Nonstationary Wavelets Related to the Walsh Functions

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ABSTRACT

Using the Walsh-Fourier transform, we give a construction of compactly supported nonstationary dyadic wavelets on the positive half-line. The masks of these wavelets are the Walsh polynomials defined by finite sets of parameters. Application to compression of fractal functions are also discussed.

Keywords: Walsh Functions; Nonstationary Dyadic Wavelets; Fractal Functions; Adapted Multiresolution Analysis

1. Introduction

As usual, let $\mathbb{R}_+ = [0, +\infty)$ be the positive half-line, $\mathbb{N} = \{0, 1, 2, \cdots\}$ be the set of all nonnegative integers, and let $\mathbb{Z} = \{1, 2, \cdots\}$ be the set of all positive integers. The first examples of orthogonal wavelets on $\mathbb{R}_+$ related to the Walsh functions and the corresponding wavelets on the Cantor dyadic group have been constructed in [1]; then, in [2] and [3], a multifractal structure of this wavelets is observed and conditions for wavelets to generate an unconditional basis in $L^p$-spaces are studied. In the present paper, using the Walsh-Fourier transform, we construct orthogonal and biorthogonal wavelets on $\mathbb{R}_+$.

The masks of these wavelets are the Walsh polynomials defined by finite sets of parameters. Application to compression of fractal functions are also discussed.

The Walsh-Fourier transform of every function $f$ that belongs to $L^p(\mathbb{R}_+) \cap L^q(\mathbb{R}_+)$ is defined by

$$\hat{f}(\omega) = \int_{\mathbb{R}_+} f(x) \chi(x, \omega) dx,$$

and extent to the whole space $L^2(\mathbb{R}_+)$ in a standard way. The intervals

$$\Delta_k = \left[k2^{-n}, (k+1)2^{-n}\right), \ k \in \mathbb{N}_+,$$

are called the dyadic intervals of range $n$. The dyadic topology on $\mathbb{R}_+$ is generated by the collection of dyadic intervals. A subset $E$ of $\mathbb{R}_+$ which is compact in the dyadic topology will be called $W$-compact.

For any $j \in \mathbb{N}_+$ we define $\varphi_j$ and $\psi_j$ by the following algorithm:

**Step 1.** For each $j \in \mathbb{N}_+$ choose $n_j \in \mathbb{N}_+$ and $b_k(j) \in \mathbb{N}_+$, $k = 0, 1, \cdots, 2^{n_j} - 1$, such that

$$b_{k}(j)^2 = 1, |b_{k}(j)|^2 + |b_{k}(j')|^2 = 1$$

for all $k = 0, 1, \cdots, 2^{n_j} - 1$.

**Step 2.** Define the masks

$$m_{0}^{(j)}(\omega) = \frac{1}{2^{-\sum_{k=0}^{n_j}}} \sum_{k=0}^{2^{-n_j}-1} c_{k}^{(j)} w_{k}(\omega)$$

with the coefficients

$$c_{k}^{(j)} = \frac{1}{2^{n_j}} \sum_{k=0}^{2^{-n_j}-1} b_{k}(j) w_{k}(2^{-n_j} k), \ k = 0, 1, \cdots, 2^{n_j} - 1,$$

so that $m_{0}^{(j)}(\omega) = b_{k}(j)$ for all $\omega \in \Delta_k^{(j)}$ (cf. [15, Sect. 9.7]).

**Step 3.** For each $j \in \mathbb{N}_+$ put

$$\tilde{\phi}_{j}(\omega) = 2^{-j/2} \prod_{k=1}^{\infty} m_{0}^{(j)}(2^{-j} \omega),$$
so that
\[ \varphi_j(x) = \frac{1}{\sqrt{2}} \sum_{k=0}^{2^j-1} c_{j+1}^{(k)} \varphi_{j+1}(x \oplus 2^{-j-1}k). \] (4)

**Step 4.** Define \( \psi_j \) by the formula
\[ \psi_j(x) = \frac{1}{\sqrt{2}} \sum_{k=0}^{2^j-1} (-1)^{k} \psi_{j+1}(x \oplus \frac{k}{2^{j+1}}). \] (5)

Further, let us define subspaces \( V_j \) and \( W_j \) in \( L^2(\mathbb{R}_+) \) as follows
\[ V_j = \text{span}\{ \varphi_{j,k} : k \in \mathbb{Z}_+ \}, \]
\[ W_j = \text{span}\{ \psi_{j,k} : k \in \mathbb{Z}_+ \} \]
for all \( j \in \mathbb{Z}_+ \).

We say that a polynomial \( n \) satisfies the modified Cohen condition if there exists a \( W \)-compact subset \( E \) of \( \mathbb{R}_+ \) such that
\[ \int E \geq 0, \mu(E) = 1, E = [0,1)(\text{mod}\, \mathbb{Z}_+) \]
and
\[ \inf \inf_{n \in \mathbb{Z}_+} \mu(2^{-j}/2) > 0. \] (6)

**Theorem.** Suppose that the masks \( m_n^{(0)} \) satisfy the modified Cohen condition with a subset \( E \) and there exists \( j_0 \in \mathbb{Z}_+ \) such that
\[ m_n^{(0)}(\omega) = 1 \text{ for all } \omega \in [0,2^{-h}], \, n \in \mathbb{Z}_+. \] (7)

Then for any \( j \in \mathbb{Z}_+ \), the following properties hold:

a) \( \varphi_j, \psi_j \in L^2(\mathbb{R}_+) \) and \( \varphi_j \subseteq (0,1) \);

b) \( \{ \varphi_{j,k} : k \in \mathbb{Z}_+ \} \) and \( \{ \psi_{j,k} : k \in \mathbb{Z}_+ \} \) are orthonormal bases in \( V_j \) and \( W_j \), respectively;

c) \( V_j \subseteq V_{j+1}, \quad V_j \oplus W_j = V_{j+1} \).

Moreover, we have
\[ \bigcup_{j=0}^\infty V_j = L^2(\mathbb{R}_+). \]

**Corollary.** The system
\[ \{ \varphi_0(\cdot \oplus k) : k \in \mathbb{Z}_+ \} \cup \{ \varphi_{j,k} : j, k \in \mathbb{Z}_+ \} \]
is an orthonormal basis in \( L^2(\mathbb{R}_+) \).

We prove this theorem in the next section. Then using the notion of an adapted multiresolution analysis suggested by Sendoz [12], we discuss an application of the nonstationary dyadic wavelets to compression of the Weierstrass function and the Schwartz function.

**2. Proof of the Theorem**

At first we prove the orthonormality of \( \{ \varphi_{j,k} \}_{k \in \mathbb{Z}_+} \). In view of
\[ \langle \varphi_{j,0}, \varphi_{j,n} \rangle = \langle \hat{\varphi}_{j,0}, \hat{\varphi}_{j,n} \rangle = \int_0^\infty \hat{\varphi}_{j,0}(\omega) \hat{\varphi}_{j,n}(\omega) w_n(2^{-j}\omega) d\omega, \]
we let show that
\[ \int_0^\infty \hat{\varphi}_{j,0}(\omega) \hat{\varphi}_{j,n}(\omega) w_n(2^{-j}\omega) d\omega = \delta_{n,0}, \, n \in \mathbb{Z}_+. \]

Denote by \( 1_E \) the characteristic function of \( E \). For each \( j \) we define
\[ \hat{\varphi}_{j}^{(1)}(\omega) = 2^{-j/2} \sum_{n=1}^E m_n^{(j)}(2^{-j}\omega)1_E(2^{-j}\omega) \]
for \( s = j+1, j+2, \cdots \). Since \( 0 \in \text{int} E \) and, for all \( j \in \mathbb{Z}_+ \), \( m_n^{(j)}(\omega) = 1 \) in some neighbourhood of zero, we obtain from Equation (3)
\[ \lim_{k \to \infty} \hat{\varphi}_{j}^{(1)}(\omega) = \hat{\varphi}_{j}(\omega) \quad \text{for all } \omega \in \mathbb{Z}_+. \] (8)

Let
\[ I_{j}^{(1)}[n] = \int_0^\infty \hat{\varphi}_{j}(\omega) \hat{\varphi}_{j}(\omega) w_n(2^{-j}\omega) d\omega, \]
where \( k > j, \quad n \in \mathbb{Z}_+ \). Letting \( \zeta = 2^{-j}\omega \), we have
\[ I_{j}^{(1)}[k] = 2^{-j/2} \int_0^\infty \sum_{n=1}^E m_n^{(j)}(2^{-j}\zeta) w_n(2^{-j}\zeta) d\zeta \]
\[ = 2^{-j/2} \int_0^\infty \sum_{n=1}^E m_n^{(j)}(\zeta) w_n(2^{-j}\zeta) d\zeta \]
\[ = 2^{-j/2} \left[ \sum_{n=1}^E \sum_{n=1}^E m_n^{(j)}(\zeta) \right] + m_0^{(j)}(\zeta + 1/2) \]
\[ \times \sum_{n=1}^E m_n^{(j)}(2^{-j}\zeta) w_n(2^{-j}\zeta) d\zeta, \]
that yields \( I_{j}^{(1)}[k] = I_{j}^{(1)}[k] \). By induction, we obtain
\[ I_{j}^{(1)}[k] = I_{j}^{(1)}[k] = \cdots = I_{j}^{(1)}[k] = \delta_{k,0}. \]

According to Equation (8), by Fatou’s lemma, we have
\[ \int_0^\infty \hat{\varphi}_{j}(\omega) \hat{\varphi}_{j}(\omega) d\omega \leq \lim_{j \to \infty} \int_0^\infty \hat{\varphi}_{j}^{(1)}(\omega) \hat{\varphi}_{j}^{(1)}(\omega) d\omega = \lim_{j \to \infty} I_{j}^{(1)}[0] = 1. \] (9)

Consequently, \( \varphi_j \in L^2(\mathbb{R}_+) \) and, in view of Equation (5), \( \psi_j \in L^2(\mathbb{R}_+) \). It is known that if \( f \in L^2(\mathbb{R}_+) \) is constant on dyadic intervals of range \( n \), then \( \text{supp} f \subseteq [0,2^n] \) (see [16, Sect. 6.2]). Therefore, each function \( \varphi_j \) is constant on \( [k,k+1), \, k \in \mathbb{Z}_+ \), which implies \( \text{supp} \varphi_j \subseteq [0,1] \).

In view of Equation (7), there exists \( j_0 \in \mathbb{Z}_+ \) such that
\[ m_n^{(j)}(2^{-j}\omega) = 1 \text{ for all } j > j_0, \, \omega \in \mathbb{R}_+ \]

Hence, for \( \omega \in E \),
\[ \hat{\varphi}_{j}(\omega) = 2^{-j/2} \sum_{n=1}^E m_n^{(j)}(2^{-j}\omega). \]

It follows from Equation (6) that for some \( c_i > 0 \).
\[ |m_{ij}(\omega')| \geq c_{i} \quad \text{for} \quad j \in \mathbb{N}, \quad \omega \in E. \]

Since \[ c_{i}^{j-h} |\hat{\phi}_{j}(\omega)| \geq 2^{-j/2} \mathbf{1}_{E}(\omega), \quad \omega \in \mathbb{N}_{+}. \]

We have
\[ |\hat{\phi}_{j}^{(i)}(\omega)| \leq c_{i}^{j-h} |\hat{\phi}_{j}(\omega)|, \quad \omega \in \mathbb{N}_{+}. \]

or, taking into account Equation (3),
\[ |\hat{\phi}_{j}^{(i)}(\omega)| \leq c_{i}^{j-h} |\hat{\phi}_{j}(\omega)|, \quad \omega \in \mathbb{N}_{+}. \]

for \( s > j, \quad j \in \mathbb{N}_{+}. \)

Applying the dominated convergence theorem we obtain
\[
\left| \int_{\omega} \hat{\phi}_{j}(\omega) \int_{j}^{s} |\hat{\phi}_{j}^{(i)}(\omega)| w_{i} \left( 2^{-j} \omega \right) d\omega \right|
= \lim_{x \to 0} \int_{x}^{\omega} \hat{\phi}_{j}^{(i)}(\omega) \int_{j}^{s} w_{i} \left( 2^{-j} \omega \right) d\omega
= \delta_{0,k},
\]

which means that \( \{ \phi_{j,k} \}_{k \in \mathbb{N}_{+}} \) is an orthonormal system.

Now, let us prove an orthonormality of \( \{ \psi_{j,k} \}_{k \in \mathbb{N}_{+}} \).

For any \( k \in \mathbb{N}_{+} \) denote \( d_{ij}^{(j)} = (-1)^{i-k} c_{i}^{j} \). Then
\[
j_{,k}(x) = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{N}_{+}} d_{ij}^{(j)} \phi_{j+1,k}(x). \tag{10}\]

Since
\[
\psi \sum_{k \in \mathbb{N}_{+}} d_{ij}^{(j)} \mathbf{1}_{2^{i}\mathbb{N}_{+}} = 2\delta_{0,k},
\]

We have
\[
\{ \psi_{j,k}, \psi_{j,k}^{*} \} = \frac{1}{2} \sum_{k \in \mathbb{N}_{+}} d_{ij}^{(j)} d_{ij}^{(j)} \{ \phi_{j+1,k}, \phi_{j+1,k}^{*} \}
= \delta_{k,k}.
\]

Then from Equation (10)
\[
V_{j} \subset V_{j+1}, \quad W_{j} \subset V_{j+1}. \tag{11}\]

Let us define
\[
m_{ij}^{(j)}(\omega) = \frac{1}{2} \sum_{k \in \mathbb{N}_{+}} d_{ij}^{(j)} w_{k}(\omega).
\]

Denote \( \omega' = 2^{-j} \omega \). Under the unitarity of the matrices
\[
\begin{bmatrix}
m_{ij}^{(j)}(\omega') & m_{ij}^{(j)}(\omega' + 1/2) \\
m_{ij}^{(j)}(\omega') & m_{ij}^{(j)}(\omega' + 1/2)
\end{bmatrix},
\]

We can write
\[
\hat{\phi}_{j+1}(\omega) = \hat{\phi}_{j+1}(\omega) \\
= \sqrt{2} \sum_{k \in \mathbb{N}_{+}} d_{ij}^{(j)} \phi_{j+1,k}(\omega) \\
+ \sqrt{2} \sum_{k \in \mathbb{N}_{+}} d_{ij}^{(j)} \phi_{j+1,k}(\omega).
\]

Using the inverse Fourier-Walsh transform, we have
\[
\phi_{j+1}(x) = \sqrt{2} \sum_{k \in \mathbb{N}_{+}} \sqrt{2} \sum_{j \in \mathbb{N}_{+}} d_{ij}^{(j)} \phi_{j+1,k}(x) + d_{j}^{(j)} \phi_{j+1,k}(x)
\]
or,
\[
\phi_{j+1}(x) = \sqrt{2} \sum_{k \in \mathbb{N}_{+}} \sqrt{2} \sum_{j \in \mathbb{N}_{+}} d_{ij}^{(j)} \phi_{j+1,k}(x) + d_{j}^{(j)} \phi_{j+1,k}(x) 
\]

With Equation (11) it yields \( V_{j} \subset V_{j+1} \). To conclude the proof it remains to show that
\[
\bigcup_{j=0}^{\infty} V_{j} = L_{2}(\mathbb{N}_{+}). \tag{12}\]

Note, that by Equation (7) for any \( \omega \in \mathbb{N}_{+} \) there exist \( J \in \mathbb{N}_{+} \) such that \( \hat{\phi}_{j}(\omega) = 2^{-j/2} \) and, consequently,
\[
\bigcup_{j=0}^{\infty} \text{supp} \hat{\phi}_{j} = \mathbb{N}_{+}. \tag{13}\]

For any \( x \in \mathbb{N}_{+} \) the subspace \( \bigcup_{j=0}^{\infty} V_{j} \) is invariant with respect to the shift \( f(\cdot) \mapsto f(\cdot + x) \). Actually, an arbitrary \( x \in \mathbb{N}_{+} \) can be approximated by fractions \( 2^{-j} \), with arbitrary large \( j \). Besides, each \( V_{j} \) is invariant with respect to the shifts \( 2^{-i} \). By Equation (4) it is clear that \( V_{j} \subset V_{j+1} \).

Let \( f \in \bigcup_{j=0}^{\infty} V_{j} \). There exist \( j_{i} \) such that \( f \in V_{h} \) and hence \( f(\cdot + 2^{-j}) \in V_{j} \) for all \( j \geq j_{i} \). The continuity of \( \| f(\cdot + x) \| \) implies that \( f(\cdot + x) \in \bigcup_{j=0}^{\infty} V_{j} \). If \( g \in \bigcup_{j=0}^{\infty} V_{j} \), then approximating \( g \) with \( f \) from \( \bigcup_{j=0}^{\infty} V_{j} \) and using the invariance of a norm with respect to the shift, we obtain \( g(\cdot + x) \in \bigcup_{j=0}^{\infty} V_{j} \).
Denote by $\left( \bigcup_{j=0}^{\infty} V_j \right)^{\perp}$ the set of all $\hat{f}$ such that $f \in \bigcup_{j=0}^{\infty} V_j$. By the Weiner’s theorem we can write
\[
\left( \bigcup_{j=0}^{\infty} V_j \right)^{\perp} = L_2(\Omega) , \quad \text{for some measurable } \Omega \subset \mathbb{R}^+ . \]
It is clearly that $\bigcup_{j=0}^{\infty} \text{supp } \phi_j \subset \Omega$ and, in view of Equation (13), we have $\Omega = \mathbb{R}^+$. Hence, the Equation (12) holds. The theorem is proved.

3. Numerical Experiments

For any $N \in \mathbb{R}$, let $\Delta_j(N) := \left[0,(2N-1)2^{-j}\right]$, \quad $j \in \mathbb{R}$.

According to [12] an adapted multiresolution analysis (AMRA) of rank $N$ in $L_2(\mathbb{R})$ is a collection of closed subspaces $V_j \subset L_2(\mathbb{R})$, \quad $j \in \mathbb{R}$, which satisfies the following conditions:
1. $V_j \subset V_{j+1}$ for all \quad $j \in \mathbb{R}$;
2. $\bigcup_{j=0}^{\infty} V_j = L_2(\mathbb{R})$;
3. For every \quad $j \in \mathbb{R}$ there is a function $\phi_j$ in $L_2(\mathbb{R})$ with a finite support \quad $\Delta_j(N)$ such that $\{\phi_j(\cdot - k2^{-j}):k \in \mathbb{N}\}$ is an orthonormal basis of $V_j$;
4. For every \quad $j \in \mathbb{R}$ there exists a filter $c(j) = \{c_k(j)\}_{k=0}^{2N-1}$ such that
\[
\phi_{j-1}(x) = \sum_{k=0}^{2N-1} c_k(j) \phi_j(x - k2^{-j}) , \quad j \in \mathbb{R} . \tag{14}\]

The sequence $\{\phi_j\}$ from condition (4) is called a scaling sequence for given an AMRA. The corresponding wavelet sequence $\{\psi_j\}$ can be defined by
\[
\psi_{j-1}(x) = \sum_{k=0}^{2N-1} (-1)^k c_{2N-k-1}(j) \phi_j(x - k2^{-j}) . \tag{15}\]

Denote by $W_j$ the orthogonal complement of $V_j$ in $V_{j+1}$. It is known that, under some conditions, the system $\{\psi_j(\cdot - k2^{-j}):k \in \mathbb{N}\}$ is an orthonormal basis of $W_j$ (for more details, see, e.g., [14, Sect. 8.1]). Moreover, if $f \in L_2(\mathbb{R})$ is a function in the subset $A \subset L_2(\mathbb{R})$, then $f$ can be written as $f = f_{R} + f_{W}$, where $f_{R}$ is the restriction of $f$ in $A$, and $f_{W}$ is the orthogonal projection of $f$ in $W_j$. The expression for the size of $W_j$ is given by
\[
\|f\|^2 = \|f_{R}\|^2 + \sum_{j=0}^{\infty} \|f_{W_j}\|^2 ,
\]

and
\[
\|f_{W_j}\|^2 = \|f_{j-1}\|^2 + \|f_{W_{j-1}}\|^2 . \tag{16}\]

Let us denote
\[
h_k(j) = c_k(j)/\sqrt{2}
\]
and
\[
g_k(j) = (-1)^k h_k(j) .
\]

For a given array
\[
A(j) = \{a,j,0,a,j,1,\ldots,a,j,2^j-1\},
\]
the direct non-stationary discrete wavelet transform
\[
a_{j-1,k} = \sum_{l \in \mathbb{Z}} h_{l-2^j}(j)a_{j,l} , \quad d_{j-1,k} = \sum_{l \in \mathbb{Z}} g_{l-2^j}(j)a_{j,l} ,
\]
maps it into
\[
A(j-1) = \{a,j-1,0,a,j-1,1,\ldots,a,j-1,2^{j-1}-1\}
\]
and
\[
D(j-1) = \{a,j-1,0,a,j-1,1,\ldots,a,j-1,2^{j-1}-1\} .
\]

The inverse transform is defined as follows
\[
a_{j,l} = \sum_{k \in \mathbb{Z}} h_{l-2^j}(j)a_{j-1,k} + g_{l-2^j}(j)d_{j-1,k}
\]
and reconstructs $A(j)$ by $A(j-1)$ and $D(j-1)$. According to [12] in order to choose the filter $c(j)$ to maximize $\|f_{j-1}\|^2$ in Equation (16), we must solve the following problem.

Problem 1. Let $U_N^{(i)}$ be the subset of the $2N$-dimensional Euclidean space $\mathbb{R}^{2N}$, which consists of the points $u = (u_0,u_1,\ldots,u_{2N-1})$ satisfying the conditions
\[
\sum_{k=0}^{2N-1} u_k^2 = 1 , \quad \sum_{k=0}^{2N-1} u_k^2 = 0 . \tag{17}\]
for $i = 0,1,\ldots,N-1$. Find a point $u^*$ for which
\[
\sum_{m,k=0}^{2N-1} u_m^* u_k^* F_{m,k} = \sup_{u \in U_N^{(i)}} \left\{ \sum_{m,k=0}^{2N-1} u_m u_k F_{m,k} \right\} , \tag{18}\]
where $F_{m,k}$ is a $2N \times 2N$ symmetric matrix.

Problem 1 has a solution since $U_N$ is a compact. But, as noted in [12], the numerical solution of this problem is not trivial even for $N = 2$.

Concerning the standard Haar and Daubechies (with 4 coefficients) discrete transforms see, e.g., [17]; we will denote them as SWTH and SWTD, respectively. We write NSWH for the simplest case of a multiresolution analysis of rank 1 which is considered in [12, Sect. 3] (see also [13]). The nonstationary Daubechies discrete wavelet transform which corresponds an AMRA of rank $N$ are defined in [12] and we will use the symbol NSWTHD2 to denote this transform (see NSWTD1 and NSWTD2 in the tables below).

Method $A$ associated with one of the mentioned above discrete wavelet transforms (cf. [17, Chap. 7]) consists of the following steps:
Step 1. Apply the discrete wavelet transform \( j \) times to an input array \( A(j) \) and get the sequence
\[
A(0), D(0), D(1), \ldots, D(j-1).
\]

Step 2. Allocate a certain percentage of the wavelet coefficients with largest absolute value (we choose 10%) and nullify the remaining coefficients.

Step 3. Apply the inverse wavelet transform to the modified arrays of the wavelet coefficients.

Step 4. Calculate \( \|A(j) - A(j)\| \), where \( A(j) \) is a reconstructed array.

In Method B the second step is replaced on the uniform quantization and the forth step is replaced on the calculation of the entropy of a vector, obtained in the third step.

We recall that \( y = \{y_1, \ldots, y_n\} \) is a vector uniform quantization for given vector \( x = \{x_1, \ldots, x_n\} \), if
\[
y_j = \begin{cases} 
0, & |x_j| < \Delta, \\
\Delta \left( \frac{x_j}{\Delta} + \text{sign}(x_j) \frac{\Delta}{2} \right) |x_j| \geq \Delta,
\end{cases}
\]
where \( \Delta \) is the length of the quantization interval.

The value \( \Delta \) will be calculated by
\[
\Delta = \left( \max_{1 \leq j \leq n} x_j - \min_{1 \leq j \leq n} x_j \right) / 50.
\]

The Shannon entropy of \( x \) is defined by the formula
\[
H(x) = -\sum_{j=1}^{n} p_j \log_2(p_j),
\]
where \( p_j \) is frequency of the value \( x_j \).

Let us consider a similar approach associated with the following problem:

**Problem 2.** Let \( N = 2^{n+1} \). Denote by \( U(N) \) the set of all points \( u = (u_0, u_1, \ldots, u_{2^{n+1}-1}) \in \mathbb{R}^{2N} \) such that
\[
(u_i)^2 + (u_{i+N})^2 = 1, i = 1, 2, \ldots, N - 1.
\]

For every \( u \in U(N) \), we define
\[
c_k(u) = \frac{1}{N} \sum_{j=0}^{2N-1} u_j w_j(k/(2N))
\]
for \( k = 0, 1, \ldots, 2N-1 \). Find a point \( u^* \) for which
\[
\sum_{m,k=0}^{2N-1} c_m(u^*) c_k(u^*) F_{m,k} = \sup_{u \in U(N)} \left\{ \sum_{m,k=0}^{2N-1} c_m(u) c_k(u) F_{m,k} \right\}.
\]
Table 1. Values of the square error corresponding to Method A.

<table>
<thead>
<tr>
<th></th>
<th>SWTH</th>
<th>NSWTH</th>
<th>NSWTL1</th>
<th>SWTD</th>
<th>NSWTD1</th>
<th>NSWTD2</th>
<th>NSWTL2</th>
</tr>
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<tbody>
<tr>
<td>( s )</td>
<td>0.166547</td>
<td>0.123983</td>
<td>0.123980</td>
<td>0.248311</td>
<td>0.167071</td>
<td>0.128120</td>
<td>0.122886</td>
</tr>
<tr>
<td>( W_{n+5} )</td>
<td>16.813738</td>
<td>15.932313</td>
<td>15.932307</td>
<td>15.378600</td>
<td>15.171461</td>
<td>14.782221</td>
<td>15.130797</td>
</tr>
<tr>
<td>( W_{n+7} )</td>
<td>15.887306</td>
<td>13.631379</td>
<td>13.631383</td>
<td>15.595433</td>
<td>16.649683</td>
<td>12.724437</td>
<td>12.674001</td>
</tr>
</tbody>
</table>

Table 2. Values of the entropy obtained by Method B.

<table>
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<tr>
<th></th>
<th>SWTH</th>
<th>NSWTH</th>
<th>NSWTL1</th>
<th>SWTD</th>
<th>NSWTD1</th>
<th>NSWTD2</th>
<th>NSWTL2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s )</td>
<td>0.320865</td>
<td>0.327626</td>
<td>0.310639</td>
<td>0.863949</td>
<td>0.299818</td>
<td>0.304861</td>
<td>0.241210</td>
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<tr>
<td>( W_{n+3} )</td>
<td>4.486757</td>
<td>3.810555</td>
<td>3.772764</td>
<td>4.152313</td>
<td>3.822598</td>
<td>3.525294</td>
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<tr>
<td>( W_{n+5} )</td>
<td>4.688737</td>
<td>3.874187</td>
<td>3.848227</td>
<td>4.224801</td>
<td>4.106692</td>
<td>3.766994</td>
<td>3.707672</td>
</tr>
<tr>
<td>( W_{n+7} )</td>
<td>4.392570</td>
<td>3.371864</td>
<td>3.344916</td>
<td>4.001358</td>
<td>4.435942</td>
<td>3.232151</td>
<td>3.197167</td>
</tr>
</tbody>
</table>
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denote these discrete transforms as NSWTL1 if \( N = 1 \)
and as NSWTL2 if \( N = 2 \).

Let us recall that the Weierstrass function is defined as
\[
W_{\alpha, \beta}(x) = \sum_{n=1}^{\infty} a^n \cos(\beta^n \pi x), \quad 0 < \alpha < 1, \beta \geq \frac{1}{\alpha},
\]
and the Swartz function is defined as
\[
S(x) = \sum_{n=-\infty}^{\infty} \frac{h(2^n x)}{4^n},
\]
where \( h(x) = [x] - \sqrt{x} [x] \). We will consider arrays \( A(8) \) with elements \( a_{nk} = W_{1/2}(k/128) \) or \( a_{nk} = S(k/256), \quad k = 0, \ldots, 255 \). Then we use the Matlab function fminsearch to solve the optimization problems in Equations (18) and (19). The results of these numerical experiments are presented in Tables 1 and 2. We see that in several cases the introduced nonstationary dyadic wavelets have an advantage over the classical Haar and Daubechies wavelets.

REFERENCES


