The Theory of Membership Degree of $\Gamma$-Conclusion in Several $n$-Valued Logic Systems*

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Received April 14, 2012; revised May 18, 2012; accepted May 30, 2012

ABSTRACT

Based on the analysis of the properties of $\Gamma$-conclusion by means of deduction theorems, completeness theorems and the theory of truth degree of formulas, the present papers introduces the concept of the membership degree of formulas $A$ is a consequence of $\Gamma$ (or $\Gamma$-conclusion) in Łukasiewicz $n$-valued propositional logic systems, Gödel $n$-valued propositional logic systems and the $R_0$ $n$-valued propositional logic systems. The condition and related calculations of formulas $A$ being $\Gamma$-conclusion were discussed by extent method. At the same time, some properties of membership degree of formulas $A$ is a $\Gamma$-conclusion were given. We provide its algorithm of the membership degree of formulas $A$ is a $\Gamma$-conclusion by the constructions of theory root.

Keywords: $N$-Valued Propositional Logic; $\Gamma$-Conclusion; Theory; Root; Membership Degree

1. Introduction

Fuzzy logic is the theoretical foundation of fuzzy control. Spurred by the success in its applications, especially in fuzzy control, fuzzy logic has aroused the interest of many famous scholars, a series of important results have been created in documents [1-5]. For the sake of reasoning, we have to choose a subset $\Gamma$ of well-formed formulas, which can reflect come essential properties, as the axioms of the logical system and we then deduce the so-called $\Gamma$-conclusion through some reasonable inference rules [6-9]. So, a natural question then arises: how to judge whether or not a general formula $A$ is a conclusion of a given theory $\Gamma$, or to what extend the formula $A$ is a conclusion of $\Gamma$? It is basic problem to judge one thing belong to one kind in artificial intelligence. As is well known, human reasoning is approximate rather than precise in nature. we basic starting point is to establish graded version of basic logical notions. In order to establish a solid foundation for fuzzy reasoning, professor G. J. Wang proposed the concept of root of theory [3], J. C. Zhang proposed the concept of generalized root of theory [10,11], in propositional logic systems. The graded description and properties of formulas $A$ being $\Gamma$-conclusion were discussed. And provide its algorithm of membership degree of formulas $A$ is a $\Gamma$-conclusion, by the constructions of theory root in the above-mentioned logic systems.

2. Preliminaries

It is well known that different implication operators and valuation lattices $L$ (i.e., the set of truth degrees for logic) determine different logic systems (see [12]). Here valuation lattices is $L_n = \left\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\right\}$ and three popularly used implication operators and the corresponding t-norms defined as follows:

$$R_{L_n}(x, y) = \begin{cases} 1, & x \leq y, \\ (1-x+y), & x > y, \end{cases}$$

$$x \otimes_{L_n} y = \max(0, x + y - 1), x, y \in L_n;$$

$$R_{G_n}(x, y) = \begin{cases} 1, & x \leq y, \\ y, & x > y, \end{cases}$$

$$x \otimes_{G_n} y = \min(x, y), x, y \in L_n;$$

$$R_0(x, y) = \begin{cases} 1, & x \leq y, \\ (1-x) \vee y, & x > y, \end{cases}$$

$$x \otimes_0 y = \begin{cases} x \wedge y, & x + y > 1, \\ 0, & x + y \leq 1, \end{cases}$$

$$x, y \in L_n.$$

These three implication operators $R_{L_n}, R_{G_n},$ and $R_0,$ are called Łukasiewicz implication operator $R_{L_n},$ Gödel implication operator $R_{G_n},$ and the $R_0$-implication operator $R_0,$ respectively. The t-norm, which corresponds to $R_0$-implication operator $R_0,$ is called also Nilpotent implication operator.

*The work was supported by the Science and Technology Item of the Education Department of Fujian Province of China (No. 2010JA10235).
Minimum norm [6]. If we fix a t-norm \( \otimes \) above we then fix a propositional calculus (whose set of truth values is \( L_\otimes \)): \( \otimes \) is taken for the truth function of the strong conjunction \( \& \), the residuum \( R \) of \( \otimes \) becomes the truth function of the implication operator and \( R(\cdot, 0) \) is the truth function of the negation. In more details, we have the following definitions.

**Definition 1** [7, 8]. The propositional calculus \( PC(\otimes) \) given by a t-norm \( \otimes \) has the set \( S \) of propositional variables \( p_1, p_2, \cdots, p_n \) and connectives \( \neg, \& \), \( \rightarrow \). The set \( F(S) \) of well-formed formulas in \( PC(\otimes) \) is defined inductively as follows: each propositional variable is a formula; if \( A, B \) are formulas, then \( \neg A, A \& B \) and \( A \rightarrow B \) are all formulas.

**Definition 2** [8, 9, 13]. The formal deductive systems of \( PC(\otimes) \) given by \( \otimes \) corresponding to \( R_\otimes, R_{\neg \otimes} \), and \( R_\rightarrow \), are called Lukasiewicz \( n \)-valued logic systems \( L_n \), G\ödel \( n \)-valued logic systems \( G_n \), and the \( R_\rightarrow \) \( n \)-valued logic systems \( L'_n \), respectively.

Define in the above-mentioned logic systems
\[
A^m := A \& A \& \cdots \& A, \\
\text{and in the corresponding algebras} \quad L_n,
\]
where \( \otimes \) is the t-norm defined on \( L_n \).

**Remark 1.** It is easy to verify that the following assertions are true:

1. (1) in \( G_n \), \( a^m = a \) for every \( m \in N \).
2. (2) in \( L_n \), \( a^m = a^{(2)} \) for every \( m \in N \), and \( m \geq 2 \).
3. (3) in \( L'_n \), \( a^m = (na - (n - 1)) \lor 0 \), for every \( n \in N \).

**Definition 3** [7, 8]. (1) A homomorphism \( \varphi: F(S) \rightarrow L_n \) of type \( (\neg, \& \), \( \rightarrow) \) from \( F(S) \) into the valuation lattice \( L_n \), i.e., \( \varphi(\neg A) = \neg \varphi(A) \), \( \varphi(A \& B) = \varphi(A) \& \varphi(B) \), \( \varphi(A \rightarrow B) = \varphi(A) \rightarrow \varphi(B) \), is called an R-valuation of \( F(S) \). The set of all R-valuations will be denoted by \( \Omega_R \).

2. A formula \( A \in F(S) \) is called a tautology w.r.t. \( R \) if \( \forall v \in \Omega_R, \varphi(A) = 1 \) holds.

**Remark 2** [8, 13]. It is not difficult to verify in the above-mentioned logic systems that
\[\varphi(A) = \max \{ \varphi(A), \varphi(B) \}, \quad \text{and} \]
\[\varphi(A \& B) = \max \{ \varphi(A), \varphi(B) \} \quad \text{for every valuation} \quad \varphi \in \Omega_R. \]
Moreover, one can check in \( L_n \) and \( L'_n \) that \( A \& B \) and \( \neg(A \rightarrow \neg B) \) are logically equivalent.

**Definition 4** [8]. Assume that \( A = A(p_1, p_2, \cdots, p_m) \) is a formula generated by propositional variables \( p_1, p_2, \cdots, p_m \) through connectives \( \neg, \& \), and \( \rightarrow \). Substitute \( x_i \) for \( p_i \) in \( A(i = 1, 2, \cdots, m) \) and keep the logic connectives in \( A \) unchanged but explain them as the corresponding operators defined on the valuation lattice \( L_n \). We get a function \( \overline{A}: L_n \rightarrow L_n \) and call \( \overline{A}(x_1, x_2, \cdots, x_m) \) the truth degree function of \( A \).

**Definition 5** [7, 8]. (1) A subset of \( F(S) \) is called a theory.

2. Let \( \Gamma \) be a theory, \( A \in F(S) \). A deduction of \( A \) from \( \Gamma \), in symbols, \( \Gamma \vdash A \), is a finite sequence of formulas \( A_1, \cdots, A_n = A \) such that for each \( 1 \leq i \leq m \), \( A_i \) is an axiom of \( L_n \), or \( A_i \in \Gamma \), or there are \( j, k \in \{1, \cdots, i-1\} \) such that \( A_i \) follows from \( A_j \) and \( A_k \) by MP. Equivalently, we say that \( A \) is a conclusion of \( \Gamma \) (or \( \Gamma \)-conclusion). The set of all conclusions of \( \Gamma \) is denoted by \( D(\Gamma) \). By a proof of \( A \) we shall henceforth mean a deduction of \( A \) from the empty set. We shall also write \( \vdash \) in place of \( \otimes \) and call \( A \) a theorem.

It is easy for the reader to check the following Proposition 1.

**Proposition 1.** Let \( \Gamma \) be a theory and \( A \in F(S) \). If \( \Gamma \vdash A \) then there exist a finite subset of \( \Gamma \) say, \( \{A_1, A_2, \cdots, A_m\} \) such that \( \{A_1, A_2, \cdots, A_m\} \vdash A \).

**Theorem 1** (Generalized deduction theorems) [7, 8, 12]. Suppose that \( \Gamma \) is a theory, \( A, B \in F(S) \), then

1. (1) in \( L_n \),
\[\Gamma \cup \{A\} \vdash B \iff \exists s \in N \quad \text{s.t.} \quad \Gamma \vdash A' \rightarrow B.\]

2. in \( G_n \),
\[\Gamma \cup \{A\} \vdash B \iff \Gamma \vdash A \rightarrow B.\]

3. in \( L'_n \),
\[\Gamma \cup \{A\} \vdash B \iff \Gamma \vdash A' \rightarrow B.\]

**Definition 6** [8, 13]. Suppose that \( A = A(p_1, p_2, \cdots, p_m) \) is a formula of \( F(S) \) containing \( m \) atomic formulas \( p_1, p_2, \cdots, p_m \), and \( \overline{A}(x_1, x_2, \cdots, x_m) \) be the truth degree function of \( A \). Then
\[\tau(A) = \sum_{i=0}^{n-1} \frac{i}{n-1} \frac{A}{n^m} \]
is called the truth degree of \( A \), where \(| B | \) is the cardinal of set \( B \).

**Theorem 2.** Suppose that \( A = A(p_1, p_2, \cdots, p_i) \in F(S) \) and \( n \geq 2 \), then in \( L_n \) and \( G_n \),
\[T(A) = 1 \quad \text{iff} \quad A \quad \text{is a tautology i.e.,} \quad \vdash A.\]

**Proof.** Assume that \( T(A) = 1 \). Since
\[T(A) \leq \sum_{i=0}^{n-2} \frac{A^{-1}(n-1)}{n^m} + \frac{1}{n-1} \frac{A^{-1}(1)}{n^m},\]
then \( A^{-1}(1) \geq n'. \) By definite, \( A^{-1}(1) \leq n' \), thus
\[A^{-1}(1) = n' \quad \text{i.e.,} \quad \forall v \in \Omega_R, \quad \nu(A) = 1, \quad \text{then} \quad A \quad \text{is a} \]

tautology. Conversely, assume that $A$ is a tautology i.e.,
\[ A \vdash n' \] then \[ A \vdash \left( i \over n-1 \right) = 0 \] \( i = 0,1,\ldots,n-2 \), so
\( T(A) = 1 \). This completes the proof.

**Theorem 3** [8]. Suppose that $A \in F(S)$, then in \( L_n \), $\forall A$, is a tautology, i.e., $\models A$.

**Theorem 4.** Suppose that $A, B \in F(S)$. If for every $\nu \in \Omega_2, \nu(A) \geq \nu(B)$, then $\tau(A) \geq \tau(B)$.

**Proof.** Suppose that $A = A(p_1, p_2, \ldots, p_m)$ and $B = B(p_1, p_2, \ldots, p_n)$ are all formulas of $F(S)$ containing $m$ atomic formulas $p_1, p_2, \ldots, p_m$, it follows from $\nu(A) \geq \nu(B)$ that
\[
\overline{A}(x_1, x_2, \ldots, x_m) = \nu(A)
\]
and
\[
\overline{A}(x_1, x_2, \ldots, x_m) = \nu(A)
\]
\[
\overline{B}(x_1, x_2, \ldots, x_n)
\]
\[
(i = n-1, n-2, \ldots, 1)
\]

If for every $B \in D(\Gamma)$ we have $\vdash A \rightarrow B$, then $A$ is called the root of $\Gamma$.

**Theorem 5.** Suppose that $\Gamma$ is a finite theory, say $\Gamma = \{ A, A_1, \ldots, A_m \}$, then
\[
(1) \quad \text{in } L_n,
\]
\[
A^{n-1}_1 \land A^{n+1}_2 \land \cdots \land A^{n-1}_m \text{ is a root of } \Gamma;
\]
\[
(2) \quad \text{in } L^*_n,
\]
\[
A^{n-1}_1 \land A^{n+1}_2 \land \cdots \land A^{n-1}_m \text{ is a root of } \Gamma;
\]
\[
(3) \quad \text{in } G_n, \quad A \land A_1 \land \cdots \land A_m \text{ is a root of } \Gamma.
\]

**Proof.** (1) It following form references [4] that $A^{n-1}_1 \land A^{n+1}_2 \land \cdots \land A^{n-1}_m \in D(\Gamma)$, for every $B \in D(\Gamma)$, there exist $n_1, n_2, \ldots, n_m \in N$ such that
\[
\vdash A^{n_1}_1 \land A^{n_2}_2 \land \cdots \land A^{n_m}_m \rightarrow B \text{ by Theorem 1.}
\]
It is easy to check that $A^{n_1}_1 \land A^{n_2}_2 \land \cdots \land A^{n_m}_m \rightarrow B$ by Hypothetical, this shows that $A^{n_1}_1 \land A^{n_2}_2 \land \cdots \land A^{n_m}_m$ is a root of $\Gamma$.

(2) It following form references [4] that $A^{n_1}_1 \land A^{n_2}_2 \land \cdots \land A^{n_m}_m \in D(\Gamma)$, for every $B \in D(\Gamma)$, it follows from Theorem 1 that
\[
\vdash A^{n_1}_1 \rightarrow \left( A^{n_1}_1 \rightarrow \left( \cdots \left( A^{n_1}_1 \rightarrow B \right) \right) \right),
\]
since
\[
A^{n_1}_1 \land A^{n_2}_2 \land \cdots \land A^{n_m}_m \rightarrow B
\]
and
\[
A^{n_1}_1 \rightarrow \left( A^{n_1}_1 \rightarrow \left( \cdots \left( A^{n_1}_1 \rightarrow B \right) \right) \right)
\]
are provably equivalent, and so is $\vdash A^{n_1}_1 \land A^{n_2}_2 \land \cdots \land A^{n_m}_m \rightarrow B$. This shows $A^{n_1}_1 \land A^{n_2}_2 \land \cdots \land A^{n_m}_m$ is a root of $\Gamma$.

(3) It following from references [4] that $A \land A_1 \land \cdots \land A_m \in D(\Gamma)$, for every $B \in D(\Gamma)$, we get
\[
\vdash A \rightarrow \left( A \rightarrow \left( \cdots \left( A \rightarrow B \right) \right) \right)
\]
by Theorem 1, it is easy to verify that $A \rightarrow (B \rightarrow C)$ and $A \land B \rightarrow C$ are provably equivalent, hence $A \land A_1 \land \cdots \land A_m \rightarrow B$ and $A \rightarrow \left( A \rightarrow \left( \cdots \left( A \rightarrow B \right) \right) \right)$ are provably equivalent, and so is $\vdash A \land A_1 \land A_2 \land \cdots \land A_m \rightarrow B$. This shows $A \land A_1 \land A_2 \land \cdots \land A_m$ is a root of $\Gamma$.

**4. Membership Degree of Formulas $A$ Is $\Gamma$-Conclusion**

In following, let us first take an analysis on the conditions of formulas $A$ is a $\Gamma$-conclusion in $L_n$. Suppose that $\Gamma$ is a theory and $A$ is a $\Gamma$-conclusion, it follows from Proposition 1 and Theorem 1 that there exist a finite string of formulas $A_1, A_2, \ldots, A_m \in \Gamma$ and $n_1, n_2, \ldots, n_i \in N$ such that $\vdash A_1^{n_1} \land A_2^{n_2} \land \cdots \land A_m^{n_m} \rightarrow A$ holds, i.e., the formula $A^{n_1}_1 \land A^{n_2}_2 \land \cdots \land A^{n_m}_m \rightarrow A$ is a theorem of $L_n$, let $B = A^{n_1}_1 \land A^{n_2}_2 \land \cdots \land A^{n_m}_m$, hence $B \rightarrow A$ is a tautology, it follows from Theorem 2 that
\[
\tau(B \rightarrow A) = 1.
\]
Conversely, if there exist a $\Gamma$-conclusion $B$ such that $\tau(B \rightarrow A) = 1$, then following from

**3. Properties of the Roots of Theories**

**Definition 7** [3]. Suppose that $\Gamma$ is a theory, $A \in D(\Gamma)$.
Theorem 2 that $B \rightarrow A$ is a tautology, thus $B \rightarrow A$ is
a theorem of $L_s$, i.e., $\vdash B \rightarrow A$ holds and $\Gamma \vdash B$, we
have that $\Gamma \vdash A$ by MP, i.e., $A$ is a $\Gamma$-conclusion.
Moreover, the larger the membership degree of such formulas are, the
more closer $A$ is to be $\Gamma$-conclusion.
Hence it is natural and reasonable for us using the supremum of true degree of all formulas with the form $B \rightarrow A$ to measure $A$ is a $\Gamma$-conclusion.

**Definition 8.** Suppose that $\Gamma$ is a theory, $A \in F(S)$.
Define

$$T(A | \Gamma) = \sup \{ \tau(B \rightarrow A) | B \in D(\Gamma) \},$$

then $T(A | \Gamma)$ is called the membership degree of formulas $A$ is a $\Gamma$-conclusion.

It is easy to verify that $0 \leq T(A | \Gamma) \leq 1$ and following
Proposition 2 by Definition 8.

**Proposition 2.** In $L_s$, $G_s$, and $L^*_s$.
If $A$ is a $\Gamma$-conclusion, then $T(A | \Gamma) = 1$.

**Theorem 6.** In $L_s$, $G_s$, and $L^*_s$, if $\Gamma$ is a finite
theory, say $\Gamma = \{A_1, A_2, \ldots, A_n\}$, then $A$ is a $\Gamma$ -conclusion iff $T(A | \Gamma) = 1$.

**Proof.** The necessity part by proposition 2, it is only
necessary to prove the sufficiency. Let $T(A | \Gamma) = 1$. For
every number $\varepsilon > 0$, there exist a formulas $B \in D(\Gamma)$
such that $\tau(B \rightarrow A) > 1 - \varepsilon$ by Definition 8.

(1) In $L^*_s$, it follows from Theorem 5 that

$$A_1 ^{\varepsilon} \land A_2 ^{\varepsilon} \land \cdots \land A_n ^{\varepsilon} \rightarrow B \land \vdash A \land A_2 \land \cdots \land A_n \rightarrow B.$$

Hence for every $\forall \in \Omega_s$, we have $\forall(A_1 ^{\varepsilon} \land A_2 ^{\varepsilon} \land \cdots \land A_n ^{\varepsilon} \rightarrow \tau(B)) \leq \forall(B)$, it
follows from properties of implication operators that $\forall(A_1 ^{\varepsilon} \land A_2 ^{\varepsilon} \land \cdots \land A_n ^{\varepsilon} \rightarrow \tau(B)) = \forall(B) \rightarrow A > 1 - \varepsilon$, since $\varepsilon$ is arbitrary, we have $\forall(A_1 ^{\varepsilon} \land A_2 ^{\varepsilon} \land \cdots \land A_n ^{\varepsilon} \rightarrow A) = 1$, thus $A_1 ^{\varepsilon} \land A_2 ^{\varepsilon} \land \cdots \land A_n ^{\varepsilon} \rightarrow A$ is a tautology, and $A_1 ^{\varepsilon} \land A_2 ^{\varepsilon} \land \cdots \land A_n ^{\varepsilon} \rightarrow A$ is a theorem , together with the result $\Gamma \vdash A_1 ^{\varepsilon} \land A_2 ^{\varepsilon} \land \cdots \land A_n ^{\varepsilon}$, then $\Gamma \vdash A$ by MP, i.e.,

$A \in D(\Gamma)$.

(2) In $L_s$, notice that $A_1 ^{\varepsilon-1} \land A_2 ^{\varepsilon-1} \land \cdots \land A_n ^{\varepsilon-1}$ is a root of $\Gamma$ by Theorem 5, hence the proof of (2) is similar to that of the proof (1) and so is omitted.

(3) In $G_s$, notice that $A_1 \land A_2 \land \cdots \land A_n \rightarrow B$ is a root of $\Gamma$ by Theorem 5, hence the proof of (2) is similar to that of the proof of (1).

**Theorem 7.** Suppose that $\Gamma = \{A_1, A_2, \ldots, A_n\}$, then
(1) in $L^*_s$, 

$$T(A | \Gamma) = \tau(A_1 \land A_2 \land \cdots \land A_n \rightarrow A);$$

(2) in $L_s$, 

$$T(A | \Gamma) = \tau(A_1 ^{\varepsilon-1} \land A_2 ^{\varepsilon-1} \land \cdots \land A_n ^{\varepsilon-1} \rightarrow A);$$

(3) in $G_s$, 

$$T(A | \Gamma) = \tau(A_1 \land A_2 \land \cdots \land A_n \rightarrow A).$$

**Proof.** (1) Since $A_1 ^{\varepsilon} \land A_2 ^{\varepsilon} \land \cdots \land A_n ^{\varepsilon}$ is a root of $\Gamma$ by Theorem 5, hence for every $B \in D(\Gamma)$, we have $\vdash A_1 ^{\varepsilon} \land A_2 ^{\varepsilon} \land \cdots \land A_n ^{\varepsilon} \rightarrow B$. Thus for every $\forall \in \Omega_s$, we have $\forall(A_1 ^{\varepsilon} \land A_2 ^{\varepsilon} \land \cdots \land A_n ^{\varepsilon} \leq \forall(B)$, and $\forall(A_1 ^{\varepsilon} \land A_2 ^{\varepsilon} \land \cdots \land A_n ^{\varepsilon} \rightarrow A) \geq \forall(B \rightarrow A)$ holds, then $\tau(A_1 ^{\varepsilon} \land A_2 ^{\varepsilon} \land \cdots \land A_n ^{\varepsilon} \rightarrow A) \geq \tau(B \rightarrow A)$ by Theorem 4. It follows from $A_1 ^{\varepsilon} \land A_2 ^{\varepsilon} \land \cdots \land A_n ^{\varepsilon} \in D(\Gamma)$ that $\tau(A_1 ^{\varepsilon} \land A_2 ^{\varepsilon} \land \cdots \land A_n ^{\varepsilon} \rightarrow A) = \tau(B \rightarrow A)$, 

(2) Notice that in $L_s$, $A_1 ^{\varepsilon-1} \land A_2 ^{\varepsilon-1} \land \cdots \land A_n ^{\varepsilon-1}$ is a root of $\Gamma$ by Theorem 5, the proof of (2) is similar to that of the proof (1) and so is omitted.

(3) Notice that in $G_s$, $A_1 \land A_2 \land \cdots \land A_n$ is a root of $\Gamma$ by Theorem 5, the proof of (2) is similar to that of the proof (1) and so is omitted.

**Theorem 8.** Suppose that $\Gamma$ is a infinite theory. Then

(1) in $L^*_s$, 

$$T(A | \Gamma) = \tau(A_1 ^{\varepsilon} \land A_2 ^{\varepsilon} \land \cdots \land A_n ^{\varepsilon} \rightarrow A);$$

(2) in $L_s$, 

$$T(A | \Gamma) = \tau(A_1 ^\varepsilon \land A_2 ^\varepsilon \land \cdots \land A_n ^\varepsilon \rightarrow A);$$

(3) in $G_s$, 

$$T(A | \Gamma) = \tau(A_1 \land A_2 \land \cdots \land A_n \rightarrow A).$$

**Proof.** (1) For every $B \in D(\Gamma)$, it following from
Proposition 1 that there exist a finite string of formulas $A_1 \land A_2 \land A_3 \land \cdots \land A_n \in \Gamma$ such that $\{A_1, A_2, \ldots, A_n\} \in \Gamma$. It follows from Theorem 1 that $A_1 ^\varepsilon \land A_2 ^\varepsilon \land \cdots \land A_n ^\varepsilon \in D(\Gamma)$ and is a tautology by completeness theorem, and for every $\forall \in \Omega_s$, we have $\forall(A_1 ^\varepsilon \land A_2 ^\varepsilon \land \cdots \land A_n ^\varepsilon \rightarrow A) = 1$, holds, then $A_1 ^\varepsilon \land A_2 ^\varepsilon \land \cdots \land A_n ^\varepsilon \rightarrow A$ is a theorem , together with the result $\Gamma \vdash A_1 ^\varepsilon \land A_2 ^\varepsilon \land \cdots \land A_n ^\varepsilon$, we have $\Gamma \vdash A$ by MP. The proof is completed.

**Theorem 9.** Suppose that $\Gamma = \{A_1, A_2, \ldots, A_n\}$, then
(1) in $L^*_s$, 

$$T(A | \Gamma) = \tau(A_1 ^\varepsilon \land A_2 ^\varepsilon \land \cdots \land A_n ^\varepsilon \rightarrow A);$$

(2) Notice that in $L_s$, $A_1 ^{\varepsilon-1} \land A_2 ^{\varepsilon-1} \land \cdots \land A_n ^{\varepsilon-1}$ is a root of $\Gamma$ by
Remark 1, the proof of (2) is similar to that the Proof of (1)
and so is omitted.

(3) Notice that in \( G_s \), \( A_1 \land A_2 \rightarrow B \) and \( A_1 \rightarrow (A_2 \rightarrow B) \) is Provably equivalent, the Proof of (3) is similar to that the Proof of (1) and so is omitted.

**Theorem 9.** Suppose that \( \Gamma \) is a theory, \( T(A \mid \Gamma) \geq \alpha \) and \( T(A \rightarrow C \mid \Gamma) \geq \beta \), then \( T(C \mid \Gamma) \geq (\alpha + \beta - 1) \lor 0 \).

**Proof.** (1) If \( \alpha + \beta \leq 1 \), we get \( (\alpha + \beta - 1) \lor 0 = 0 \), then \( T(C \mid \Gamma) \geq (\alpha + \beta - 1) \lor 0 \).

(2) If \( \alpha + \beta > 1 \), we get \( \alpha > 0 \) and \( \beta > 0 \), for any given positive number \( \varepsilon \) such that \( \alpha - \varepsilon > 0 \) and \( \beta - \varepsilon > 0 \), there exists formulas \( B_i \in D(\Gamma), i = 1, 2 \), such that \( \tau(B_i \rightarrow A) > \alpha - \varepsilon \), and \( \tau(B_i \rightarrow (A \rightarrow C)) > \beta - \varepsilon \). It follows from properties of Regular implication operators that

\[
\tau(B_i \land B_j \rightarrow (A \rightarrow C)) \geq (\alpha + \beta - 2\varepsilon - 1) \lor 0.
\]

Bucas \( B_i \land B_j \rightarrow (A \rightarrow C) \) and \((B_i \land B_j) \rightarrow C \) are provably equivalent (i.e., logically equivalent), hence

\[
\tau((B_i \land B_j) \rightarrow C) \geq (\alpha + \beta - 2\varepsilon - 1) \lor 0,
\]

it is easy to verify that \( B_i \land B_j \rightarrow C \) and \( A \rightarrow (B_i \land B_j) \rightarrow C \) are provably equivalent (i.e., logically equivalent), hence

\[
\tau(A \rightarrow (B_i \land B_j) \rightarrow C) \geq (\alpha + \beta - 2\varepsilon - 1) \lor 0,
\]

and if \( \varepsilon \) is small enough, then \( T(C \mid \Gamma) \geq (\alpha + \beta - 1) \lor 0 \) by the defination of the membership degree of formulas.

**Example 1.** Suppose that \( \Gamma = \{p_1, p_2\} \subset S, p_i \in S \). In \( L_1, L_2 \) and \( G_s \), compute \( T(p_1 \mid \Gamma) \), respectively.

**Solution.** (1) In \( L_3 \), assume that \( A = p_1 \land p_2 \rightarrow p_3 \).

Since \( \overline{A(x_1, x_2)} = x_1 \otimes x_2 \rightarrow x_3 \) and

\[
x_i \in \{0, 1\},
\]

thus

\[
x_i = \begin{cases} 1, & x_i = 1, \\ 0, & \text{otherwise}; \end{cases}
\]

and

\[
x_i \otimes x_j \rightarrow x_k = \begin{cases} 1, & x_i = x_j = 1, x_k = 1, \\ 0, & x_i = x_j = 1, x_k = 0; \\ 1, & \text{otherwise}; \end{cases}
\]

We have \( \overline{A}^{-1} \left( \frac{1}{2} \right) = 1 \) and \( \overline{A}^{-1} (1) = 3^2 - 2 = 25 \), hence

\[
\tau(A) = \tau(p_1 \land p_2 \rightarrow p_3) = 1 - \frac{1}{2} \left[ \frac{3}{2} \left( \frac{1}{2} \right)^3 - 2 \right] = \frac{17}{18},
\]

then \( T(p_1 \mid \Gamma) = \tau(p_1 \land p_2 \rightarrow p_3) = \frac{17}{18} \).

(2) In \( G_s \), assume that \( A = p_1 \land p_2 \rightarrow p_3 \). Since \( \overline{A(x_1, x_2)} = x_1 \otimes x_2 \rightarrow x_3 \), and

\[
x_i \otimes x_j \rightarrow x_k = \begin{cases} 1, & x_i = x_j = 1, x_k = 1, \\ 0, & x_i = x_j = 1, x_k = 0; \\ 0, & \text{otherwise}; \end{cases}
\]

thus

\[
\tau(A) = \tau(p_1 \land p_2 \rightarrow p_3) = 1 - \frac{1}{2} \left[ \frac{3}{2} \left( \frac{1}{2} \right)^3 - 2 \right] = \frac{17}{18},
\]

then \( T(p_1 \mid \Gamma) = \tau(p_1 \land p_2 \rightarrow p_3) = \frac{17}{18} \).

(3) In \( L_2 \), assume that \( A = p_1 \land p_2 \rightarrow p_3 \). Since \( \overline{A(x_1, x_2)} = x_1 \otimes x_2 \rightarrow x_3 \), and

\[
x_i \otimes x_j \rightarrow x_k = \begin{cases} 1, & x_i = x_j = 1, x_k = 1, \\ 0, & x_i = x_j = 1, x_k = 0, \\ 1, & \text{otherwise}; \end{cases}
\]

thus

\[
\tau(A) = \tau(p_1 \land p_2 \rightarrow p_3) = 1 - \frac{1}{2} \left[ \frac{3}{2} \left( \frac{1}{2} \right)^3 - 2 \right] = \frac{17}{18},
\]
then $T(p_3 | \Gamma) = \frac{17}{18}$.

**Example 2.** Suppose that $\Gamma = \{ p_1, p_1 \rightarrow p_2 \}$, $p_i, p_2 \in S$, in $L_1$, compute $T(p_2 | \Gamma)$.

**Solution.** (1) Assume that $A = p_2^A \land (p_1 \rightarrow p_2) \rightarrow p_2$. Since $A(x_1, x_2) = x_1 \otimes (x_1 \rightarrow x_2)^2 \rightarrow x_2$, and

$x_1^2 = (2x_1 - 1) \lor 0 = \begin{cases} 1, & x_1 = 1, \\ 0, & \text{otherwise}; \end{cases}$

$(x_1 \rightarrow x_2) = \begin{cases} \frac{1}{2}, & x_1 = 1, x_2 = \frac{1}{2} \text{ or } x_1 = \frac{1}{2}, x_2 = 0, \\ 1, & \text{otherwise}; \end{cases}$

$(x_1 \rightarrow x_2)^2 = \begin{cases} 0, & x_1 = 1, x_2 = 0, \\ \text{or } x_1 = 1, x_2 = \frac{1}{2} \text{ or } x_1 = \frac{1}{2}, x_2 = 0, \\ 1, & \text{otherwise} \end{cases}$

$x_1 \otimes (x_1 \rightarrow x_2)^2 \rightarrow x_2 = 1$, $
\tau(A) = \tau\left[p_2^A \land (p_1 \rightarrow p_2)^2 \rightarrow p_2\right] = \frac{1}{3^4} \left[\left\lfloor \frac{1}{2} \right\rfloor + \left\lfloor \frac{1}{2} \right\rfloor \right] = \frac{1}{9}(0 + 9) = 1$
thus $T(p_2 | \Gamma) = \tau\left[p_2^A \land (p_1 \rightarrow p_2)^2 \rightarrow p_2\right] = 1$, then $p_2$ is a $\Gamma$ -conclusion.

**REFERENCES**


