The Best Finite-Difference Scheme for the Helmholtz Equation

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ABSTRACT

The best finite-difference scheme for the Helmholtz equation is suggested. A method of solving obtained finite-difference scheme is developed. The efficiency and accuracy of method were tested on several examples.

Keywords: The Best Finite-Difference Scheme for the Helmholtz and Laplace’s Equations

1. Introduction

The finite-difference method is a standard numerical method for solving boundary value problems. Recently, considerable attention has been attracted to construct a best (or exact) difference approximation for some ordinary and partial differential equations [1-3]. In this paper a best finite-difference method is developed for Helmholtz equation with general boundary conditions on the rectangular domain in $\mathbb{R}^2$. The method proposed here comes out from [4] and is based on separation of variables method and expansion of one-dimensional three-point difference operators for sufficiently smooth solution. The paper is organized as follows. The statement of problem and the separation of variables method are considered in Section 2. A detailed description of the best difference approximation to the Helmholtz equation in rectangular domain is given in Section 3.

Section 4 is devoted to derive the best approximation for the given third kind boundary conditions. The method of solution for the obtained difference equations is considered in Section 5 and numerical examples are given last Section 6.

2. Statement of Problem

Let $\Omega = (a,b) \times (c,b) \times (a,b)$ be an open rectangular domain in Euclidean $\mathbb{R}^2$ space with boundary given by $\partial \Omega$. The aim is to determine a function $u(x,y)$, satisfying equation

$$\Delta u + Cu = 0, (x,y) \in \Omega$$

(2.1)

with boundary condition

$$\alpha u + \beta \frac{\partial u}{\partial n} = \psi (x,y), (x,y) \in \partial \Omega,$$

(2.2)

where $C$ in (2.1) is a given number and $\frac{\partial}{\partial n}$ is the outward normal on $\partial \Omega$.

It is well known that the stabilized oscillation problems and diffusing processes in gas lead to the so called Helmholtz Equation (2.1) with a positive coefficient $C = \lambda^2$. The diffusing process in the moving field leads to the Equation (2.1) with negative coefficient $C = -\lambda^2$. If $C = 0$ the Equation (2.1) leads to Laplace’s ones. Obviously, the properties of the solution of Equation (2.1) depend essentially upon the sign of the coefficient $C$ in (2.1). We will assume that the problem (2.1), (2.2) has an unique and sufficiently smooth solution.

By virtue of variables method looking for the solution $u(x,y)$ of Equation (2.1), (2.2) in the form

$$u(x,y) = U_1(x)U_2(y)$$

(2.3)

we arrive at equation

$$U_1''(x) + U_2''(y) = -C,$$

which is splitted into two independing equations

$$U_1''(y) = \omega U_1(x)$$

(2.4a)

and

$$U_2''(y) = \beta U_2(y), \beta = -C - \omega,$$

(2.4b)

where the unknown separation constant $\omega$ is to be found.

By virtue of (2.3) the boundary condition (2.2) is splitted into ones for $U_1(x)$ and $U_2(y)$

$$\alpha_1 u_1 - \beta U_1'(a) = x_{10},$$

$$\alpha_2 U_2(b) + \beta U_2'(b) = x_{2N},$$

(2.5)
and

\[ \alpha x U_2(c) - \beta u U_2'(c) = x_0, \]
\[ \alpha x U_2(d) + \beta u U_2'(d) = x_{4N} \]  
(2.6)

The solution of boundary value problem (2.4a), (2.5) is found in a closed form

\[ U_1(x) = \frac{x_0 F_1(\omega) + x_{4N} F_2(\omega)}{F_3(\omega) + \sqrt{\omega} F_2(\omega)}, \]  
(2.7)

where

\[ F_1(\omega) = \alpha sh(\sqrt{\omega}(b-x)) + \beta ch(\sqrt{\omega}(b-x)), \]
\[ F_2(\omega) = \alpha sh(\sqrt{\omega}(a-x)) + \beta ch(\sqrt{\omega}(a-x)), \]
\[ \omega > 0 \]

and

\[ F_3(\omega) = (\alpha \alpha + \beta \beta) sh(\sqrt{\omega}(b-a)), \]
\[ F_4(\omega) = (\alpha \alpha + \beta \beta) ch(\sqrt{\omega}(b-a)). \]
\[ \omega > 0 \]

When \( \omega < 0 \) the functions \( sh \) and \( ch \) in (2.7) are to be replaced by \( sin \) and \( cos \) respectively and \( \omega \) replaced by \( -\omega \). Analogously, we can find the solutions of the boundary value problem (2.4b) and (2.6) in closed form. Then from (2.3) and (2.7) clear, that the problem consists in determining the separation constant \( \omega \).

3. Construction of the Best Finite-Difference Equations

For the numerical solution of problem (2.1), (2.2) is introduced the uniform rectangular grid \( \Omega_h \):

\[ \Omega_h = \{(x, y) | x = x_0 + i h_x, y = y_0 + j h_y, \]
\[ i = 0, 1, \ldots, N; j = 0, 1, \ldots, M \} \]

where \( h_x = \frac{(b-a)}{N} \) and \( h_y = \frac{(d-c)}{M} \) are the mesh sizes in the x and y directions respectively. Usually, the Equation (2.1) is approximated by the five-point difference equation

\[ \frac{y_{i+1,j} - 2 y_{i,j} + y_{i-1,j}}{h_x^2} + \frac{y_{i,j+1} - 2 y_{i,j} + y_{i,j-1}}{h_y^2} + C y_{i,j} = 0, \]
\[ i = 1, \ldots, N-1; j = 1, \ldots, M-1. \]  
(3.1)

The local discretization error of the Equation (3.1) is of \( O(h_x^2 + h_y^2) \) order. Now we describe how to derive the best difference scheme for Equation (2.1). To this end, we consider expression

\[ (\Lambda_1 + \Lambda_2) u_{ij} = \frac{u_{i+1,j} - 2 u_{i,j} + u_{i-1,j}}{h_x^2} + \frac{u_{i,j+1} - 2 u_{i,j} + u_{i,j-1}}{h_y^2} \]  
(3.2)

where \( u_{ij} = u(x_i, y_j) \). If we denote by \( U_1, U_2, U_3 \) the values of \( u_{ij} \), \( u_{i,j} \), the values of \( U_{1}(x_i) \) and \( U_{2}(y_j) \), respectively, the using (2.3) the Equation (3.2) may be written as

\[ (\Lambda_1 + \Lambda_2) U_{ij} = U_{i+1,j} + U_{i-1,j} + U_{i,j+1} - 2 U_{i,j} \]  
(3.3)

Due to smoothness assumption of solution \( u(x, y) \), as well as, functions \( U_1(x) \) and \( U_2(y) \), the Taylor series expansion yields

\[ \Lambda_1 U_{ii} = \frac{h_x^2}{(2k+2)!} \sum_{k=1}^{4} \frac{2k U_{x}^{2k+2}(x_i)}{(2k+2)^k}, \]  
(3.4a)

\[ \Lambda_1 U_{jj} = \frac{h_y^2}{(2k+2)!} \sum_{k=1}^{4} \frac{2k U_{y}^{2k+2}(y_j)}{(2k+2)^k}, \]  
(3.4b)

Because of (2.4) we have

\[ U_{i,j}^{(2k)} = \alpha^k U_1, U_{j,j}^{(2k)} = \beta^k U_2, \]
\[ k = 1, 2, \ldots \]  
(3.5)

Taking into account (3.4), (3.5) in (3.3) it follows that

\[ \left\{ \Lambda_1 + \Lambda_2 + \left( C - 2 \sum_{k=1}^{4} \frac{h_x^2}{(2k+2)!} \sum_{k=1}^{4} \frac{2k U_{x}^{2k+2}(x_i)}{(2k+2)^k} \right) E \right\} u_{ij} = 0, \]  
(3.6)
\[ i = 1, 2, \ldots, N-1; j = 1, 2, \ldots, M-1, \]

where \( E \) is unit operator. The difference Equation (3.6) contains unknown nonzero parameter \( \omega \) and therefore it may be considered as a nonlinear equation with respect to the parameter \( \omega \) and \( u_{ij} \). The series in (3.6) may be expressed through analytical functions depending on the sign of quantities \( \omega \) and \( u_{ij} \). The series in (3.6) can be rewritten as

\[ (\Lambda_1 + \Lambda_2 - 2D(\omega) E) u = 0, (x, y) \in \Omega_h. \]  
(3.7)

There are three cases:

1) Let \( C = \lambda^2 > 0 \). Then it is easy to show that

\[ \begin{cases} \cos(\sqrt{\omega}h_x) - 1 & \cos(\sqrt{\beta h_x} - 1) \quad \text{if} \quad \omega \in (-\infty, -\lambda^2), \\ \frac{\cos(\sqrt{\omega}h_x) - 1}{h_x^2} & \cos(\sqrt{\beta h_x} - 1) \quad \text{if} \quad \omega \in (-\lambda^2, 0), \\ \frac{\cos(\sqrt{\omega}h_x) - 1}{h_x^2} & \cos(\sqrt{\beta h_x} - 1) \quad \text{if} \quad \omega \in (0, +\infty), \end{cases} \]  
(3.8)

2) Let \( C = 0 \). In this case \( D \) is given by
Thus we obtain the best (or exact) five-point difference boundary condition for \( U(x) \) at point \( x = b \) as given by

\[
U_{1N} = \frac{X_{2N}}{\alpha_z}, \quad \text{when } \beta_2 = 0, \quad (4.4a)
\]

\[
U_{1N-1} = \theta_3(\omega) U_{1N} + \theta_4(\omega), \quad \text{when } \beta_2 \neq 0, \quad (4.4b)
\]

where \( \theta_3(\omega) \) and \( \theta_4(\omega) \) are given by

\[
\theta_3(\omega) = \begin{cases} 
  \frac{\ch(\sqrt{\omega}h_1) + \frac{\alpha_2}{\beta_2} \sh(\sqrt{\omega}h_1)}{\sqrt{\omega}}, & \omega > 0 \\
  \frac{\cos(\sqrt{-\omega}h_1) + \frac{\alpha_2}{\beta_2} \sin(\sqrt{-\omega}h_1)}{\sqrt{-\omega}}, & \omega < 0 
\end{cases} \quad (4.5a)
\]

and

\[
\theta_4(\omega) = \begin{cases} 
  \frac{X_{2N}}{\alpha_z}, & \omega > 0 \\
  \frac{X_{2N}}{\beta_2} \sin(\sqrt{-\omega}h_1), & \omega < 0 
\end{cases} \quad (4.5b)
\]

Analogously, it is easy to verify that the exact difference boundary condition for \( U(x) \) at point \( x = b \) is given by

\[
U_{1N} = \frac{X_{2N}}{\alpha_z}, \quad \text{when } \beta_2 = 0, \quad (4.4a)
\]

\[
U_{1N-1} = \theta_3(\omega) U_{1N} + \theta_4(\omega), \quad \text{when } \beta_2 \neq 0, \quad (4.4b)
\]

where \( \theta_3(\omega) \) and \( \theta_4(\omega) \) are given by

\[
\theta_3(\omega) = \begin{cases} 
  \frac{\ch(\sqrt{\omega}h_1) + \frac{\alpha_2}{\beta_2} \sh(\sqrt{\omega}h_1)}{\sqrt{\omega}}, & \omega > 0 \\
  \frac{\cos(\sqrt{-\omega}h_1) + \frac{\alpha_2}{\beta_2} \sin(\sqrt{-\omega}h_1)}{\sqrt{-\omega}}, & \omega < 0 
\end{cases} \quad (4.5a)
\]

and

\[
\theta_4(\omega) = \begin{cases} 
  \frac{X_{2N}}{\alpha_z}, & \omega > 0 \\
  \frac{X_{2N}}{\beta_2} \sin(\sqrt{-\omega}h_1), & \omega < 0 
\end{cases} \quad (4.5b)
\]

In the same way, as before, one can construct the best difference boundary conditions for \( U(x) \). We omit the evaluation and present only the final results:

\[
U_{20} = \frac{X_{20}}{\alpha_3}, \quad \text{when } \beta_3 = 0, \quad (4.6a)
\]

\[
U_{21}\overline{\theta}_1(\omega) U_{20} + \overline{\theta}_2(\omega), \quad \text{when } \beta_3 \neq 0, \quad (4.6b)
\]

and

\[
U_{2M} = \frac{X_{4M}}{\alpha_4}, \quad \text{when } \beta_4 = 0 \quad (4.7a)
\]

\[
U_{2M-1}\overline{\theta}_3(\omega) U_{2M} + \overline{\theta}_4(\omega), \quad \text{when } \beta_4 \neq 0 \quad (4.7b)
\]

where \( \overline{\theta}_i(\omega) \) are defined by
\( \omega \) are known. Thanks to (4.2) it is possible to find \( U_{10} \) or \( U_{11} \) depending on the \( \beta \). For example, if \( \beta = 0 \) then \( U_{10} \) is determined by (4.2a) and \( U_{11} \) and \( \omega \) to be chosen arbitrary. Otherwise, \( U_{11} \) is determined by (4.2b) and \( U_{10} \) and \( \omega \) to be chosen arbitrary.

Note, that when \( C \neq 0 \) one of the boundary conditions (2.5), (2.6) is assumed to be homogeneous. For Laplace’s equation we always can leads to equation with homogeneous boundary conditions by change of variables. The exact value of parameter \( \omega \) must satisfy

\[
\Phi(\omega) = 0, \quad (5.3)
\]

where \( \Phi(\omega) \), for examples, when \( X_{10} = 0 \) defined by

\[
\Phi(\omega) = \begin{cases} 
U_{1N} - \frac{1}{\alpha_x} X_{2N}, & \text{when } \beta > 0 \\
U_{1N} - \theta_1(\omega) U_{1N} - \theta_4(\omega), & \text{when } \beta \neq 0
\end{cases} \quad (5.4)
\]

The nonlinear Equation (5.3) can be solved by Newton’s method:

\[
\omega_{k+1} = \omega_k - \frac{\Phi(\omega_k)}{\Phi'(\omega_k)}, k = 0, 1, 2, \ldots \quad (5.5)
\]

The value \( \Phi'(\omega_k) \) in the dominate of (5.5) is found by differentiating the Equation (5.4) and (5.2a) with respect to \( \omega \). The iteration process (5.5) is terminated by criterion

\[
|\omega_{k+1} - \omega_k| \leq \mathcal{E} \quad (5.6)
\]

where \( \mathcal{E} \) is a reassigned accuracy.

If the evaluation of \( \Phi'(\omega) \) causes some difficulty we can use secant method instead of Newton’s ones. After finding \( \omega \) the three-point finite-difference scheme (3.7) on the several examples.

**Example 1.**

\[
\Delta u = 0, \quad 0 \leq x, y \leq 1
\]

with boundary condition

\[
u(0, y) - \nu(0, y) = 6\alpha \frac{\sin(6\pi y)}{\sin(6\pi)}, \quad u(1, y) = 0,
\]

\[
u(x, 0) = 0, \quad u(x, 1) = \sin(6\pi x).
\]

The exact solution is given by

\[
u(x, y) = \sin(6\pi x) \frac{\sin(6\pi y)}{\sin(6\pi)}.
\]

In Table 1 we present the computed values of \( u_y = U_{1i} U_{2j} \) (exact values of \( u_y = U_{ix} U_{jy} \) present in

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brackets) for $N = 5$ and $M = 4$. In order to use secant method we need two first approximations $\omega_0$ and $\omega_1$ to $\omega$. The iteration was terminated by criterion (5.6) with $\mathcal{E} = 10^{-7}$.

Example 2.

$$\Delta u = 0, \quad 0 \leq x, \ y \leq 1$$

with boundary condition

$$u(0, y) = u(1, y) = 0, \quad u(x, 0) = 0, \quad u(x, 1) = \sin(\pi x).$$

The exact solution is given by

$$u(x, y) = \sin(\pi x), \frac{\sinh(\pi y)}{\sinh(\pi)}.$$

In Table 2 we present the computed values of $u(y) = U_{ij}$ (exact values of $u(x, y)$) present in brackets) for $N = 6$ and $M = 6$. In order to use secant method we need two first approximations $\omega_0$ and $\omega_1$ to $\omega$. In this example choose $\omega_0 = -5$ and $\omega_1 = -6$. The exact value of $\omega$ is $\omega = -\pi^2$. The convergence of $\omega$ was tabulated in Table 3. The iteration was terminated by criterion (5.6) with $\mathcal{E} = 10^{-7}$.

Example 3.

$$\Delta u + 2u = 0, \quad 0 \leq x, \ y \leq 1$$

with boundary condition

$$u(0, y) - u'(0, y) = 0, \quad u(1, y) = \exp(1 + y),$$

$$u(x, 0) = \exp(x), \quad u(x, 1) = \exp(1 + x).$$

The exact solution is given by

$$u(x, y) = \exp(x + y),$$

In Table 4 we present the computed values of $u(y) = U_{ij}$ (exact values of $u(x, y)$) present in brackets) for $N = 6$ and $M = 4$. In order to use secant

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**Table 1. Computed values of $u_j = U_{i_1}U_{i_2}$ for $N = 5$ and $M = 4.$**

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**Table 2. Computed values of $u_j = U_{i_1}U_{i_2}$ for $N = 6$ and $M = 6.$**

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Table 3. The convergence of $\omega_1$.

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<th>$\epsilon$</th>
<th>$\lambda$</th>
</tr>
</thead>
</table>

Table 4. Computed values of $u_j = U_jU_{i,j}$ for $N = 6$ and $M = 4$.

<table>
<thead>
<tr>
<th>$u_{i,j}$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4$</td>
<td>2.7182818 (2.7182818)</td>
<td>3.2112705 (3.2112705)</td>
<td>3.7936679 (3.7936679)</td>
<td>4.4816891 (4.4816891)</td>
<td>5.2944901 (5.2944901)</td>
<td>6.2547010 (6.2547010)</td>
<td>7.3890561 (7.3890561)</td>
</tr>
<tr>
<td>$3$</td>
<td>2.1170000 (2.1170000)</td>
<td>2.5009400 (2.5009400)</td>
<td>2.9545115 (2.9545115)</td>
<td>3.4903430 (3.4903430)</td>
<td>4.1233530 (4.1233530)</td>
<td>4.8711660 (4.8711660)</td>
<td>5.7546027 (5.7546027)</td>
</tr>
<tr>
<td>$2$</td>
<td>1.6487213 (1.6487213)</td>
<td>1.9477340 (1.9477340)</td>
<td>2.3009759 (2.3009759)</td>
<td>2.7182818 (2.7182818)</td>
<td>3.2112705 (3.2112705)</td>
<td>3.7936679 (3.7936679)</td>
<td>4.4816891 (4.4816891)</td>
</tr>
<tr>
<td>$1$</td>
<td>1.2840254 (1.2840254)</td>
<td>1.5168968 (1.5168968)</td>
<td>1.7920018 (1.7920018)</td>
<td>2.1170000 (2.1170000)</td>
<td>2.5009400 (2.5009400)</td>
<td>2.9545115 (2.9545115)</td>
<td>3.4903430 (3.4903430)</td>
</tr>
<tr>
<td>$0$</td>
<td>1.0000000 (1.0000000)</td>
<td>1.1813604 (1.1813604)</td>
<td>1.3956124 (1.3956124)</td>
<td>1.6487213 (1.6487213)</td>
<td>1.9477340 (1.9477340)</td>
<td>2.3009759 (2.3009759)</td>
<td>2.7182818 (2.7182818)</td>
</tr>
</tbody>
</table>

Table 5. The convergence of $\omega_4$.

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.1269983$</td>
<td>$1.0086468$</td>
<td>$1.0000778$</td>
<td>$1.0000000$</td>
<td></td>
</tr>
</tbody>
</table>

method we need two first approximations $\omega_0$ and $\omega_1$ to $\omega$. In this example we were choose $\omega_0 = 3$ and $\omega_1 = 2$. The exact value of $\omega$ is $\omega = 1$. The convergence of $\omega$ was tabulated in Table 5. The iteration was terminated by criterion (5.6) with $\varepsilon = 10^{-7}$.

REFERENCES


