An Existence Theorem of Solutions for the System of Generalized Vector Quasi-Variational-Like Inequalities

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ABSTRACT
In this paper, we introduce and study the system of generalized vector quasi-variational-like inequalities in Hausdorff topological vector spaces, which include the system of vector quasi-variational-like inequalities, the system of vector variational-like inequalities, the system of vector quasi-variational inequalities, and several other systems as special cases. Moreover, a number of C-diagonal quasiconvexity properties are proposed for set-valued maps, which are natural generalizations of the g-diagonal quasiconvexity for real functions. Together with an application of continuous selection and fixed-point theorems, these conditions enable us to prove unified existence results of solutions for the system of generalized vector quasi-variational-like inequalities. The results of this paper can be seen as extensions and generalizations of several known results in the literature.

Keywords: The System of Generalized Vector Quasi-Variational-Like Inequalities; Fixed Point Theorem; Open Lower Section; Upper Semicontinuous; C-Diagonal Quasiconvexity

1. Introduction and Formulation
In recent years, the system of generalized vector quasi-variational-like inequality, which is a unified model for the system of vector quasi-variational-like inequalities, the system of vector variational-like inequalities, the system of vector variational inequalities, the system of vector equilibrium problems and the system of variational inequalities etc., has been studied (see [1-18] and references therein).

In this paper, we consider the systems of four kinds of generalized vector quasi-variational-like inequalities with set-valued mappings and discuss the existence of its solutions in locally convex topological vector space (l.c.s. in short), motivated and inspired by the recent works of Peng [1] and Ansari et al. [2].

Throughout this paper, unless otherwise specified, assume that I be an index set. For each i ∈ I , let Zᵢ be a locally convex topological vector space (l.c.s., in short) and Kᵢ be a nonempty convex subset of Hausdorff topological vector space (t.v.s., in short) Eᵢ. Let Yᵢ be a subset of continuous function space L(Eᵢ,Zᵢ) from Eᵢ into Zᵢ , where L(Eᵢ,Zᵢ) is equipped with a σ - topology. Let int A and coA denote the interior and convex hull of a set A respectively. Let Cᵢ : Kᵢ → 2ᵢ be a set-valued mapping such that intCᵢ(x) ≠ ∅ for each x ∈ Kᵢ. Denote that K = ∏ᵢ∈I Kᵢ and E = ∏ᵢ∈I Eᵢ.

For each i ∈ I , let ηᵢ : Kᵢ × Kᵢ → Eᵢ be a vector-valued mapping, Gᵢ : L(E,Z) → 2ᵢ(Eᵢ,Zᵢ), Sᵢ : K × K → 2ᵢ , Tᵢ : K → 2ᵢ and Dᵢ : K → 2ᵢ be four set-valued mappings. Then,

1) Strong type I system of generalized vector quasi-variational-like inequalities which is to find (x, T) ∈ K × Y such that x ∈ Dᵢ(x), T ∈ Tᵢ(x) and

\[ \{Gᵢ(x, x, x) + Sᵢ(x, y, x)\} \in Cᵢ( x, y, x) \] \quad ∀ y ∈ Dᵢ(x), (1.1)

2) Strong type II system of generalized vector quasi-variational-like inequalities which is to find (x, T) ∈ K × Y such that x ∈ Dᵢ(x), T ∈ Tᵢ(x) and

\[ \{Gᵢ(x, x, x) + Sᵢ(x, y, x)\} \cap Cᵢ( x, y, x) \neq ∅ \] \quad ∀ y ∈ Dᵢ(x), (1.2)

3) Weak type I system of generalized vector quasi-variational-like inequalities which is to find (x, T) ∈ K × Y such that x ∈ Dᵢ(x), T ∈ Tᵢ(x) and
where \( \langle l, x \rangle \) denotes the evaluation of \( l \in L(E, Z) \) at \( x \in E \). By the corollary of the Schaefer [3], \( L(E, Z) \) becomes a l.c.s.. By Ding and Tarafdar [4], the bilinear map \( \langle \cdot, \cdot \rangle: L(K, Z) \times K \to Z \) is continuous.

The following problems are the special cases of above four kinds of systems of generalized vector quasi-variational-like inequalities.

The above system of generalized vector quasi-variational-like inequalities encompass many models of system of variational inequalities. The following problems are the special cases of problem (1.4).

1) If for each \( i \in I \), let \( G_i \) be an identity mapping, \( S_i = 0 \), problem (1.4) reduces to the system of generalized quasi-variational-like inequalities of finding \( \vec{x} \in K \) such that for each \( i \in I \), \( \vec{x} \in D_i(\vec{x}) \) and

\[
\forall y_i \in D_i(\vec{x}), \exists \bar{T}_i \in T_i(\vec{x}) : \langle \bar{T}_i, \eta \rangle(y_i, \vec{x}) \not\in -int C_i(\vec{x}),
\]

which was introduced and studied by Peng [1].

2) If for each \( i \in I \), let \( G_i \) be an identity mapping, \( S_i = 0 \) and \( D_i(x) = K_i \), problem (1.5) reduces to the system of generalized variational-like inequalities of finding \( \vec{x} \in K \) such that for each \( i \in I \), \( \vec{x} \in K_i \) and

\[
\forall y_i \in K_i, \exists \bar{T}_i \in T_i(\vec{x}) : \langle \bar{T}_i, \eta \rangle(y_i, \vec{x}) \not\in -int C_i(\vec{x}),
\]

In addition, let \( Z_\theta = \mathbb{R}^\theta = \{r \in \mathbb{R} | r \geq 0\} \) for all \( x \in K \), then problem (1.5) reduces to the system of generalized vector quasi-variational inequalities studied by Ansari and Yao [5].

3) If for each \( i \in I \), \( G_i \) be an identity mapping, \( S_i = 0 \), \( \eta \rangle(y_i, \vec{x}) = y_i - \vec{x} \) and \( D_i(x) = K_i \), then problem (1.5) reduces to the system of generalized vector variational inequalities of finding \( \vec{x} \in K \) such that for each \( i \in I \), \( \vec{x} \in K_i \) and

\[
\forall y_i \in K_i, \exists \bar{T}_i \in T_i(\vec{x}) : \langle \bar{T}_i, \eta \rangle(y_i, \vec{x}) \not\in -int C_i(\vec{x}),
\]

4) If \( I = \{1\} \), problem (1.4) reduces to generalized vector quasi-variational-like inequalities of finding \( \vec{x} \in K \) such that \( \vec{x} \in D(\vec{x}) \) and

\[
\langle G(\vec{x}), \eta \rangle(y, \vec{x}) + S(\vec{x}, y) \not\in -int C(\vec{x}), \forall y_i \in K, \quad (1.8)
\]

such type of problem studied in [6-10].

5) If \( I = \{1\} \) and \( \eta \rangle(y, \vec{x}) = y - \vec{x}, \ T \) is single valued mapping, \( G \) be an identity mapping, \( S = 0 \), and \( C(x) = \mathbb{R}^+ \) for all \( x \in K \), then problem (1.4) reduces to classical variational inequality problem of finding \( \vec{x} \in K \) such that \( \vec{x} \in D(\vec{x}) \) and

\[
\forall y \in D(\vec{x}), \exists T \in T(\vec{x}) : \langle T(\vec{x}), (y - \vec{x}) \rangle \not\in -int C(\vec{x}),
\]

which was introduced and studied by Hartman and Stampacchia [11].

### 2. Preliminaries

**Definition 2.1.** [12] Let \( E \) and \( Z \) be two t.v.s. and \( K \) be a convex subset of t.v.s. \( E \). Let \( C: K \to 2^Z \) and \( \theta: K \times K \to 2^Z \) be two set-valued mappings. Assume given any finite subset \( \Lambda = \{x_i, x_2, \cdots, x_n\} \) in \( K \), any \( x = \sum_{i=1}^n \alpha_i x_i \), with \( \alpha_i \geq 0 \) for \( i = 1, \cdots, n \), and \( \sum_{i=1}^n \alpha_i = 1 \).

Then, 1) \( \theta \) is said to be strong Type I C-diagonally quasiconvex (SIC-DQC, in short) in the second argument if for some \( x_i \in \Lambda \),

\[
\theta(x, x_i) \subseteq C(x);
\]

2) \( \theta \) is said to be strong Type II C-diagonally quasiconvex (SIIC-DQC, in short) in the second argument if for some \( x_i \in \Lambda \),

\[
\theta(x, x_i) \cap C(x) \neq \emptyset;
\]

3) \( \theta \) is said to be weak Type I C-diagonally quasiconvex (WIC-DQC, in short) in the second argument if for some \( x_i \in \Lambda \),

\[
\theta(x, x_i) \cap -int C(x) \neq \emptyset;
\]

4) \( \theta \) is said to be weak Type II C-diagonally quasiconvex (WIIC-DQC, in short) in the second argument if for some \( x_i \in \Lambda \),

\[
\theta(x, x_i) \not\subseteq -int C(x).
\]

It is easy to verify that the following proposition, 1) SIC-DQC implies SIIC-DQC; 2) SIIC-DQC implies WIC-DQC; 3) WIC-DQC implies WIIC-DQC. The converse is not true. Following example shows that the converse is not true.

**Example 2.1.** Let \( E = Z = \mathbb{R} \) and \( \phi(x, x_2) = co\{x, x_2\} \). Then \( \phi \) is SIIC-DQC, but it is not SIC-DQC.

**Definition 2.2.** [13] Let \( E \) and \( Z \) be two t.v.s. and \( K \) be a convex subset of t.v.s. \( E \). A mapping \( \theta: K \times K \to 2^Z \) is called (generalized) vector 0-
diagonally convex if for any finite subset
\[ \Lambda = \{ x_1, x_2, \cdots, x_n \} \] of \( K \) and any \( x = \sum_{i=1}^{n} \alpha_i x_i \) with \( \alpha_i \geq 0 \) for \( i = 1, \cdots, n \), and \( \sum_{i=1}^{n} \alpha_i = 1 \),
\[ \sum_{i=1}^{n} \alpha_i \theta(x, x_i)(\mathcal{G}) \not\subseteq -\text{int } C(x). \]

**Definition 2.3.** [14] Let \( X \) and \( Y \) be two topological spaces and \( T : X \to 2^Y \) be a set-valued mapping. Then,
1) \( T \) is said to have open lower sections if the set \( T^{-1}(y) = \{ x \in X : y \in T(x) \} \) is open in \( X \) for every \( y \in Y \);
2) \( T \) is said to be upper semicontinuous (u.s.c., in short) if for each \( x_0 \in X \) and each open set \( U \) in \( Y \) with \( T(x_0) \subseteq U \), there exists an open neighborhood \( V \) of \( x_0 \) in \( X \) such that \( T(x) \subseteq U \) for each \( x \in V \);
3) \( T \) is said to be lower semicontinuous (l.s.c., in short) if for each \( x_0 \in X \) and each open set \( U \) in \( Y \) with \( T(x_0) \cap U \neq \emptyset \), there exists an open neighborhood \( V \) of \( x_0 \) in \( X \) such that \( T(x) \cap U \neq \emptyset \) for each \( x \in V \);
4) \( T \) is said to be continuous if it is both upper and lower semicontinuous;
5) \( T \) is said to be closed if for any net \( \{ x^\alpha \} \) in \( X \) such that \( x^\alpha \to x^\alpha \) and any net \( \{ y^{\alpha_i} \} \) in \( Y \) such that \( y^{\alpha_i} \to y^* \) and \( y^{\alpha_i} \in T(x^{\alpha_i}) \) for any \( \alpha \), we have \( y^* \in T(x^*) \).

**Lemma 2.1.** [15] Let \( X \) and \( Y \) be two topological spaces. If \( T : X \to 2^Y \) is u.s.c. set-valued mapping with closed values, then \( T \) is closed.

**Lemma 2.2.** [16] Let \( X \) and \( Y \) be two topological spaces and \( T : X \to 2^Y \) is u.s.c. mapping with compact values. Suppose \( \{ x^\alpha \} \) is a net in \( X \) such that \( x^\alpha \to x^* \). If \( y^* \in T(x^*) \) for each \( \alpha \), then there are a \( y^* \in T(x^*) \) and a subnet \( \{ y^{\alpha_i} \} \) of \( \{ y^\alpha \} \) such that \( y^{\alpha_i} \to y^* \).

**Lemma 2.3.** [17] Let \( X \) and \( Y \) be two topological spaces. Suppose that \( T : X \to 2^Y \) and \( K : X \to 2^Y \) are set-valued mappings having open lower sections, then
1) A set-valued mapping \( F : X \to 2^Y \) defined by, for each \( x \in X \), \( F(x) = \text{co} T(x) \) has open lower sections;
2) A set-valued mapping \( J : X \to 2^Y \) defined by, for each \( x \in X \), \( J(x) = T(x) \cap K(x) \) has open lower sections.

For each \( i \in I \), \( E_i \) a Hausdorff t.v.s. Let \( \{ K_i \} \) be a family of nonempty compact convex subsets with each \( K_i \) in \( E_i \). Let \( K = \bigcap_{i \in I} K_i \) and \( E = \bigcap_{i \in I} E_i \). The following system of fixed-point theorem is needed in this paper.

**Lemma 2.4.** [18] For each \( i \in I \), let \( T_i : K \to 2^{K_i} \) be a set-valued mapping. Assume that the following conditions hold.
1) For each \( i \in I \), \( T_i \) is convex set-valued mapping;
2) \( K = \bigcup \{ \text{int } T_i^{-1}(x_i) : x_i \in K_i \} \).

Then there exist \( x_i \in K_i \) such that \( x_i \in T(x_i) = \bigcap_{i \in I} T_i(x_i) \), that is, \( x_i \in T_i(x) \) for each \( i \in I \), where \( x_i \) is the projection of \( x \) onto \( K_i \).

### 3. Main Results

**Theorem 3.1.** For each \( i \in I \), let \( Z_i \) be a l.c.s., \( K_i \) a nonempty compact convex subset of Hausdorff t.v.s. \( E_i \), \( Y_i \) a nonempty compact convex subset of \( L(E_i, Z_i) \), which is equipped with a \( \sigma \)-topology. For each \( i \in I \), assume that the following conditions are satisfied.
1) \( D_i : K \to 2^{K_i} \) and \( T_i : K \to 2^{Y_i} \) are two nonempty convex set-valued mappings and have open lower sections;
2) For each \( t_i \in Y_i \) and \( x_i \in \text{co} A_i \), the mapping \( \langle G_{t_i}, \eta_i(x, x_i) \rangle + S_i(x_i, y_i) : K \to 2^{K_i} \) is WIIC-DQC;
3) For each \( y_i \in K_i \), the set \( \{ (x, t) \in K \times Y : \langle G_{t_i}, \eta_i(x, y_i) \rangle + S_i(x_i, y_i) \subseteq -\text{int } C_i(x) \} \) is open.

Then there exist \( x_i \in D_i(x) \) and \( T_i(x) \) such that
\[ \langle G_{T_i}, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \subseteq -\text{int } C_i(x), \quad \forall y_i \in D_i(x). \]

**Proof.** Define a set-valued mapping \( P_i : K \times Y \to 2^{K_i} \) by
\[ P_i(x, t) = \{ y_i \in K_i : \langle G_{t_i}, \eta_i(x, x_i) \rangle + S_i(x_i, y_i) \subseteq -\text{int } C_i(x) \}, \quad \forall (x, t) \in K \times Y. \]

We first prove that \( x_i \notin \text{co} \{ P_i(x, t) \} \) for all \( (x, t) \in K \times Y \). To see this, suppose, by way of contradiction, that there exist some \( i \in I \) and some point \( \langle x_i, T_i x_i \rangle \in K \times Y \) such that \( x_i \in \text{co} \{ P_i(x_i, T_i x_i) \} \). Then, there exist finite points \( y_{i_1}, y_{i_2}, \cdots, y_{i_n} \) in \( K_i \) and \( \alpha_j \geq 0 \) with \( \sum_{j=1}^{n} \alpha_j = 1 \) such that \( x_i = \sum_{j=1}^{n} \alpha_j y_{i_j} \) and \( y_{i_j} \in P_j(x_i, T_i x_i) \) for all \( j = 1, \cdots, n \) such that
\[ \langle G_{T_i}, \eta_j(x_j, x_i) \rangle + S_j(x_i, y_{i_j}) \subseteq -\text{int } C_i(x), \quad j = 1, \cdots, n, \]
which contradicts the hypothesis 2). Hence, \( x_i \notin \text{co} \{ P_i(x, t) \} \).

By hypothesis 3), for each \( i \in I \) and each \( y_i \in K_i \), we known that
\[ Q^{-1} (y) = \{(x,t) \in K \times Y : \langle G_{\ell}, \eta_i (y, x) \rangle + S_i (x, y) \} \subseteq - \text{int } C_i (x) \]

is open and so \( P_i \) has open lower sections.

For each \( i \in I \), consider a set-valued mapping \( Q_i : K \times Y \to 2^K \) defined by

\[
Q_i (x,t) = \text{co} (P_i (x,t)) \cap D_i (x), \quad \forall (x,t) \in K \times Y.
\]

Since \( D_i \) has open lower sections by hypothesis 1), we may apply Lemma 2.3 to assert that the set-valued mapping \( Q_i \) has also open lower sections. Let

\[
W_i = \{(x,t) \in K \times Y : Q_i (x,t) \neq \emptyset \} \subset K \times Y.
\]

There are two cases to consider. In the case \( W_i = \emptyset \), we have

\[
\text{co} (P_i (x,t)) \cap D_i (x) = \emptyset, \quad \forall (x,t) \in K \times Y.
\]

This implies that \( \forall (x,t) \in K \times Y \),

\[
P_i (x,t) \cap D_i (x) = \emptyset.
\]

On the other hand, by condition 1), and the fact \( K_i \) is a compact convex subset of \( E_i \), we can apply Lemma 2.4 to assert the existence of a fixed point \( x^*_i \in D_i (x^*) \).

Since \( T_i (x^*) \neq \emptyset \), picking \( t^*_i \in T_i (x^*) \), we have

\[
P_i (x^*,t^*_i) \cap D_i (x^*) = \emptyset.
\]

This implies \( \forall y_i \in D_i (x^*_i), y_i \notin P_i (x^*, t^*_i) \). Hence, in this particular case, the assertion of the theorem holds.

We now consider the case \( W_i \neq \emptyset \). Define a set-valued mapping \( S_i : K \times Y \to 2^K \) by

\[
S_i (x,t) = \begin{cases} Q_i (x,t), & (x,t) \in W_i \\ D_i (x), & (x,t) \in K \times Y \setminus W_i. \end{cases}
\]

Then, \( S_i (x,t) \) is a convex set-valued mapping and for each \( u \in K \), \( S_i (u) = Q_i (u) \cup D_i (u) \times Y \) is open. For each \( i \in I \), consider the set-valued mapping \( H_i : K \times Y \to 2^{K \times Y} \) where \( H = \Pi_{i \in I} H_i \) is defined by

\[
H_i (x,t) = (S_i (x,t), T_i (x)).
\]

By condition 1) and the properties of \( S_i (x,t) \), \( H_i \) satisfies all the conditions of Lemma 2.4. Therefore, there exists \( (x^*, t^*_i) \in K \times Y \) such that

\[
(x^*_i, t^*_i) \in H_i (x^*, t^*_i).
\]

Suppose that \( (x^*, t^*_i) \in W_i \), then

\[
x^*_i \in \text{co} (P_i (x^*, t^*_i)) \cap D_i (x^*_i),
\]

so that \( x^*_i \in \text{co} (P_i (x^*, t^*_i)) \). This is a contradiction. Hence, \( (x^*, t^*_i) \notin W_i \). Therefore,

\[
(x^*_i, t^*_i) \in D_i (x^*_i, T_i (x^*_i)), \quad \text{and } Q_i (x^*, t^*_i) = \emptyset.
\]

Thus

\[
x^*_i \in D_i (x^*_i), t^*_i \in T_i (x^*_i), \quad \text{co} (P_i (x^*, t^*)) \cap D_i (x^*_i) = \emptyset.
\]

This implies

\[
P_i (x^*,t^*_i) \cap D_i (x^*_i) = \emptyset.
\]

Consequently, the assertion of the theorem holds in this case.

**Corollary 3.2.** For each \( i \in I \), let \( Z_i \) be a l.c.s., \( K_i \) a nonempty compact convex subset of Hausdorff t.v.s. \( E_i \), \( Y_i \) a nonempty compact convex subset of \( L(E_i, Z_i) \), which is equipped with a \( \sigma \)-topology. For each \( i \in I \), assume that the following conditions are satisfied:

1) \( D_i : K \to 2^Y \) and \( T_i : K \to 2^Y \) are two nonempty convex set-valued mappings and have open lower sections;

2) For all \( y_i \in K_i \), the mapping \( \{G_i, \eta_i (y, \bullet)\} + S_i (\bullet, y_i) : K \times Y \to 2^K \) is an u.s.c. set-valued mapping;

3) \( C_i : K \to 2^Z \) is a convex set-valued mapping with \( \text{int } C_i (x) \neq \emptyset \) for all \( x \in K ; \)

4) \( \eta_i : K \times K_i \to E_i \) is affine in the first argument and for all \( y_i \in K_i \), \( \eta_i (x_i, y_i) = 0 \);

5) \( S_i : K \times K \to 2^Z \) is a generalized vector 0-diagonally convex set-valued mapping;

6) For a given \( x_i \in K_i \), and a neighborhood \( U_i \) of \( x \), for all \( u \in U_i \), \( \text{int } C_i (x) = \text{int } C_i (u) \).

Then there exists \( \bar{x}_i \in D_i (\bar{x}_i) \) and \( \bar{t}_i \in T_i (\bar{x}_i) \) such that

\[
\{G_i, \eta_i (y_i, \bar{x}_i) \} + S_i (\bullet, y_i) \subset \text{int } C_i (\bar{x}_i), \quad \forall y_i \in D_i (\bar{x}_i).
\]

**Proof.** Define a set-valued mapping \( P_i : K \times Y \to 2^K \) by

\[
P_i (x,t) = \{y_i \in K_i : \{G_i, \eta_i (y_i, x_i) \} + S_i (x_i, y_i) \}
\]

\[
\subseteq \text{int } C_i (x_i), \quad \forall (x,t) \in K \times Y.
\]

We first prove that \( x_i \notin \text{co} (P_i (x,t)) \) for all \( (x,t) \in K \times Y \). By contradiction, for each \( i \in I \), suppose there exists some point \( (\bar{x}_i, \bar{t}_i) \in K \times Y \) such that

\[
\bar{x}_i \in \text{co} (P_i (\bar{x}_i, \bar{t}_i)).
\]

Then, there exist finite points \( y_{i_1}, y_{i_2}, \ldots, y_{i_n} \in K_i \), such that

\[
\{G_i, \eta_i (y_i, x_i) \} + S_i (x_i, y_i) \subset \text{int } C_i (\bar{x}_i), \quad i = 1, 2, \ldots, n.
\]

Since \( \eta_i (x_i, y_i) \) is affine and \( \text{int } C_i (\bar{x}_i) \) is convex, for \( \alpha_j \geq 0 \) with \( \sum_{j=1}^{n} \alpha_j = 1 \) such that \( \bar{x}_i = \sum_{j=1}^{n} \alpha_j y_{i_j} \) and \( y_{i_j} \in P_i (\bar{x}_i, \bar{t}_i) \) for all \( j = 1, \ldots, n \) such that

\[
\{G_i, \eta_i \left( \sum_{j=1}^{n} \alpha_j y_{i_j}, \bar{x}_i \right) \} + \sum_{j=1}^{n} \alpha_j S_i (\bar{x}_i, y_{i_j}) \subset \text{int } C_i (\bar{x}_i), \quad j = 1, \ldots, n.
\]

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Since \( \eta_i(x_i, x_j) = 0 \) for all \( x_i \in K_j \)
\[
\sum_{j=1}^{n} \alpha_j S_i(\xi, y_i) \subseteq -\text{int } C_i(\xi)
\]
which contradicts the hypothesis 5). Therefore \( x_j \not\in \text{co} \{ P_i(x, t) \} \).

We now prove that for each
\[
y_i \in K_i, P_i^{-1}(y_i)
\]
and \( \{ (x, t) \in K \times Y : \langle G_t, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \} \)
\[
\subseteq -\text{int } C_i(x)
\]
is open. Indeed, let \( (\xi, \tau) \in P_i^{-1}(y_i) \), that is
\[
\langle G_t, \eta_i(y_i, \xi) \rangle + S_i(\xi, y_i) \subseteq -\text{int } C_i(\xi).
\]
Since
\[
\langle G_t, \eta_i(y_i, \xi) \rangle + S_i(\xi, y_i) : K \times Y \rightarrow 2^{\mathbb{Z}}
\]
is an u.s.c. set-valued mapping, there exists a neighborhood \( U_i \) of \( (\xi, \tau) \) such that
\[
\langle G_t, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \subseteq -\text{int } C_i(x), \forall (x, t) \in U_i.
\]

By 6),
\[
\langle G_t, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \subseteq -\text{int } C_i(x), \forall (x, t) \in U_i.
\]

Hence, \( U_i \subseteq P_i^{-1}(y_i) \). This implies, \( P_i^{-1}(y_i) \) is open for each \( y_i \in K_i \), and so \( P_i \) open lower sections. For the remainder of the proof, we can just follow that of Theorem 3.1. This completes the proof.

**Corollary 3.3.** For each \( i \in I \), let \( Z_i \) be a l.c.s., \( K_i \) a nonempty compact convex subset of Haussdorff t.v.s. \( E_i \), \( Y_i \) a nonempty compact convex subset of \( L(E_i, Z_i) \), which is equipped with a \( \sigma \)-topology. For each \( i \in I \), assume that \( S_i \) and \( G_i \) are single valued mappings and the following conditions are satisfied.

1) \( D_i : K \rightarrow 2^{Z_i} \) and \( T_i : K \rightarrow 2^{Z_i} \) are two nonempty convex set-valued mappings and have open lower sections;

2) For all \( y_i \in K_i \), the mapping
\[
\langle G_i, \eta_i(y_i, \xi) \rangle + S_i(\xi, y_i) : K \times Y \rightarrow 2^{\mathbb{Z}}
\]
is a convex set-valued mapping such that for each \( x_i \in K_i \), \( C_i(x) = C_i \) is a convex cone with \( \text{int } C_i(x) \neq \emptyset \);

3) \( \eta_i : K_i \times K_i \rightarrow E_i \) is affine in the first argument and for all \( x_i \in K_i \), \( \eta_i(x_i, x_i) = 0 \);

4) \( S_i : K_i \times K_i \rightarrow 2^{Z_i} \) is a generalized vector 0-diagonally convex set-valued mapping;

5) \( S_i : K_i \times K_i \rightarrow 2^{Z_i} \) is a generalized vector 0-diagonally convex set-valued mapping;

6) For a given \( x_i \in K_i \), and a neighborhood \( U_i \) of \( x_i \), for all \( u \in U_i \), \( \text{int } C_i(u) = \text{int } C_i(u) \).

Then, there exist \( \xi_i \in D_i(\xi) \) and \( \tau_i \in T_i(\xi) \) such that
\[
\langle G_i, \eta_i(y_i, \xi) \rangle + S_i(\xi, y_i) \subseteq -\text{int } C_i, \forall y_i \in D_i(\xi).
\]

**Proof.** By hypothesis 3), the condition 4) in Corollary 3.2 is satisfied. Hence, all the conditions are satisfied as in Corollary 3.2.

**Corollary 3.4.** For each \( i \in I \), let \( Z_i \) be a l.c.s., \( K_i \) a nonempty compact convex subset of Haussdorff t.v.s. \( E_i \), \( Y_i \) a nonempty compact convex subset of \( L(E_i, Z_i) \), which is equipped with a \( \sigma \)-topology. For each \( i \in I \), assume that \( S_i \) and \( G_i \) are single valued mappings and the following conditions are satisfied.

1) \( D_i : K \rightarrow 2^{Z_i} \) and \( T_i : K \rightarrow 2^{Z_i} \) are two nonempty convex set-valued mappings and have open lower sections;

2) For all \( y_i \in K_i \), the mapping
\[
\langle G_i, \eta_i(y_i, \xi) \rangle + S_i(\xi, y_i) : K_i \times Y_i \rightarrow Z_i
\]
is continuous;

3) \( C_i : K_i \rightarrow 2^{Z_i} \) is a convex set-valued mapping with \( \text{int } C_i(x) \neq \emptyset \) for all \( x_i \in K_i \);

4) \( \eta_i : K_i \times K_i \rightarrow E_i \) is affine in the first argument and for all \( x_i \in K_i \), \( \eta_i(x_i, x_i) = 0 \);

5) \( S_i : K_i \times K_i \rightarrow 2^{Z_i} \) is a vector 0-diagonally convex mapping;

6) \( Z_i \backslash \{ -\text{int } C_i(x) \} \) is an u.s.c. set-valued mapping.

Then, there exist \( \xi_i \in D_i(\xi) \) and \( \tau_i \in T_i(\xi) \) such that
\[
\langle G_i, \eta_i(y_i, \xi) \rangle + S_i(\xi, y_i) \subseteq -\text{int } C_i, \forall y_i \in D_i(\xi).
\]

**Proof.** Define a set-valued mapping \( P_i : K \times Y \rightarrow 2^{\mathbb{Z}} \) by
\[
P_i(x, t) = \{ y_i \in K_i : \langle G_i, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \}
\]
\[
\subseteq -\text{int } C_i(x), \forall (x, t) \in K \times Y.
\]

We now prove that for each
\[
y_i \in K_i, P_i^{-1}(y_i)
\]
\[
= \{ (x, t) \in K \times Y : \langle G_i, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \}
\]
\[
\subseteq -\text{int } C_i(x)
\]
is open, that is, the set
\[
\{ (x, t) \in K \times Y : \langle G_i, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \}
\]
\[
\subseteq -\text{int } C_i(x)
\]
is closed. Indeed, let \( \{ x^{*}, t^{*} \} \) be a net in \( K \times Y \) such that \( x^{*}, t^{*} \) converges to \( x^{*}, t^{*} \) and
\[
\{ G_i, \eta_i(y_i, x^{*}) \} \backslash \{ S_i(x^{*}, y_i) \} \subseteq Z_i \backslash \{ -\text{int } C_i(x^{*}) \}.
\]

Since \( \{ G_i, \eta_i(y_i, x^{*}) \} \backslash \{ S_i(x^{*}, y_i) \} \subseteq Z_i \backslash \{ -\text{int } C_i(x^{*}) \} \) is continuous, hence
\[
\{ G_i, \eta_i(y_i, x^{*}) \} \backslash \{ S_i(x^{*}, y_i) \} \subseteq Z_i \backslash \{ -\text{int } C_i(x^{*}) \}.
\]
Since $Z \setminus \{ -\text{int } C_i(x) \}$ is an u.s.c. set-valued mapping with closed values, by Lemma 2.1, we have

$$
\left\{ G_{t^*} \eta_i (y, x_{t^*}) + S_i (x_{t^*}, y_i) \right\} \in Z \setminus \{ -\text{int } C_i(x^*) \},
$$

and hence $(x^*, t^*)$ in the set

$$
\{(x, t) \in K \times Y : \left\{ G_{t} \eta_i (y, x_{t}) + S_i (x_{t}, y_i) \right\} \in -\text{int } C_i(x) \}.
$$

This implies $P_i^{-1}(y)_{i}$ is open for each $y_{i} \in K_{i}$ and so $P_i$ has open lower sections. For the remainder of the proof, we can just follow that of Theorem 3.1 and Corollary 3.2. This completes the proof.

**Theorem 3.5.** For each $i \in I$, let $Z_i$ be a l.c.s., $K_i$ a nonempty compact convex subset of Hausdorff t.v.s. $E_i$, $Y_i$ a nonempty compact convex subset of $L(E_i, Z_i)$, which is equipped with a $\sigma$-topology. For each $i \in I$, assume that the following conditions are satisfied.

1) $D_i : K \to 2^{E_i}$ and $T_i : K \to 2^{K_i}$ are two nonempty convex set-valued mappings and have open lower sections;
2) For each $i \in I$, $x_{i} \in \text{co} \Lambda_{i}$, the mapping $\left\{ G_{t_i} \eta_i (y, x_{t_i}) + S_i (x_{t_i}, y_i) : K \to 2^{E_i} \right\}$ is WIC-DQC;
3) $Z_i \setminus \{ -\text{int } C_i(x) \}$ is an u.s.c. set-valued mapping.

Then there exist $\bar{x}_i \in D_i(\bar{x})$ and $\bar{t}_i \in T_i(\bar{x})$ such that

$$
\left\{ G_{\bar{t}_i} \eta_i \left( y_{i}, \bar{x}_i \right) \right\} + S_i \left( x_{t_i}, y_i \right) \subseteq Z_i \setminus \{ -\text{int } C_i(x^*) \}.
$$

**Proof.** Define a set-valued mapping $P : K \times Y \to 2^{K_i}$ by

$$
P_i(x, t) = \left\{ y_i \in K_i : \left\{ G_{t} \eta_i (y, x_{t}) + S_i (x_{t}, y_i) \right\} \in -\text{int } C_i(x) \right\},
$$

$$
\forall (x, t) \in K \times Y.
$$

For the remainder proof, we just follow that of Theorem 3.1.

**Corollary 3.6.** For each $i \in I$, let $Z_i$ be a l.c.s., $K_i$ a nonempty compact convex subset of Hausdorff t.v.s. $E_i$, $Y_i$ a nonempty compact convex subset of $L(E_i, Z_i)$, which is equipped with a $\sigma$-topology. For each $i \in I$, assume that the following conditions are satisfied.

1) $D_i : K \to 2^{E_i}$ and $T_i : K \to 2^{K_i}$ are two nonempty convex set-valued mappings and have open lower sections;
2) For each $i \in I$, $x_{i} \in \text{co} \Lambda_{i}$, the mapping $\left\{ G_{t_i} \eta_i (y, x_{t_i}) + S_i (x_{t_i}, y_i) : K \to 2^{E_i} \right\}$ is WIC-DQC;
3) $Z_i \setminus \{ -\text{int } C_i(x) \}$ is an u.s.c. set-valued mapping.

Then there exist $\bar{x}_i \in D_i(\bar{x})$ and $\bar{t}_i \in T_i(\bar{x})$ such that

$$
\left\{ G_{\bar{t}_i} \eta_i \left( y_{i}, \bar{x}_i \right) \right\} + S_i \left( x_{t_i}, y_i \right) \subseteq Z_i \setminus \{ -\text{int } C_i(x^*) \}.
$$
Hence, for all \((x', t') \in U_i(x', y') \cap U_i(x', y_i)\), there exists \(w^0 \in \bigcup G_{t,i}(y_i, x_i) + S_i(x_i, y_i)\) such that \(w^0 \notin Z_i \setminus \text{int} C_i(x^0)\), which is contradiction. Therefore, the set
\[
\{ (x, t) \in K \times Y : \bigcup G_{t,i}(y_i, x_i) + S_i(x_i, y_i) \}
\]
is closed. Hence, all the conditions of Theorem 3.5 satisfied. Consequently, the assertion of the theorem holds.

**Theorem 3.7.** For each \(i \in I\), let \(Z_i\) be a l.c.s., \(K_i\) a nonempty compact convex subset of Hausdorff t.v.s. \(E_i\), \(Y_i\) a nonempty compact convex subset of \(L(E_i, Z_i)\), which is equipped with a \(\sigma\)-topology. For each \(i \in I\), assume that the following conditions are satisfied.

1) \(D_i: K \to 2^{K_i}\) and \(T_i: K \to 2^Y_i\) are two nonempty convex set-valued mappings and have open lower sections;
2) For each \(t_i \in Y_i\) and \(x_i \in \text{co}A_i\), the mapping \(G_{t,i}(\cdot, x_i) + S_i(\cdot, y_i)\) is SIC-DQC;
3) for each \(y_i \in K_i\), the set \(\{ (x, t) \in K \times Y : \bigcup G_{t,i}(y_i, x_i) + S_i(x_i, y_i) \}
\]
is open.

Then there exist \(\pi_i \in D_i(\pi)\) and \(\tilde{t}_i \in T_i(\pi)\) such that \(\bigcup G_{t,i}(\cdot, x_i) + S_i(\cdot, y_i) \cap C_i(\pi) \neq \emptyset, \forall y_i \in D_i(\pi)\).

**Proof.** Define a set-valued mapping \(P_i: K \times Y \to 2^{K_i}\) by
\[
P_i(x, t) = \{ y_i \in K_i : \bigcup G_{t,i}(y_i, x_i) + S_i(x_i, y_i) \}
\]
\[
\cap C_i(x) = \emptyset, \forall (x, t) \in K \times Y.
\]

For the remainder proof, we just follow that of Theorem 3.1.

**Corollary 3.8.** For each \(i \in I\), let \(Z_i\) be a l.c.s., \(K_i\) a nonempty compact convex subset of Hausdorff t.v.s. \(E_i\), \(Y_i\) a nonempty compact convex subset of \(L(E_i, Z_i)\), which is equipped with a \(\sigma\)-topology. For each \(i \in I\), assume that the following conditions are satisfied.

1) \(D_i: K \to 2^{K_i}\) and \(T_i: K \to 2^Y_i\) are two nonempty convex set-valued mappings and have open lower sections;
2) For each \(t_i \in Y_i\) and \(x_i \in \text{co}A_i\), the mapping \(G_{t,i}(\cdot, x_i) + S_i(\cdot, y_i)\) is SIC-DQC;
3) for all \(x \in K\), \(C_i(x)\) is closed convex cone \(C_i\).

Then there exist \(\pi_i \in D_i(\pi)\) and \(\tilde{t}_i \in T_i(\pi)\) such that
\[
\bigcup G_{t,i}(\cdot, x_i) + S_i(\cdot, y_i) \cap C_i(x) = \emptyset, \forall (x, t) \in K \times Y.
\]

**Proof.** Define a set-valued mapping \(P_i: K \times Y \to 2^{K_i}\) by
\[
P_i(x, t) = \{ y_i \in K_i : \bigcup G_{t,i}(y_i, x_i) + S_i(x_i, y_i) \}
\]
\[
\cap C_i(x) = \emptyset, \forall (x, t) \in K \times Y.
\]

The rest of the proof is similar to that of Theorem 3.1.

**Corollary 3.10.** For each \(i \in I\), let \(Z_i\) be a l.c.s., \(K_i\) a nonempty compact convex subset of Hausdorff t.v.s. \(E_i\), \(Y_i\) a nonempty compact convex subset of \(L(E_i, Z_i)\), which is equipped with a \(\sigma\)-topology. For each \(i \in I\), assume that the following conditions are satisfied.

1) \(D_i: K \to 2^{K_i}\) and \(T_i: K \to 2^Y_i\) are two nonempty convex set-valued mappings and have open lower sections;
2) For each \(t_i \in Y_i\) and \(x_i \in \text{co}A_i\), the mapping
\( \{ G_t, \eta_y (x, y) \} + S_x (x, y) : K \to 2^{\mathcal{Z}} \) is SIC-DQC;

3) \( C_x (x) \) is an u.c. mapping with closed values.

Then there exist \( \xi_x \in D_x (x) \) and \( \xi \in T_x (x) \) such that

\[
G_t \xi_y (y, \xi) + S_x (x, y) \subseteq C_x (x), \quad \forall y \in D_x (x).
\]

**Proof.** Let \( P : K \times Y \to 2^{KP} \) a set-valued mapping defined in Theorem 3.9. We prove that for each \( y_i \in K \),

the set \( \{(x, y) \in K \times Y : \{ G_t, \eta_y (y, x) \} \cap S_x (x, y) \not\subseteq C_x (x) \} \) is open, that is, the set

\( \{(x, y) \in K \times Y : \{ G_t, \eta_y (y, x) \} \cap S_x (x, y) \subseteq C_x (x) \} \) is closed. Indeed, let \( \{(x^0, t^0) \} \) be a net in \( K \times Y \) such that \( (x^0, t^0) \to (x^0, t^0) \) and

\[
G_t^0 \eta_y (y, x^0) \cap S_x (x^0, y) \subseteq C_x (x^0).
\]

We claim that

\[
G_t^0 \eta_y (y, x^0) \cap S_x (x^0, y) \subseteq C_x (x^0).
\]

To prove this assertion, we can just follow that of Corollary 3.6. Hence, the set \( \{(x, y) \in K \times Y : \{ G_t, \eta_y (y, x) \} \cap S_x (x, y) \not\subseteq C_x (x) \} \) is open. Therefore, all the conditions of Theorem 3.9 are satisfied. Consequently, the assertion of the corollary hold.

**REFERENCES**


