}{t-1} \right)^2 \), \( \psi_{G_5}(0) = \psi_{G_5}(0) = 1 \), \( \psi_{G_5}(t) \neq \psi_{G_5}(t) \). Thus, \( G_5 \) and \( H \)-shape are not cospectral.

**Theorem 3.11** Let \( H = H(l_1, l_2, l_3, l_4, l_5) \) \( (l_i \geq 0, l_i \geq 1, i = 2, 3, 4, 5) \), if a graph \( \Gamma \) (\( \Gamma \neq W_n \)) cospectral to an \( H \)-shape, then either \( \Gamma \) is \( U \) (Definition 1) uniting some even cycles \( C_{n_i} (n_i \geq 6) \), denoted by \( U' \), and \( U' \), \( H \) satisfying one of the following conditions.

1) There are one 6-cycle in \( U' \) and \( l_i \geq 1, l_i \geq 1, i = 2, 3, 4, 5 \).
2) There are one 6-cycle in \( U' \) and \( l_i = 0, \) have 1 element is 1 in \( \{ l_2, l_3, l_4, l_5 \} \).
3) No 6-cycle in \( U' \) and \( l_i \geq 1, \) have 2 elements are 1 in \( \{ l_2, l_3, l_4, l_5 \} \).
4) No 6-cycle in \( U' \) and \( l_i = 0, \) have 3 elements are 1 in \( \{ l_2, l_3, l_4, l_5 \} \), or \( \Gamma \) is the graph \( G_3 \) and \( G_4 \) in Figure uniting some even cycles \( C_{n_i} (n_i \geq 6) \), respectively.

**Proof.** This result is contained from Lemma 3.2 up to Lemma 3.10.

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