The Conditions for the Convergence of Power Scaled Matrices and Applications

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Abstract

For an invertible diagonal matrix \( D \), the convergence of the power scaled matrix sequence \( D^{-N} A_N \) is investigated. As a special case, necessary and sufficient conditions are given for the convergence of \( D^{-N} T N \), where \( T \) is triangular. These conditions involve both the spectrum as well as the digraph of the matrix \( T \). The results are then used to provide a new proof for the convergence of subspace iteration.

Keywords: Convergence, Iterative Method, Triangular Matrix, Gram-Schmidt

1. Introduction

The aim of iterative methods both in theory as well as in numerical settings, is to produce a sequence of matrices \( A_0, A_1, \cdots \), that converges to hopefully, something useful. When this sequence diverges, the natural question is how to produce a new converging sequence from this data. One of these convergence producing methods is to diagonally scale the numbers \( A_0 \) and form the sequence \( \{D_0 A_0\} \). Examples of this are numerous, such as the Krylov sequence (\( x, \ A x, \ A^2 x, \cdots \)), which when divergent can be suitably scaled to yield a dominant eigenvector.

The convergence of power scaled iterative methods and more general power scaled Cesaro sums were studied by Chen and Hartwig [4,6]. In this paper, we continue our investigation of this iteration and derive a formula for the powers of an upper triangular matrix, and use this to investigate the convergence of the sequence \( \{D_0^{-N} T^N\} \).

We also investigate the subspace iterations, which has been started by numerous authors [1,3,10,11,15], and turn our attention to the case of repeated eigenvalues.

The main contributions of this paper are:

- We present the necessary and sufficient conditions for convergence of power scaled triangular matrices with the explicit expression for the G-S factors of \( D^{-N} T^N \) [3] and present a new proof of the convergence of simultaneous iteration for the case where the eigenvalues of the matrix \( A \) satisfy

\[
| \lambda_i \geq | \lambda_j | \geq \cdots \geq | \lambda_{n-1} | \geq | \lambda_n |
\]

\[
| \lambda_i | = | \lambda_j | \Rightarrow \lambda_i = \lambda_j.
\]

Because of the explicit expression for the GS factors, and the exact convergence results, our discussion is more precise than that given previously [12,17].

One of the needed steps in our investigation is the derivation a formula for the powers of a triangular matrix \( T \), which in turn will allow us to analyze the convergence of \( D_0^{-N} T^N \).

Throughout this note all our matrices will be complex and, as always, we shall use \( \| \| \) and \( \rho(\cdot) \) to denote the Euclidean norm and spectral radius of \( \cdot \).

This paper is arranged as follows. As a preliminary result, a formula for the power of an upper triangular matrix is presented in Section 2. It is shown in Section 3 that the convergence of \( D_0^{-N} T^N \) is closely related to the digraph induced by \( T \). Section 4 is the main section in which convergence of general power scaled sequence \( D^{-N} A_N \) is investigated and this, combined with path conditions in Section 3, is then used to discuss the convergence of \( D^{-N} T^N \). As an application we analyze the convergence results for subspace iterations, in which the eigenvalues are repeated, but satisfy a peripheral constraint.
2. Preliminary Results

We first need a couple of preliminary results.

Lemma 2.1. If $\rho(A) < 1$ and $0 < \epsilon < 1$, then

$$
\sum_{k=0}^{N} A^k \epsilon_k
$$

converges.

Proof. For $f(z) = \sum_{k=0}^{\infty} c_k z^k$, we have

$$
|f(z)| = |\sum_{k=0}^{\infty} c_k z^k| \leq \sum_{k=0}^{\infty} |z|^k.
$$

As the geometric summation on the right-hand side has radius of convergence 1, $f(z)$ converges for all $z$ such that $|z|<1$, which in turn tells us that the radius of convergence of $f(z)$ is no less than 1. Therefore, from Theorem 6.2.8. of [8], $f(A)$ converges.

Next consider the triangular matrix

$$
U = \begin{bmatrix}
\mu_1 & u_{12} & \cdots & u_{1n} \\
0 & \mu_2 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \mu_n
\end{bmatrix},
$$

which is used in the following characterization of the powers of a triangular matrix.

Lemma 2.2. Let $T = \begin{bmatrix} \lambda & a^T & \beta \\ 0 & U & c \\ 0 & 0 & \nu \end{bmatrix}$ where $a$ and $c$ are column vectors and suppose that

$$
T^N = \begin{bmatrix} \lambda^N & a^T \beta_N \\ 0 & U^N & c_N \\ 0 & 0 & \nu^N \end{bmatrix},
$$

then

$$
\beta_N = \beta \left( \sum_{k=0}^{N-1} \lambda^N a^T \nu^k \right)
$$

in particular,

1) if $\nu = 0$, then

$$
\beta_N = \beta \nu^{N-1} \left( \sum_{k=0}^{N-2} (a^T U^k c) \nu^{N-k-2} \right),
$$

2) if $\nu = 0$, then

$$
\beta_N = \beta \lambda^N \left( \sum_{k=0}^{N-2} (a^T U^k c) \lambda^{N-k-2} \right),
$$

3) if $\lambda \neq 0$ and $\lambda \neq \nu$, then

$$
\beta_N = \lambda^N \beta \left( \frac{1-(\nu / \lambda)^N}{\lambda-\nu} \right) + \lambda^{N-1} \sum_{k=0}^{N-2} (a^T U^k c) \frac{1-(\nu / \lambda)^{N-k-1}}{\lambda-\nu},
$$

4) if $\lambda \neq 0$ and $\lambda = \nu$, then

$$
\beta_N = N \beta \lambda^{N-1} + \lambda^{N-2} \sum_{k=0}^{N-2} (a^T U^k c) c(N-k-1)
$$

5) if $\lambda = \nu = 0$, then

$$
\beta_N = a^T U^{N-2} c.
$$

Proof. It is easily verified by induction that $T^N = \begin{bmatrix} T_N^0 & y_N \\ O & \nu^N \end{bmatrix}$, where

$$
T_N^0 = \begin{bmatrix} \lambda & a^T \nu \nu^N \end{bmatrix},
$$

and

$$
y_N = \begin{bmatrix} c_N \\ \sum_{k=0}^{N-1} T_{N-k-1} \beta_c \nu^k \end{bmatrix}. \tag{11}
$$

Now

$$
y_N = \sum_{k=0}^{N-1} \lambda^{N-k-1} \sum_{j=0}^{N-1} \lambda^{N-k-j-1} a^T U^j c \nu^k
$$

Hence

$$
\beta_N = \beta \left( \sum_{k=0}^{N-1} \lambda^{N-k-1} \nu^k \right) + \sum_{j=0}^{N-2} (a^T U^j c) \lambda^{N-j-2} \lambda^{N-j-2} \nu^k
$$

completing the proof of (4). The special cases (1) - (5) are easy consequences of (4).

Let us now illustrate how the power of $T$ is related to its digraph.
3. The Digraph of $T$

Suppose $T = \begin{bmatrix} \lambda & a^T \\ O & U & c \end{bmatrix}$ is an $(n+2) \times (n+2)$ upper triangular matrix. Correspondingly we select $n+2$ nodes $S_0, S_1, \ldots, S_n, S_{n+1}$, and consider the assignment

$$
\begin{bmatrix}
S_0 & S_1 & \cdots & S_n & S_{n+1} \\
S_0 & \lambda & a^T & \beta \\
S_1 & \mu_1 & u_{12} & \cdots & u_{1n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
S_n & 0 & \cdots & 0 & \mu_n \\
S_{n+1} & 0 & \cdots & O & c
\end{bmatrix}
$$

with $a = [a_1, a_2, \ldots, a_n]^T$ and $c = [c_1, c_2, \ldots, c_n]^T$.

We next introduce the digraph induced by $T$, i.e. $G = (V, E)$ where $V = \{S_0, S_1, \ldots, S_{n+1}\}$ is the vertex set and $E = \{(S_i, S_j) \mid t_{ij} \neq 0\}$ is the edge set. As usual we say $(S_i, S_j) \in E$ if and only if $t_{ij} \neq 0$. A path from $S_i$ to $S_j$ in $G$ is a sequence of vertices $S_i = S_k, S_2, \ldots, S_l = S_j$ with $(S_k, S_{k+1}) \in E$, for $i = 1, \ldots, l-1$, for some $l$. If there is a path from $S_i$ to $S_j$, we say that $S_i$ has access to $S_j$ and $S_k$ can be reached from $S_j$. We write

- $S_i \rightarrow S_j$ if $(S_i, S_j) \in E$,
- $S_i \rightarrow S_j$ if there is a path from $S_i$ to $S_j$,
- $S_i \leftrightarrow S_j$ if $S_i \rightarrow S_j$ and $S_j \rightarrow S_i$.

Let $\pi = S_0, S_{n+1} = S_0, S_1, \ldots, S_{n+1}$ be the sandwich set of $S_0$ and $S_{n+1}$, i.e., $\{S_0, S_1, \ldots, S_{n+1}\}$ is the set of all the nodes from $\{S_0, \ldots, S_n\}$ such that $S_0 \rightarrow S_n \rightarrow S_{n+1}$, i.e., $S_n$ can be reached from $S_0$ and has access to $S_{n+1}$. Let us now introduce the notation

- $a = [a_1, a_2, \ldots, a_n]^T$,
- $U = \{\mu_1, \mu_2, \ldots, \mu_n\}$,
- $c = [c_1, c_2, \ldots, c_n]^T$.

Then we have the following result.

**Lemma 3.1.** $a^T U c = a^T \hat{U} c$.

**Proof.** If $a_i u_j c_j \neq 0$, then $(S_i, S_j), (S_j, S_i)$, $(S_j, S_{n+1}) \in E$, thus $S_i, S_j \in \pi$, which implies that

$$
a^T U c = \sum_{i=1}^{n} a_i u_j c_j = \sum_{i=1}^{n} a_i u_i c_i.
$$

This completes the proof. □

This following corollaries are the direct consequences of the above lemma.

**Corollary 3.2.** If $S_0 \rightarrow S_{n+1}$ and there is no intermediate node that lies in $\{S_1, \ldots, S_n\}$ on any path from $S_0$ to $S_{n+1}$, then

1) $S_0 \rightarrow S_{n+1}$, i.e. $\beta \neq 0$,

2) $a^T U c = a^T \hat{U} c = 0$.

**Corollary 3.3.** $a^T U c = a^T \hat{U} c$, for $i = 1, 2, \ldots$.

We now turn to the main theorem of this section.

**Theorem 3.4.** Let $T = \begin{bmatrix} \lambda_1 & \beta_1 \\ \vdots & \ddots & \vdots \\
0 & \cdots & \lambda_{n+1} \end{bmatrix}$ be nonsingular and $D_T = \text{diag}(T) = \text{diag}(\lambda_1, \ldots, \lambda_{n+1})$. Then, the following statement are equivalent

1) $D_T^{-N} T^N$ converges.

2) if $S_i \rightarrow S_j$, then $|\lambda_i | \leq |\lambda_j |$, i.e. if there is a path from $S_i$ to $S_j$, then $|\lambda_i | < |\lambda_j |$.

**Proof.** We prove the theorem by induction on $n$. For $n = 2$,

$$
T = \begin{bmatrix} \lambda_1 & \beta_1 \\ 0 & \lambda_2 \end{bmatrix}
$$

and $M^{(2)}_T = D_T^{-N} T^N = \begin{bmatrix} \beta_1 / \lambda_1 & 0 \\ 0 & 1 \end{bmatrix}$

where

$$
\beta_1 = \left[ \prod_{i=1}^{n} (\lambda_i / \lambda_1)^{N} \right] / (\lambda_i / \lambda_1 - 1) \text{ and } \lambda_i / \lambda_1 \cdot N / \lambda_1
\lambda_1 = \left[ \prod_{i=1}^{n} (\lambda_i / \lambda_1)^{N} \right] / (\lambda_i / \lambda_1 - 1) \text{ and } \lambda_i / \lambda_1 \cdot N / \lambda_1
\lambda_1 = \left[ \prod_{i=1}^{n} (\lambda_i / \lambda_1)^{N} \right] / (\lambda_i / \lambda_1 - 1) \text{ and } \lambda_i / \lambda_1 \cdot N / \lambda_1
$$

It is easily seen that the convergence of $M^{(2)}$ implies that of $\beta_1$. Hence if $\lambda_i = \lambda_j$, then $\beta = 0$. Conversely, if $\beta \neq 0$, then $|\lambda_i / \lambda_j | < 1$ which implies that $M^{(2)}$ converges.

Next, assume that the result holds for all triangular matrices of size $n+1$ or less. Let $T$ be defined as in (12) and set $D_T = \text{diag}(\lambda_1, \ldots, \lambda_{n+1}) = \text{diag}(\lambda, \Delta, \nu)$ which is nonsingular. Consider the vertex set $V = \{S_0, \ldots, S_{n+1}\}$ and the assignment

$$
M^{(n+1)}_T = \begin{bmatrix} S_0 & S_1 & \cdots & S_n & S_{n+1} \\
S_0 & 1 & a_2 / \lambda_1 & \cdots & a_n / \lambda_1 \\
S_1 & \Delta_2 U & \Delta_3 U & \cdots & \Delta_{n+1} U \\
S_n & 0 & \cdots & O & 1
\end{bmatrix}
$$

and $M^{(n+1)}_T$ converges. Then by induction, both $\begin{bmatrix} \lambda & a^T \\ 0 & U & c \end{bmatrix}$ and $\begin{bmatrix} \lambda & a^T \\ 0 & U & c \end{bmatrix}$ obey the theorem. Suppose $S_i \rightarrow S_j$ in $V$. If $|i-j| < n+1$ we are done since then both endpoints lie in $\{S_0, \ldots, S_n\}$ or $\{S_0, \ldots, S_{n+1}\}$. So we only need to consider the case where $S_i = S_0$ and $S_j = S_{n+1}$, i.e. $S_0 \rightarrow S_{n+1}$.

**Subcase (a):** There is an intermediate node from...
\[ \{S_1, \ldots, S_n\}, \text{ say } S_0 \rightarrow S_1 \rightarrow S_{n+1}, \quad (1 \leq p \leq n). \]

Then by the induction hypothesis \( |\lambda| > |\mu_p| > |\nu| \), and we are done.

**Subcase (b):** There is no intermediate node between \( S_0 \) and \( S_{n+1} \). In this case \( S_0 \rightarrow S_{n+1} \), and by Corollary 3.2., \( \beta \neq 0 \) and \( a^T U^c c = a^T U^c c = 0 \) for arbitrary \( i \).

Since the sandwich set \( \pi \) is empty, we see from Lemma 2.2.,

\[ \lambda^{-N} \beta_N = \begin{cases} \frac{1}{\beta} \left( 1 - \frac{v}{\lambda} \right)^{N+1} / (\lambda - v) & \text{if } \lambda \neq v \\ \beta N / \lambda & \text{if } \lambda = v \end{cases} \]  

(14)

Now because we are given that \( \lambda^{-N} \beta_N \) converges and \( \beta \neq 0 \), we must have \( |v| < 1 \).

Conversely, assume that \( S_0 \rightarrow S_1 \Rightarrow |\lambda_j| > |\lambda_i| \) and assume that the hypothesis holds for matrices of size \( n+1 \) or less. Since the graph condition also hold for \( \{S_1, \ldots, S_n\} \) and \( \{S_1, \ldots, S_{n+1}\} \), it follows by the hypothesis that all the entries in \( \mathbf{M}^{(N+1)} \) converges, with the possible exception of \( \beta_N \lambda^{-N} \). Consequently, all we have to show is that \( \lambda^{-N} \beta_N \) also converges, given the path conditions. Consider

\[ \lambda^{-N} \beta_N = \begin{cases} \frac{1}{\lambda - v} \sum_{i=0}^{N-1} \left( \frac{U^c}{\lambda} \right)^i \left( 1 - \frac{v}{\lambda} \right)^{N-i-1} & \text{if } \lambda \neq v, \beta \neq 0 \\ \beta N / \lambda & \text{if } \lambda = v \end{cases} \]  

(15)

If \( S_0 \rightarrow S_{n+1} \), then \( S_0 \rightarrow S_{n+1} \) and therefore \( \beta = 0 \). Moreover, \( \pi \) is empty and the right hand side of (15) is zero, i.e. \( \lambda^{-N} \beta_N = 0 \) and we are done. Suppose \( S_0 \rightarrow S_{n+1} \) and thus \( |\lambda| > |\nu| \). In this case

\[ \frac{1}{\lambda - v} \sum_{i=0}^{N-1} \left( \frac{U^c}{\lambda} \right)^i \left( 1 - \frac{v}{\lambda} \right)^{N-i-1} \]

converges (possibly to 0 when \( \beta = 0 \)).

Now if \( \pi = \emptyset \) then the second term of (15) vanishes by Lemma 2.2. Lastly suppose \( \pi \neq \emptyset \), i.e. there are intermediate nodes \( S_1, \ldots, S_n \). From Lemma 2.2., we recall that \( a^T U^c c = a^T U^c c \), where

\[ \hat{U} = \begin{bmatrix} S_{p_1} \\ \vdots \\ S_{p_j} \\ O \end{bmatrix} \]

(16)

Since for each \( i \), \( S_0 \rightarrow S_1 \rightarrow S_{n+1} \), we know that \( |\lambda| > |\mu_p| > |\nu| \) and thus \( |\lambda| > |\rho(\hat{U})| \). Hence \( \rho (\hat{U} / \lambda) < 1 \) which implies that

\[ a^T \sum_{i=0}^{N-2} \left( \frac{\hat{U}}{\lambda} \right)^i \rightarrow a^T \left( I - \frac{\hat{U}}{\lambda} \right)^{-1} c. \]

To complete the proof we observe that

\[ \sum_{i=0}^{N-1} \left( \frac{\hat{U}}{\lambda} \right)^i \]

also converges because of Lemma 2.1. with \( A = \hat{U} / \lambda \) and \( \beta_i = (v/\lambda)^{N-i} \).

We at once have, as seen in [3].

**Corollary 3.5.** Let \( T \) be an upper triangular matrix and \( D = \text{diag}(T) = \text{diag}(\lambda_1, \ldots, \lambda_n) \). If

\[ |\lambda_i| > |\lambda_j| > \cdots > |\lambda_n|, \]

then \( D^{-N} T^{-N} \) converges to an upper triangular matrix of diagonal 1.

We now turn to the main result in this paper. Our aim is to characterize the convergence of \( D^{-N} A_N \) in terms of the GS factorization of \( A_N \).

**4. Main Theorem**

Let us denote the set of increasing sequences of \( p \) elements taken from \( \{1, 2, \ldots, m\} \) by

\[ Q_{p,n} = \{(i_1, \ldots, i_p) | 1 \leq i_1 < \cdots < i_p \leq m\} \]

and assume this set \( Q_{p,n} \) is ordered lexicographically. Suppose \( (s,t) := (s, s+1, \ldots, t) \) is a subsequence of \( \{1, 2, \ldots, m\} \) and we define

\[ Q_{p,n}(s,t) = \{U = (u_1, \ldots, u_p) | s < u_1 < \cdots < u_p < t\}. \]

Clearly, \( Q_{p,n}(1,m) \).

Suppose \( B \) is an \( m \times n \) matrix of rank \( r \). The determinant of a \( p \times p \) submatrix of \( A \) (\( 1 \leq p \leq \min(m,n) \)), obtained from \( A \) by striking out \( m-p \) rows and \( n-p \) columns, is called a minor of order \( p \) of \( A \). If the rows and columns retained are given by subscripts (see Householder [9]) \( I = (i_1, \ldots, i_p) \in Q_{p,n} \), and \( J = (j_1, \ldots, j_p) \in Q_{p,n} \), respectively, then the corresponding \( p \times p \) submatrix and minor are respectively denoted by \( A_{I,J} \) and \( \text{det}(A_{I,J}) \).

The minors for which \( I = J \) are called the principal minors of \( A \) of order \( p \), and the minors with \( I = J = (1, 2, \ldots, p) \) are referred to as the leading principal minors of \( A \).

Let \( I = (i_1, \ldots, i_p) \in Q_{p,n} \) and \( J = (j_1, \ldots, j_p) \in Q_{p,n} \). For convenience, we denote by \( I[i_p] \in Q_{p-1,n} \) the sequences of \( p-1 \) elements obtained by striking out the \( kth \) element \( i_k \); while \( I(j) \) denotes the sequences of \( p+1 \) elements obtained by adding a new element \( j \) after \( i_p \), i.e.\( I(j) = (i_1, \ldots, i_p, j) \). Note that if \( i_p > j \), then \( I(j) \) is not an element of \( Q_{p+1,n} \) because it is no longer an increasing sequence. If \( p + q \leq m \), we denote the concatenation \((i_1, \ldots, i_p, j, \ldots, j) \).
... of I and J by IJ. It has p + q elements. Again, IJ may not be an element of Qp,q,n.

Since the natural sequence (1, 2, ..., p) of p elements will be used frequently, we particularly denote this sequence by (p) = (1, 2, ..., p); while \( (p) \) and \( (p) \) are simply denoted by \( (p) \).

Next recall [2] that the volume \( Vol(B) \) of a real matrix \( B \), is defined as the product of all the nonzero singular values of \( B \). It is known [2] that

\[ Vol(B) = \sqrt{\sum |\det(B_j^T)|^2} \]

where \( B_j^T \) are all \( r \times r \) submatrices of \( B \). In particular, if \( B \) has full column rank, then

\[ Vol(B) = \sqrt{|\det(B'B)|} \]  \hspace{1cm} (17)

Lastly, suppose \( A = [a_1, a_2, ..., a_r] \) is an \( n \times r \) matrix of full column rank and

\[ A = YG \]  \hspace{1cm} (19)

is its GS factorization so that the columns of \( Y = [y_1, y_2, ..., y_r] \) are orthogonal and \( G \) is \( r \times r \) upper triangular matrix of diagonal 1. For \( k \leq r \), we define \( A_k = [a_1, ..., a_k] \) and

\[ V_k = Vol(A_k) \].  \hspace{1cm} (20)

It follows directly that

\[ V_k = \frac{\sum_{i=1}^{k} |\det(A_{ik})|^2}{\sqrt{\det(A_k'A_k)}} \]  \hspace{1cm} (21)

Theorem 4.1. Let \( A \) be an \( n \times r \) matrix of rank \( r \) and let \( A = YG \) be its GS factorization. Then

\[ y_{ij} = \sum_{l=0}^{n} \det(A_{jl}) \cdot |\det(A_{l+1})|/V_{l+1} \]  \hspace{1cm} (22)

and

\[ g_{ik} = \frac{|\det(A'A)(j,j-k)|}{V_j^2} \]  \hspace{1cm} (23)

Proof. The result of (22) follows from Theorem 2.1. in [3], while on account of Corollary 2.1. in [3], \( G = (Y^*Y)^{-1}Y^*A \). Hence we arrive at

\[ g_{ik} = \frac{V_j^2}{V_j^2} y_{ik} \sum_{l=0}^{n} a_{lk} = \frac{V_j^2}{V_j^2} \sum_{l=0}^{n} y_{lk}d_{ln} \]

\[ = \sum_{l=1}^{n} (-1)^{i+l} a_{lk} \det(A'A)(j,l)/V_j^2 \]

Because \( \sum_{l=1}^{n} a_{lk} \) is just the \( (t, k) \) element of matrix \( A'A \), we see that

\[ g_{ik} = \sum_{l=1}^{n} (-1)^{i+l} \sum_{l=1}^{n} a_{lk} \det(A'A)(j,l)/V_j^2 \],

which is the Laplace expansion along column \( j \) of \( \det(A'A)(j,j-k) \). Thus

\[ g_{ik} = \frac{|\det(A'A)(j,j-k)|}{V_j^2} \]

completing the proof.

Remark: A different proof of (23) was given in [9, § 1.4].

For a diagonal matrix \( D = diag(d_1, ..., d_n) \), we say that \( D \) is decreasing if

\[ |d_i| \geq |d_j| \]

Moreover, \( D \) is called locally primitive, if it is decreasing and

\[ d_i = d_j \]

It is obvious that we can partition a decreasing matrix \( D \) as

\[ D = diag(D^{(i)}, ..., D^{(i)}) \]  \hspace{1cm} (26)

where each \( D^{(i)} = \delta_i diag(e^{d_1}, ..., e^{d_n}) \) with \( \delta_i > 0 \). As a special case, if \( D \) is locally primitive, then \( D \) can be written as

\[ D = diag(\delta_11_{n}, ..., \delta_{n}1_{n}) \]

Now let us define \( q_{ij} = \sum_{s=1}^{n} q_{ij} \) and \( \Omega_n = \{q_{ij} : q_{ij} = \sum_{s=1}^{n} q_{ij} \} \). Suppose \( A_N = [a_{ij}]_{n \times r} \) is a sequence of \( n \times r \) matrices and let

\[ A_N = Y_N G_N = [y_{ij}^{(N)}]_{n \times r}, [g_{ij}^{(N)}]_{n \times r} \]  \hspace{1cm} (28)

be their GS factorization. Suppose \( B \) is a \( n \times r \) matrix, we can partition \( B \) conformally as \( D \) in (26). It is easily verified that the \( (u, v) \) element of \( (i, j) \)

the block \( B_{ij} \) of \( B \) is exactly the \( (q_{i+u}, q_{j+v}) \) element of the whole matrix \( B \). \( B \) is said to satisfy condition \( (\beta) \) if for each \( k \) there exists \( \Omega_n = \{q_{ij} : q_{ij} \} \) such that

\[ det B_{k}^{(N)} \neq 0 \]

We now have the following theorem.

Theorem 4.2. Let \( A_N \) be a sequence of \( n \times r \) matrices of full column rank with GS factor \( A_N = Y_N G_N \). Also suppose \( D \) is a diagonal matrix and \( D \) is a \( r \times r \)

leading submatrix of \( D \). Then

1) \( D^{-1}A_N \) converges to \( \widetilde{B} \) which satisfies condition \( (\beta) \) \( \Rightarrow \) \( G_N \) converges and \( D^{-1}Y_N \)

converges to \( Z \) which satisfies condition \( (\beta) \)

2) If in addition \( D \) is decreasing, i.e. \( D \) satisfies (26), then for \( k = q_{i+u} \) there exists \( \Omega_n = \{q_{ij} : q_{ij} \} \) such that

\[ \frac{y_{ij}^{(N)}}{d_i} = Q \left[ \frac{\delta_i^{(N)}}{d_i} \right] \]  \hspace{1cm} (29)

Proof: 1) The sufficiency is obvious. So let us turn to the necessary part. For \( D = diag(d_1, ..., d_n) \), there exists a permutation \( Q \) such that \( D = Q' DQ \) is decreasing. Meanwhile, by hypothesis and the fact that \( D^{-1}A_N = \widetilde{Q} \)
\((Q \mathcal{D}^{-N}Q)(\tilde{Q} \mathcal{A}_y) = \mathcal{D}^{-N} \hat{\mathcal{A}}_y\) with \(\hat{\mathcal{A}}_y = \tilde{Q} \mathcal{A}_y\), it follows that \(\mathcal{D}^{-N} \hat{\mathcal{A}}_y\) converges. So without loss of generality, we assume that \(D = \text{decreasing and partition} \ D \) as (26) and simply consider \(\mathcal{D}^{-N} \mathcal{A}_y\). We shall now, without risk of confusion, abbreviate the set \(Q_{x}(q_{i-1} : q_i) = \{\Omega_x = (\alpha_1, \ldots, \alpha_k) \mid q_{i-1} < \alpha_1 < \ldots < \alpha_k < q_i\}\) to \(Q_x\) and for \(I = (i_1, \ldots, i_l)\) set \(\pi_x = d_{i_1} \ldots d_{i_l}\). It at once follows that

\[
|\pi_x| \leq |\pi_x|.
\]  

(30)

We now have from (23)

\[
\sum_{\Omega_x \in Q_x} \det(A_{x, \mathcal{A}}(\mathcal{D}^{-1})(\mathcal{D}^{-1})) / V_k^2
\]

\[
= \sum_{\Omega_x \in Q_x} \left| \det((A_{x, \mathcal{A}}(\mathcal{D}^{-1}))' \right|^2
\]

(from Cauchy-Binet)

\[
= \sum_{\Omega_x \in Q_x} \left( \sum_{\Omega_y \in Q_y} \det(A_{x, \mathcal{A}}(\mathcal{D}^{-1}))' \right)^2
\]

\[
= \sum_{\Omega_x \in Q_x} \left( \sum_{\Omega_y \in Q_y} \det(A_{x, \mathcal{A}}(\mathcal{D}^{-1}))' \right)^2
\]

On account of (30), this is equal to

\[
\sum_{\Omega_x \in Q_x} \frac{\det(A_{x, \mathcal{A}}(\mathcal{D}^{-1}))'}{(\pi_{x, \mathcal{A}}(\delta_1)^N) + o(\delta_1)^N}
\]

\[
\sum_{\Omega_x \in Q_x} \left| \det((A_{x, \mathcal{A}}(\mathcal{D}^{-1}))' \right|^2
\]

(31)

Since \(\mathcal{D}^{-N} \mathcal{A}_y\) converges, so does the submatrices \((\mathcal{D}^{-N} \mathcal{A}_y)_{(q_{i-1} : q_i)^N}\) and their determinant and hence

\[
\det(A_{x, \mathcal{A}}(\mathcal{D}^{-1}))' = \det((D_{(q_{i-1} : q_i)^N})^{-1} (A_{x, \mathcal{A}}(\mathcal{D}^{-1}))')
\]

\[
= \det(D^{-N} \mathcal{A}_y)_{(q_{i-1} : q_i)^N}
\]

converges, say, to \(\tilde{A}_{x, \mathcal{A}}(\mathcal{D}^{-1})\). We have that consequently (31) converges to

\[
\sum_{\Omega_x \in Q_x} \left( \sum_{\Omega_y \in Q_y} \det((A_{x, \mathcal{A}}(\mathcal{D}^{-1}))' \right)^2
\]

in which the denominator is nonzero as \(\tilde{A}\) satisfies condition \((\beta)\). Hence \(G_\mathcal{A}\) converges and this implies that \(D^{-N} Y_\mathcal{A} = \mathcal{D}^{-N} \mathcal{A}_y G_\mathcal{A}\) also converges.

2) Lastly, what remains is to show that \(Y_\mathcal{A} D^{-N}\) converges if \(D\) is decreasing, i.e. \(D\) satisfies (26). Now for \(k = q_{i-1} + u, \ i = q_{j-1} + v \ (i \leq j - 1)\), it follows that

\[
y_{k-1}^{(\mathcal{A})} = \frac{1}{d_k^{\mathcal{A}}}
\]

\[
\sum_{\Omega_x \in Q_x} \frac{\det(A_{x, \mathcal{A}}(\mathcal{D}^{-1}))'}{(\pi_{x, \mathcal{A}})^N} + o(\pi_{x, \mathcal{A}})^N)
\]

\[
\sum_{\Omega_x \in Q_x} \left( \sum_{\Omega_y \in Q_y} \det((A_{x, \mathcal{A}}(\mathcal{D}^{-1}))' \right)^2
\]

\[
\sum_{\Omega_x \in Q_x} \frac{\det(A_{x, \mathcal{A}}(\mathcal{D}^{-1}))'}{(\pi_{x, \mathcal{A}})^N} + o(\pi_{x, \mathcal{A}})^N)
\]
This completes the proof of 2).

As a consequence of the above theorem we have

**Corollary 4.3.** Suppose $D$ is decreasing and $A_j$'s have orthogonal columns. If $D^{-N} A_N$ converges to $\bar{B}$ which satisfies condition ($\beta$), then for $k = q_{j-1} + u$ and $l = q_{j-1} + v$ ($i \leq j - 1$)

$$
\frac{A_u^{(N)}}{d_i^{(N)}} = O \left( \frac{\delta_j^{(N)}}{\delta_i} \right)
$$

**Proof.** In this case the GS factorization of $A_N$ are $A_N = A_N I_r$. So the result is the direct consequence of Theorem 4.2.

**Lemma 4.4.** Suppose $D$ is decreasing and $A_j$'s are of full column rank. If $B_N = [B_{kq}] = D^{-N} A_N$ converges, say, to $\bar{B}$, then $A_N D_r^{-N}$ converges if and only if

1) $B_{q_{j-1} + u, q_{j-1} + v} \neq 0 \Rightarrow \theta_u^{(N)} = \theta_v^{(N)}$ ($j = 1, \ldots, t$)

2) $i < j$, then

$$
\left( \frac{\delta_i}{\delta_j} \right)^N \left[ B_{q_{j-1} + u, q_{j-1} + v} \right]^{\text{NS} \left(\theta_u^{(N)}, \theta_v^{(N)}\right)}
$$

converges.

**Proof.** It is not difficult to see that the $(q_{j-1} + u, q_{j-1} + v)$ element of $A_N D_r^{-N}$ is

$$(A_N D_r^{-N})_{q_{j-1} + u, q_{j-1} + v} = \left( \frac{\delta_i}{\delta_j} \right)^N B_{q_{j-1} + u, q_{j-1} + v}^{\text{NS} \left(\theta_u^{(N)}, \theta_v^{(N)}\right)}.
$$

As $B_{q_{j-1} + u, q_{j-1} + v}$ converges and $\left| \frac{\delta_i}{\delta_j} \right| < 1$ for $i > j$, it follows that (32) converges to zero in this case. Hence $A_N D_r^{-N}$ converges iff i) and ii) hold.

Suppose $B$ is an $n \times n$ matrix and correspondingly there are $n$ nodes $S_1, S_2, \ldots, S_n$. We say that $B$ is **indecomposable** if for every $i$ and $j$

- either $S_i \rightarrow S_j$ or $S_j \rightarrow S_i$.

Next we have

**Theorem 4.5.** Let $A_N$ be of full column rank and $A_N = Y_s G_s$ be its GS factorization. Suppose $D^{-N} A_N$ converges, say, to $\bar{B}$ which satisfies ($\beta$). Then the following statement are true

1) If $Y_s D_r^{-N}$ converges to $\bar{Z} = \text{diag}(\bar{Z}_1, \ldots, \bar{Z}_t)$ in which each block $\bar{Z}_i$ ($i = 1, \ldots, s$) is indecomposable, then $D^{(N)} = \delta_i I_{p_i}$, $s = 1, \ldots, t$.

2) If $D^{(N)} = \delta_s I_{p_s}$, $s = 1, \ldots, t$, then $Y_s D_r^{-N}$ converges.

**Proof.** From Theorem 4.2., the convergence of $B_N = D^{-N} A_N$ implies the convergence of $G_N$ and $D^{-N} Y_s$. Suppose $D^{-N} Y_s \rightarrow \bar{Z}$. Then it follows, on account of Lemma 4.4, that $Y_s D_r^{-N}$ converges to $\bar{Z} = \text{diag}(\bar{Z}_1, \ldots, \bar{Z}_t)$ if

a) $(\bar{Z}_i)_{u,v} = \bar{Z}_{q_{j-1} + u, q_{j-1} + v} \neq 0 \Rightarrow \theta_u^{(N)} = \theta_v^{(N)}$, and

b) if $i < j$, then

$$
(D^{(N)} Y_s)_{q_{j-1} + u, q_{j-1} + v} = \left( \frac{\delta_i}{\delta_j} \right)^N \left[ Y_s D_r^{-N} \right]_{q_{j-1} + u, q_{j-1} + v}^{\text{NS} \left(\theta_u^{(N)}, \theta_v^{(N)}\right)}
$$

converges to zero. Now Corollary 4.4 says that for $i < j$

$$
(Y_s D_r^{-N})_{q_{j-1} + u, q_{j-1} + v} = \frac{Y_s D_r^{-N}}{d_{q_{j-1} + u, q_{j-1} + v}} = O \left( \frac{\delta_i^{(N)}}{\delta_j} \right)
$$

and so b) is automatically satisfied in this case. Therefore $Y_s D_r^{-N}$ converges iff a) holds. Since each $\bar{Z}_i$ is indecomposable, for arbitrary $(u, v)$ there exists a path either from $S_{q_{j-1} + u}$ to $S_{q_{j-1} + v}$ or vice versa. In either case this implies that $\theta_u^{(N)} = \theta_v^{(N)}$ for any $u$ and $v$. We complete the proof of 1).

2) This time $D$ is locally primitive, so we have $\theta_u^{(N)} = \theta_v^{(N)}$ ($j = 1, \ldots, t$) and hence

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\[ (Y_N D_r^{-N})_{q_{i-1}+n, q_{j-1}+n} \]
\[ = \left( \frac{\delta_{i,j}^N}{\delta_j^N} \right) (D_r^{-N} Y_N)_{q_{i-1}+n, q_{j-1}+n} + e^{N(d(i) - d(j))} \]
\[ \quad \text{if } i \neq j \]
\[ (D_r^{-N} Y_N)_{q_{j-1}+n, q_{j-1}+n} \quad \text{if } j = i \]

By hypothesis, the above converges for \( j = i \). The convergence for \( i > j \) is obvious; while the convergence for \( i < j \) can be easily achieved by noticing that
\[ (D_r^{-N} Y_N)_{q_{i-1}+n, q_{j-1}+n} = \frac{Y_N^{(N)}}{d_n^{(N)}} = O \left( \left( \frac{\delta_i^N}{\delta_j^N} \right)^N \right) \].

**Remark.** From Theorem 4.5, we know that in the case of multiple eigenvalues, if \( k = q_{i-1} + u \), then
\[ \frac{Y_N^{(N)}}{d_n^{(N)}} \to [0, \ldots, 0, \ p_{q_{i-1}+1}, \ldots, \ p_{q_i}, \ 0, \ldots, 0]^T. \]

Let us now turn to the applications of this theorem. Our first application is the following result gives the general convergence result of power scaled triangular matrix.

**Corollary 4.6.** Let \( D \) be diagonal and \( T \) be upper triangular. Then \( D^{-N} T^{-N} \) converges if and only if
1) Either \( |\lambda_i/d_i| < 1 \) or \( \lambda_i = d_i \) for each \( i \)
2) If \( S_i \to S_j \implies |\lambda_i| > |\lambda_j| \).

**Proof:** Let \( A_N = T^{-N} \). This time the GS factorization for \( A_N = T^{-N} \) becomes \( D_r^N (D_r^{-N} T^{-N}) \) and from Theorem 4.2., \( D_r^N T^{-N} \) converges if and only if both \( G_N = D_r^{-N} T^{-N} \) and \( D_r^{-N} D_r^N \) converge.

The convergence of \( D_r^N D_r^N \) is equivalent to 1); while the convergence of \( G_N = D_r^{-N} T^{-N} \), on account of Theorem 3.4., is exactly the same as the path condition 2).

A relevant application of Theorem 3.4. is to the question of subspace iteration. Armed with Theorem 3.4., we can get a sharper theoretical result than was previously given.

### 5. Application to the Subspace Iteration

Next, suppose \( T \) is an block upper diagonal matrix of the form
\[ T = \begin{bmatrix} \lambda_1 I_{p_1} & T_{12} & \cdots & T_{1u} \\ 0 & \lambda_2 I_{p_2} & \cdots & T_{2u} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_l I_{p_l} \end{bmatrix}, \] (33)

where \( |\lambda_1| > |\lambda_2| > \cdots > |\lambda_l| \). Let \( D_r = \text{diag}(T) = \text{diag}(\lambda_1 I_{p_1}, \ldots, \lambda_l I_{p_l}) \) and denote \( D_r = (D_r)^r \). Then from Theorem 3.4., it follows that \( D_r^{-N} T^{-N} \) converges.

Assume \( B \) is \( n \times r \) matrix of full column rank. Therefore \( r \leq n = \sum_{i=1}^r p_i \) and without loss of generality we can write \( r = \sum_{i=1}^r p_i + w \) for some \( w \leq p_{r+1} \). Thus we can write \( T_r = \text{diag}(\lambda_1 I_{p_1}, \ldots, \lambda_l I_{p_l}, \lambda_{r+1} I_w) \). We now have

**Corollary 5.1.** Let \( T \) be \( n \times n \) upper triangular matrix defined as in (33), and let \( B \) be \( n \times r \) matrix whose columns are linearly independent. If
\[ T^{-N} B = Y_N G_N \]
is its GS factorization, then the followings hold
1) \( D_r^{-N} T^{-N} \) converges, say, to a limit \( A \).
2) \( Y_N D_r^{-N} \) converges to \( \begin{bmatrix} P \\ 0 \end{bmatrix} \), where \( P = \text{diag}(P_1, \ldots, P_r, \tilde{P}) \) and each \( P_i \) (\( i = 1, \ldots, s \)) is a \( p_i \times p_i \) matrix and \( \tilde{P} \) is a \( p_{r+1} \times w \) matrix.

**Proof:** The result follows by simply choosing \( A_N = T^{-N} \) in Theorem 4.2.

Let us now turn to the question of subspace iteration for a restricted class of matrices. Suppose that
\[ A = VTV^* \] (34)
is \( n \times n \) matrix, where \( V \) is unitary and \( T \) is as in (33). Then using the same \( P \) as above we have

**Corollary 5.2.** Suppose that \( A \) is an \( n \times n \) matrix which satisfies (34). Let \( Y_0 \) be an \( n \times r \) matrix whose columns are linearly independent and \( \{Y_n\} \) be sequence of matrices defined by the following factorization
\[ A^N Y_n = Y_N G_N. \]

Then
\[ Y_N D_r^{-N} \rightarrow [V_1 P_1, \ldots, V_s P_s, V_{r+1} \tilde{P}]. \] (35)

**Proof:** Since
\[ A^N Y_n = Y_N G_N, \]
it follows that
\[ VTV^*(V^* Y_n) = Y_N G_N. \]

Partition \( V = [V_1, \ldots, V_s] \) conformally to that of \( T \) in (33) and set \( B = V^* Y_0 \), then
\[ T^{-N} B = (V^* Y_n) G_N. \] (36)

It is easily seen that the columns of \( V^* Y_n \) are orthogonal. Therefore (36) can be regarded as the GS factorization of \( T^{-N} B \). From Corollary 5.1., we have that for \( V = [V_1, \ldots, V_s] \)
\[ V^* Y_n D_r^{-N} \rightarrow \begin{bmatrix} P \\ 0 \end{bmatrix} \]
which is equivalent to
\[ Y, D, \tilde{V} \rightarrow V \begin{bmatrix} P \\ 0 \end{bmatrix} = V, P = [V, P_1, \cdots, V, P_s, V, \tilde{P}] \].

6. References


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