Reconstruction of Three Dimensional Convex Bodies from the Curvatures of Their Shadows

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Abstract

In this article, we study necessary and sufficient conditions for a function, defined on the space of flags to be the projection curvature radius function for a convex body. This type of inverse problems has been studied by Christoffel, Minkwoski for the case of mean and Gauss curvatures. We suggest an algorithm of reconstruction of a convex body from its projection curvature radius function by finding a representation for the support function of the body. We lead the problem to a system of differential equations of second order on the sphere and solve it applying a consistency method suggested by the author of the article.

Keywords

Integral Geometry, Convex Body, Projection Curvature, Support Function

1. Introduction

The problem of reconstruction of a convex body from the mean and Gauss curvatures of the boundary of the body goes back to Christoffel and Minkowski [1]. Let $F$ be a function defined on 2-dimensional unit sphere $S^2$. The following problems have been studied by E. B. Christoffel: what are necessary and sufficient conditions for $F$ to be the mean curvature radius function for a convex body. The corresponding problem for Gauss curvature is considered by H. Minkovski [1]. W. Blaschke [2] provides a formula for reconstruction of a convex body $B$ from the mean curvatures of its boundary. The formula is written in terms of spherical harmonics.

A. D. Aleksandrov and A. V. Pogorelov generalize these problems for a class of symmetric functions $G(R_1, R_2)$ of principal radii of curvatures (see [3]-[5]).

Let $B \subseteq \mathbb{R}^n$ be a convex body with sufficiently smooth boundary and let $R_i(\omega), \ldots, R_{n-1}(\omega)$ signify the principal radii of curvature of the boundary of $B$ at the point with outer normal direction $\omega \in S^{n-1}$. In $n$-dimensional case, a Christoffel-Minkowski problem is posed and solved by Firay [6] and Berg [7] (see also [8]): what are necessary and sufficient conditions for a function $F$, defined on $S^{n-1}$ to be function $\sum R_i(\omega) \cdots R_p(\omega)$ for a convex body, where $1 \leq p \leq n-1$ and the sum is extended over all increasing sequences $i_1, \ldots, i_p$ of indices chosen from the set $i = 1, \ldots, n-1$.


D. Ryabogin and A. Zvavich [10] reconstruct a convex body of revolution from the areas of its shadows by giving a precise formula for the support function.

In this paper, we consider a similar problem posed for the projection curvature radius function of convex bodies. We lead the problem to a system of differential equations of second order on the sphere and solve it applying a consistency method suggested by the author of the article. The solution of the system of differential equations is itself interesting.

Let $B \subseteq \mathbb{R}^3$ be a convex body with sufficiently smooth boundary and with positive Gaussian curvature at every point of the boundary $\partial B$. We need some notations.

$S^2$—the unit sphere in $\mathbb{R}^3$, $S_\omega \subseteq S^2$—the great circle with pole at $\omega \in S^2$, $B(\omega)$—projection of $B$ onto the plane containing the origin in $\mathbb{R}^3$ and orthogonal to $\omega$,

$\partial B(\omega)$—curvature radius of $\partial B(\omega)$ at the point with outer normal direction $\omega \in S_\omega$ and call projection curvature radius of $B$.

Let $F$ be a positive continuously differentiable function defined on the space of “flags” $\mathcal{F} = \{ (\omega, \varphi) : \omega \in S^2, \varphi \in S_\omega \}$. In this article, we consider:

**Problem 1.** What are necessary and sufficient conditions for $F$ to be the projection curvature radius function $R(\omega^\perp, \varphi)$ for a convex body?

**Problem 2.** Reconstruction of that convex body by giving a precise formula for the support function.

Note that one can lead the problem of reconstruction of a convex body by projection curvatures using representation of the support function in terms of mean curvature radius function (see [7]). The approach of the present article is useful for practical point of view, because one can calculate curvatures of projections from the shadows of a convex body. Let’s note that it is impossible to calculate mean radius of curvature from the limited number of shadows of a convex body. Also let’s note that this is a different approach for such problems, because in the present article we lead the problem to a differential equation of spatial type on the sphere and solve it using a new method (so called consistency method).

The most useful analytic description of compact convex sets is by the support function (see [11]). The support function of $B$ is defined as

$$H(x) = \sup_{y \in B} \langle y, x \rangle, \quad x \in \mathbb{R}^3.$$  

Here $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^3$. The support function of $B$ is positively homogeneous and convex. Below, we consider the support function $H$ of a convex body as a function on $S^2$ (because of the positive homogeneity of $H$ the values on $S^2$ determine $H$ completely).

$C^k(S^2)$ denotes the space of $k$ times continuously differentiable functions defined on $S^2$. A convex body $B$ is $k$-smooth if its support function $H \in C^k(S^2)$.

Given a function $H$ defined on $S^2$, by $H_\omega(\varphi), \varphi \in S_\omega$ we denote the restriction of $H$ onto the circle $S_\omega$ for $\omega \in S^2$, and call the restriction function of $H$.

Below, we show (Theorem 1) that Problem 1. is equivalent to the problem of existence of a function $H$ defined on $S^2$ such that $H_\omega(\cdot)$ satisfies the differential equation

$$H_\omega(\varphi) + H_\omega(\varphi)\varphi = F(\omega, \varphi), \quad \varphi \in S_\omega$$ (1)

for every $\omega \in S^2$.

**Definition 1.** If for a given $F$ there exists $H$ defined on $S^2$ that satisfies Equation (1), then $H$ is called a solution of Equation (1).

In Equation (1), $H_\omega(\varphi)$ is a function defined on the space of an ordered pair orthogonal unit vectors, say $e_1, e_2$, (in integral geometry such a pair is a flag and the concept of a flag was first systematically employed by
There are two equivalent representations of an ordered pair orthogonal unit vectors \( e_1, e_2 \), dual each other:

\[
(\omega, \varphi) \quad \text{and} \quad (\Omega, \Phi),
\]

where \( \omega \in S^2 \) is the spatial direction of the first vector \( e_1 \), and \( \varphi \) is the planar direction in \( S_\omega \) coincides with the direction of \( e_2 \), while \( \Omega \in S^2 \) is the spatial direction of the second vector \( e_2 \), and \( \Phi \) is the planar direction in \( S_\Omega \) coincides with the direction of \( e_1 \). The second representation we will write by capital letters.

Given a flag function \( g(\omega, \varphi) \), we denote by \( g^*(\Omega, \Phi) \) the image of \( g \) defined by

\[
g^*(\Omega, \Phi) = g(\omega, \varphi),
\]

where \( (\omega, \varphi)^* = (\Omega, \Phi) \) (dual each other).

Let \( G \) be a function defined on \( \mathcal{F} \). For every \( \omega \in S^2 \), Equation (1) reduces to a differential equation on the circle \( S_\omega \).

**Definition 2.** If \( G(\omega, \cdot) \) is a solution of that equation for every \( \omega \in S^2 \), then \( G \) is called a flag solution of Equation (1).

**Definition 3.** If a flag solution \( G(\omega, \varphi) \) satisfies

\[
G(\Omega, \Phi) = G^*(\Omega) = G(\omega, \varphi),
\]

(no dependence on the variable \( \Phi \)), then \( G \) is called a consistent flag solution.

There is an important principle: each consistent flag solution \( G \) of Equation (1) produces a solution of Equation (1) via the map

\[
G(\omega, \varphi) \to G^*(\Omega, \Phi) = G^*(\Omega) = H(\Omega),
\]

and vice versa: the restriction functions of any solution of Equation (1) onto the great circles is a consistent flag solution.

Hence, the problem of finding a solution reduces to finding a consistent flag solution.

To solve the latter problem, the present paper applies the consistency method first used in [13]-[15] in an integral equations context.

We denote: \( e[\Omega, \Phi] \)—the plane containing the origin of \( \mathbb{R}^3 \), direction \( \Omega \in S^2 \), \( \Phi \) determines rotation of the plane around \( \Omega \), \( B[\Omega, \Phi] \)—projection of \( B \in \mathcal{B} \) onto the plane \( e[\Omega, \Phi] \), \( R^*(\Omega, \Phi) \)—curvature radius of \( \partial B[\Omega, \Phi] \) at the point with outer normal direction \( \Omega \in S^2 \). It is easy to see that

\[
R^*(\Omega, \Phi) = R(\omega^+); \Phi, \]

where \( (\Omega, \Phi) \) is dual to \( (\omega, \varphi) \).

Note that in the Problem 1, uniqueness (up to a translation) follows from the classical uniqueness result on Christoffel problem, since

\[
R_1(\Omega) + R_2(\Omega) = \frac{1}{\pi} \int_0^{2\pi} R^*(\Omega, \Phi) d\Phi.
\]

Equation (1) has the following geometrical interpretation.

It is known (see [11]) that 2 times continuously differentiable homogeneous function \( H \) defined on \( \mathbb{R}^3 \), is convex if and only if

\[
H_\omega(\varphi) + \left[H_\omega(\varphi)\right]^T \varphi \geq 0 \quad \text{for every} \quad \omega \in S^2 \quad \text{and} \quad \varphi \in S_\omega,
\]

where \( H_\omega(\cdot) \) is the restriction of \( H \) onto \( S_\omega \).

So in case \( F > 0 \), it follows from (7), that if \( H \) is a solution of Equation (1) then its homogeneous extension is convex.

It is known from convexity theory that if a homogeneous function \( H \) is convex then there is a unique convex body \( B \subset \mathbb{R}^3 \) with support function \( H \) and \( F(\omega, \varphi) \) is the projection curvature radius function of \( B \) (see [11]).

The support function of each parallel shifts (translation) of that body \( B \) will again be a solution of Equation (1). By uniqueness, every two solutions of Equation (1) differ by a summand \( \langle a, \cdot \rangle \) defined on \( S^2 \), where
Thus we have the following theorem.

**Theorem 1** Let $F$ be a positive function defined on $\mathcal{F}$. If Equation (1) has a solution $H$ then there exists a convex body $B$ with projection curvature radius function $F$, whose support function is $H$. Every solution of Equation (1) has the form $H(\cdot)+(a,\cdot)$, where $a \in \mathbb{R}^3$, being the support function of the convex body $B+a$.

The converse statement is also true. The support function $H$ of a 2-smooth convex body $B$ satisfies Equation (1) for $F=R$, where $R$ is the projection curvature radius function of $B$ (see [16]).

The purpose of the present paper is to find a necessary and sufficient condition that ensures a positive answer to both Problems 1,2 and suggest an algorithm of construction of the body $B$ by finding a representation of the support function in terms of projection curvature radius function. This happens to be a solution of Equation (1).

Throughout the paper (in particular, in Theorem 2 that follows) we use usual spherical coordinates $\nu, \tau$ for points $S^2$ based on a choice of a North Pole and a reference point $\tau=0$ on the equator. The point with coordinates $\nu, \tau$ we will denote by $(\nu, \tau)$, the points $(0, \tau)$ lie on the equator. On $S_n$ we choose anticlockwise direction as positive. On the plane $\omega^\perp$ containing $S_n$ we consider the Cartesian x and y-axes where the direction of the y-axis $y$ is taken to be the projection of the North Pole onto $\omega^\perp$. The direction of the x-axis $x$ we take as the reference direction on $S_n$ and call it the East direction. Now we describe the main result.

**Theorem 2** Let $B$ be a 3-smooth convex body with positive Gaussian curvature at every point of $\partial B$ and $R$ is the projection curvature radius function of $B$. Then for $\Omega \in S^2$ chosen as the North pole

$$H(\Omega) = \frac{1}{4\pi} \int_0^{2\pi} \left[ \int_0^\pi R((0, \tau)^{\perp}, \varphi) \cos \varphi \, d\varphi \right] d\tau$$

$$+ \frac{1}{8\pi} \int_0^{2\pi} \left[ \int_0^\pi R((0, \tau)^{\perp}, \varphi)((\pi+2\tau)\cos \varphi - 2\sin^3 \varphi) \, d\varphi \right] d\tau$$

$$- \frac{1}{2\pi} \int_0^{2\pi} \left[ \frac{\sin \tau}{\cos \nu} \int_0^\nu d\nu \int_0^{2\pi} R((\nu, \tau)^{\perp}, \varphi) \sin^3 \varphi \, d\varphi \right] d\tau$$

is a solution of Equation (1) for $F=R$. On $S_n$ we measure $\varphi$ from the East direction.

Remark, that the order of integration in the last integral of (8) cannot be changed.

Obviously Theorem 2 suggests a practical algorithm of reconstruction of convex body from projection curvature radius function $R$ by calculation of support function $H$.

We turn to Problem 1. Let $R$ be the projection curvature radius function of a convex body $B$. Then $F=R$ necessarily satisfies the following conditions:

a) For every $\omega \in S^2$ and any reference point on $S_n$

$$\int_0^{2\pi} F(\omega, \varphi) \sin \varphi \, d\varphi = \int_0^{2\pi} F(\omega, \varphi) \cos \varphi \, d\varphi = 0.$$ (9)

This follows from Equation (1), see also [16].

b) For every direction $\Omega \in S^2$ chosen as the North pole

$$\int_0^{2\pi} \left[ F^*(((\nu, \tau), y)) \right]_{\nu=0} \, d\tau = 0,$$ (10)

where the function $F^*$ is the image of $F$ (see (3)) and $y$ is the direction of the y-axis on $(\nu, \tau)^{\perp}$ (Theorem 5).

Let $F$ be a positive 2 times differentiable function defined on $\mathcal{F}$. Using (8), we construct a function $\overline{F}$ defined on $S^2$:

$$\overline{F}(\Omega) = \frac{1}{4\pi} \int_0^{2\pi} \left[ \int_0^\pi F((0, \tau), \varphi) \cos \varphi \, d\varphi \right] d\tau$$

$$+ \frac{1}{8\pi} \int_0^{2\pi} \left[ \int_0^\pi F((0, \tau), \varphi)((\pi+2\tau)\cos \varphi - 2\sin^3 \varphi) \, d\varphi \right] d\tau$$

$$- \frac{1}{2\pi} \int_0^{2\pi} \left[ \frac{\sin \tau}{\cos \nu} \int_0^\nu d\nu \int_0^{2\pi} F((\nu, \tau), \varphi) \sin^3 \varphi \, d\varphi \right] d\tau$$

Note that the last integral converges if the condition (10) is satisfied.
Theorem 3 A positive 2 times differentiable function $F$ defined on $\mathcal{F}$ represents the projection curvature radius function of some convex body $B$ if and only if $F$ satisfies the conditions (9), (10) and the extension (to $\mathbb{R}^3$) of the function $F$ defined by (11) is convex.

2. The Consistency Condition

We fix $\omega \in S^2$ and try to solve Equation (1) as a differential equation of second order on the circle $S_{\omega}$. We start with two results from [16].

a) For any smooth convex domain $D$ in the plane

$$h(\varphi) = \int_0^\varphi R(\psi) \sin(\varphi - \psi) d\psi,$$

(12)

where $h(\varphi)$ is the support function of $D$ with respect to a point $s \in \partial D$. In (12) we measure $\varphi$ from the normal direction at $s$, $R(\psi)$ is the curvature radius of $\partial D$ at the point with normal direction $\psi$.

b) (12) is a solution of the following differential equation

$$R(\varphi) = h(\varphi) + h'(\varphi).$$

(13)

One can easy verify that (also it follows from (13) and (12))

$$G(\omega, \varphi) = \int_0^\varphi F(\omega, \psi) \sin(\varphi - \psi) d\psi,$$

(14)

is a flag solution of Equation (1).

Theorem 4 Every flag solution of Equation (1) has the form

$$g(\omega, \varphi) = \int_0^\varphi F(\omega, \psi) \sin(\varphi - \psi) d\psi + C(\omega) \cos \varphi + S(\omega) \sin \varphi$$

(15)

where $C_n$ and $S_n$ are some real coefficients.

Proof of Theorem 4. Every continuous flag solution of Equation (1) is a sum of $G + g_0$, where $g_0$ is a flag solution of the corresponding homogeneous equation:

$$H_\omega(\varphi) + \left[H_\omega(\varphi)\right]_{\psi = \varphi} = 0, \quad \varphi \in S_{\omega},$$

(16)

for every $\omega \in S^2$. We look for the general flag solution of Equation (16) in the form of a Fourier series

$$g_0(\omega, \varphi) = \sum_{s = 0, 1, 2, \ldots} \left[C_s(\omega) \cos n\varphi + S_s(\omega) \sin n\varphi\right].$$

(17)

After substitution of (17) into (16) we obtain that $g_0(\omega, \varphi)$ satisfies (16) if and only if

$$g_0(\omega, \varphi) = C(\omega) \cos \varphi + S(\omega) \sin \varphi.$$

Now we try to find functions $C$ and $S$ in (15) from the condition that $g$ satisfies (4). We write $g(\omega, \varphi)$ in dual coordinates i.e. $g(\omega, \varphi) = g^\ast(\Omega, \Phi)$ and require that $g^\ast(\Omega, \Phi)$ should not depend on $\Phi$ for every $\Omega \in S^2$, i.e. for every $\Omega \in S^2$

$$(g^\ast(\Omega, \Phi))'_{\varphi} = (G(\omega, \varphi) + C(\omega) \cos \varphi + S(\omega) \sin \varphi)'_{\varphi} = 0,$$

(18)

where $G(\omega, \varphi)$ was defined in (14).

Here and below $(\cdot)'_{\varphi}$ denotes the derivative corresponding to right screw rotation around $\Omega$. Differentiation with use of expressions (see [14])

$$r'_{\varphi} = \frac{\sin \varphi}{\cos \nu}, \quad \varphi'_{\varphi} = -\tan \nu \sin \varphi, \quad \nu'_{\varphi} = -\cos \varphi,$$

(19)

after a natural grouping of the summands in (18), yields the Fourier series of $-(G(\omega, \varphi))'_{\varphi}$. By uniqueness of the Fourier coefficients.
\[ (C(\omega))^' + \frac{(S(\omega))^'}{\cos \nu} + \tan \nu C(\omega) = \frac{1}{\pi} \int_0^{2\pi} A(\omega, \varphi) \cos 2\varphi d\varphi \]

\[ (C(\omega))^' - \frac{(S(\omega))^'}{\cos \nu} - \tan \nu C(\omega) = \frac{1}{2\pi} \int_0^{2\pi} A(\omega, \varphi) d\varphi \]

\[ (S(\omega))^' - \frac{(C(\omega))^'}{\cos \nu} + \tan \nu S(\omega) = \frac{1}{\pi} \int_0^{2\pi} A(\omega, \varphi) \sin 2\varphi d\varphi, \]

where

\[ A(\omega, \varphi) = \int_0^\varphi \left[ F(\omega, \psi)^' \sin(\varphi - \psi) + F(\omega, \psi) \cos(\varphi - \psi) \phi'_\psi \right] d\psi. \]

3. Averaging

Let \( H \) be a solution of Equation (1), i.e. restriction of \( H \) onto the great circles is a consistent flag solution of Equation (1). By theorem 1 there exists a convex body \( B \in \mathcal{B} \) with projection curvature radius function \( R = F \), whose support function is \( H \).

To calculate \( H(\Omega) \) for a \( \Omega \in S^2 \) we take \( \Omega \) for the North Pole of \( S^2 \). Returning to the Formula (15) for every \( \omega \in S^2 \) we have

\[ H(\Omega) = \int_0^\pi R(\omega^\perp, \psi) \sin \left( \frac{\pi}{2} - \psi \right) d\psi + S(\omega). \]

We integrate both sides of (22) with respect to uniform angular measure \( d\tau \) over \([0, 2\pi]\) to get

\[ 2\pi H(\Omega) = \int_0^{2\pi} \int_0^\pi R((0, \tau)^\perp, \psi) \cos \psi d\psi d\tau + \int_0^{2\pi} S((0, \tau)) d\tau. \]

Now the problem is to calculate

\[ \int_0^{2\pi} S((0, \tau)) d\tau = \overline{S}(0). \]

We are going to integrate both sides of (20) and (21) with respect to \( d\tau \) over \([0, 2\pi]\). For \( \omega = (\nu, \tau) \), where \( \nu \in \left[0, \frac{\pi}{2}\right) \) and \( \tau \in (0, 2\pi) \) we denote

\[ \overline{S}(\nu) = \int_0^{2\pi} S((\nu, \tau)) d\tau, \]

\[ \pi A(\nu) = \int_0^{2\pi} d\tau \int_0^\pi \left[ \int_0^\phi R(\omega^\perp, \psi)^' \sin(\varphi - \psi) + R(\omega^\perp, \psi) \cos(\varphi - \psi) \phi'_\psi \right] d\psi \sin 2\varphi d\varphi. \]

Integrating both sides of (20) and (21) and taking into account that

\[ \int_0^{2\pi} (C(\nu, \tau))^' d\tau = 0 \]

for \( \nu \in [0, \pi/2) \) we get

\[ \overline{S}(\nu) + \tan \nu \overline{S}(\nu) = A(\nu), \]

i.e. a differential equation for the unknown coefficient \( \overline{S}(\nu) \).

We have to find \( \overline{S}(0) \) given by (24). It follows from (27) that

\[ \frac{\overline{S}(\nu)}{\cos \nu} = \frac{A(\nu)}{\cos \nu}. \]
Integrating both sides of (5.1) with respect to $dν$ over $[0, \pi/2)$ we obtain
\[
\overline{S}(0) - \frac{\overline{S}(\nu)}{\cos\nu} = -\int_0^{\pi/2} A(\nu) \cos\nu \, d\nu.
\] (29)

Now, we are going to calculate $\frac{\overline{S}(\nu)}{\cos\nu}$.

It follows from (15) that
\[
\pi \overline{S}(\nu) = \int_0^{2\pi} \int_0^{2\pi} \left[ H_\omega(\varphi) - \int_0^{\varphi} R(\omega^\perp, \psi) \sin(\varphi - \psi) \, d\psi \right] \sin\varphi \, d\varphi \, d\tau
\]
\[
= \int_0^{2\pi} \int_0^{2\pi} H_\omega(\varphi) \sin\varphi \, d\varphi \, d\tau - \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} R(\omega^\perp, \psi) \left( (2\pi - \psi) \cos\psi + \sin\psi \right) \, d\psi \, d\tau.
\] (30)

Let $\varphi \in S_\omega$ be the direction that corresponds to $\varphi \in [0, 2\pi)$, for $\omega = (\nu, \tau)$. As a point of $S^2$, let $\varphi$ have spherical coordinates $u, \tau$ with respect to $\Omega$. By the sinus theorem of spherical geometry
\[
\cos\nu \sin\varphi = \sin u.
\] (31)

From (31), we get
\[
(u')_{\nu, \frac{\pi}{2}} = -\sin \varphi.
\] (32)

Fixing $\tau$ and using (32) we write a Taylor formula at a neighborhood of the point $\nu = \pi/2$:
\[
H_{(\nu, \tau)}(\varphi) = H\left( (0, \varphi + \tau) \right) + H'(\left( (0, \varphi + \tau) \right) \sin \varphi \left( \frac{\pi}{2} - \nu \right) + O\left( \frac{\pi}{2} - \nu \right).
\] (33)

Similarly, for $\psi \in [0, 2\pi)$ we get
\[
R\left( (\nu, \tau)^\perp, \psi \right) = R\left( \left( \frac{\pi}{2}, \tau \right)^\perp, \psi + \tau \right)
\]
\[
+ R'\left( \left( \frac{\pi}{2}, \tau \right)^\perp, \psi + \tau \right) \sin \psi \left( \frac{\pi}{2} - \nu \right) + O\left( \frac{\pi}{2} - \nu \right).
\] (34)

Substituting (33) and (34) into (30) and taking into account the easily establish equalities
\[
\int_0^{2\pi} \int_0^{2\pi} H\left( (0, \varphi + \tau) \right) \sin \varphi \, d\varphi \, d\tau = 0
\]
and
\[
\int_0^{2\pi} \int_0^{2\pi} R\left( \left( \frac{\pi}{2}, \tau \right)^\perp, \psi + \tau \right) \left( (2\pi - \psi) \cos \psi + \sin \psi \right) \, d\psi \, d\tau = 0
\] (35)

we obtain
\[
\lim_{\nu \to \frac{\pi}{2}} \frac{\overline{S}(\nu)}{\cos\nu} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} H'\left( (0, \varphi + \tau) \right) \sin^2 \varphi \, d\varphi \, d\tau
\]
\[
- \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} R'\left( \left( \frac{\pi}{2}, \tau \right)^\perp, \psi + \tau \right) \sin \psi \left( (2\pi - \psi) \cos \psi + \sin \psi \right) \, d\psi \, d\tau
\]
\[
= \int_0^{2\pi} \left[ H'\left( (0, \tau) \right) \right] d\tau - \frac{3}{4} \int_0^{2\pi} \frac{R'\left( (\nu, \tau), \psi \right)}{\nu^2 - \nu \pi} \, d\tau.
\] (36)

**Theorem 5** For every $3$-smooth convex body $B \in B$ and any direction $\Omega \in S^2$, we have
where \( y \) is the direction of the y-axis on \( (v, \tau) \).

**Proof of Theorem 5.** Using spherical geometry, one can prove that (see also (1))

\[
\left[ R'(v, \tau), y \right]_{v=0}^{2\pi} = \left[ H\left((v, \tau)\right) + H'^{\nu}\left((v, \tau)\right) \right]_{v=0}^{2\pi} = \left[ H\left((v, \tau)\right) + H'^{\nu} \frac{1}{\cos^2 \nu} - H' \tan \nu \right]_{v=0}^{2\pi}
\]

where \( H \) is the support function of \( B \). Integrating (38), we get

\[
\int_0^{2\pi} \left[ R'(v, \tau), y \right]_{v=0}^\tau d\tau = \int_0^{2\pi} \left[ H'^{\nu} \right]_{v=0}^\tau d\tau = 0.
\]

**4. A Representation for Support Functions of Convex Bodies**

Let \( B \in \mathbb{B} \) be a convex body and \( Q \in \mathbb{R}^3 \). By \( H_Q \) we denote the support function of \( B \) with respect to \( Q \).

**Theorem 6** Given a 2-smooth convex body \( B \in \mathbb{B} \), there exists a point \( O' \in \mathbb{R}^3 \) such that for every \( \Omega \in S^2 \) chosen as the North pole

\[
\int_0^{2\pi} H_Q\left((v, \tau)\right)_{v=0}^{2\pi} d\tau = 0.
\]

**Proof of Theorem 6.** For a given \( B \) and a point \( Q \in \mathbb{R}^3 \), by \( K_Q \) we denote the following function defined on \( S^2 \)

\[
K_Q(\Omega) = \int_0^{2\pi} H_Q\left((v, \tau)\right)_{v=0}^{2\pi} d\tau.
\]

Clearly, \( K_Q \) is a continuous odd function with maximum \( \bar{K}(Q) = \max_{\Omega \in S^2} K_Q(\Omega) \).

It is easy to see that \( \bar{K}(Q) \to \infty \) for \( |Q| \to \infty \). Since \( \bar{K}(Q) \) is continuous, so there is a point \( O' \) for which

\[
\bar{K}(O') = \min \bar{K}(Q).
\]

Let \( \Omega^* \) be a direction of maximum now assumed to be unique, i.e.

\[
\bar{K}(O') = \max_{\Omega \in S^2} K_{O'}(\Omega) = K_{\Omega^*}(\Omega^*).
\]

If \( \bar{K}(O') = 0 \) the theorem is proved. For the case \( \bar{K}(O') = a > 0 \) let \( O'' \) be the point for which \( O'O'' = \varepsilon \Omega^* \). It is easy to demonstrate that \( H_{O''}(\Omega) = H_{O'}(\Omega) - \varepsilon \left(\Omega, \Omega'\right) \), hence for a small \( \varepsilon > 0 \) we find that \( \bar{K}(O'') = a - 2\pi\varepsilon \), contrary to the definition of \( O' \). So \( \bar{K}(O') = 0 \). For the case where there are two or more directions of maximum one can apply a similar argument.

Now we take the point \( O' \) of the convex body \( B \) for the origin of \( \mathbb{R}^3 \). Below \( H_{O'} \), we will simply denote by \( H \).

By Theorem 6 and Theorem 5, we have the boundary condition (see (36))

\[
\frac{\overline{S}(\nu)}{\cos \nu} \bigg|_\tau = 0.
\]

Substituting (29) into (23) we get
Using expressions (19) and integrating by \( d\varphi \) yields

\[
2\pi H(\Omega) = \int_0^{2\pi} \int_0^{\pi} R(\omega^+, \psi) \cos \psi \, d\psi \, d\Omega - \frac{1}{2} \int_0^{2\pi} \frac{d\nu}{\cos \psi} \int_0^{\pi} \left[ R(\omega^+, \psi) + R(\omega^-, \psi) \cos(\psi - \varphi) \right] \, d\psi \sin 2\varphi \, d\varphi \, d\tau.
\]

Integrating by parts (42) we get

\[
2\pi H(\Omega) = \int_0^{2\pi} \int_0^{\pi} R(\omega^+, \psi) \cos \psi \, d\psi \, d\Omega + \left[ R(\omega^-, \psi) + R(\omega^+, \psi) \tan \psi \right] I \, d\psi,
\]

where

\[
I = \int_0^{2\pi} \sin 2\varphi \cos(\psi - \varphi) \, d\varphi = \frac{(2\pi - \varphi) \cos \psi}{4} + \frac{\sin \psi \left(1 + \sin^2 \psi \right)}{4} - \sin^3 \psi,
\]

and

\[
I = \int_0^{2\pi} \sin 2\varphi \sin(\psi - \varphi) \cos \varphi \, d\varphi = \frac{(2\pi - \varphi) \cos \psi}{4} + \frac{\sin \psi \left(1 + \sin^2 \psi \right)}{4}.
\]

Integrating by parts (42) we get

\[
2\pi H(\Omega) = \int_0^{2\pi} \int_0^{\pi} R(\omega^+, \psi) \cos \psi \, d\psi \, d\Omega - \frac{1}{2} \int_0^{2\pi} \frac{d\nu}{\cos \psi} \int_0^{\pi} \left[ R(\omega^+, \psi) \sin \psi \sin^3 \psi \right] \, d\psi \sin 2\varphi \, d\varphi \, d\tau.
\]

Using (34), Theorem 5 and taking into account that

\[
\int_0^{2\pi} I \, d\psi = 0
\]

we get

\[
2\pi H(\Omega) = \int_0^{2\pi} \int_0^{\pi} R(\omega^+, \psi) \cos \psi \, d\psi \, d\Omega - \frac{1}{2} \int_0^{2\pi} \frac{d\nu}{\cos \psi} \int_0^{\pi} \left[ R(\omega^+, \psi) \sin \psi \sin^3 \psi \right] \, d\psi \sin 2\varphi \, d\varphi \, d\tau.
\]

From (44), using (9) we obtain (8). Theorem 2 is proved.

5. Proof of Theorem 3

Necessity: if \( F \) is the projection curvature radius function of a convex body \( B \in B \), then it satisfies (9) (see [16]), the condition (10) (Theorem 5) and \( F \) defined by (11) is convex since it is the support function of \( B \) (Theorem 2).

Sufficiency: let \( F \) be a positive 2 times differentiable function defined on \( \mathcal{F} \) satisfies the conditions (9), (10). We construct the function \( F \) on \( S^2 \) defined by (11). There exists a convex body \( B \) with support function \( F \) since its extension is a convex function. Also Theorem 2 implies that \( F \) is the projection curvature radius of \( B \).

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References


