An Exponential Lévy Model for Stocks Paying Discrete Dividends

Guochao JIANG¹, Hailing DONG²
¹Department of Economics and Management, HIT Shenzhen Graduate School, Shenzhen, China
²School of Mathematics and Computational Science, University of Shenzhen, Shenzhen, China
Email: guochaojiang@gmail.com, hailingdong@gmail.com

Abstract: By introducing the impact of discrete stochastic dividends on stock price dynamics, this article proposes an exponential Lévy model with dividend jump, and derives stock pricing formula in three cases respectively. One is that dividends are announced and paid at the same time; another is that dividends are announced in advance; the third is that dividend policy can be changed with regime.

Keywords: exponential Lévy model; discrete dividends; stock pricing formula

1 Introduction

So far, stock pricing has been studied from different perspectives and methods by a lot of papers, we cite in particular [1-4]. The literatures have proposed a series of stock pricing models, such as Black-Scholes model, an exponential Lévy model, dividend discount Model and so on. In most cases, the stock will pay dividends in actual financial markets, so in recent years, studying the price dynamics of stock paying discrete stochastic dividends has become a hot, see [5,6]. However, with regard to exponential Lévy model with dividend jump, and deduces stock pricing formula in three cases respectively. One is that dividends are announced and paid at the same time; another is that dividends are announced in advance; the third is that dividend policy can be changed with regime.

2 The Exponential Lévy Model

Based on the existing works of the article [1], this section considers pricing of stock paying discrete stochastic dividends and proposes an exponential Lévy model with dividend jump. Details are as follows:

(1) One riskless asset, such as short-term debt, its pricing formula is

\[ B_t = \exp(rt) \]

Namely, risk-free interest rate is a constant, denoted by \( r \).

(2) A stock price process paying discrete dividends can be denoted by \( \{S(t), \ t \geq 0\} \), where \( S(t) \) is the stock price at time \( t \).

Prior to the first payment of dividend and between the adjacent two dividend-paying, the stock price process \( \{S(t), \ t \geq 0\} \) follows the path of an exponential Lévy process.

Namely, \( \forall t_j \leq t < t_{j+1}, j = 0, 1, 2, 3, ..., \) we have

\[ S(t) = S(t_j) \exp[X_{t_j - t_j^*}] = S(t_j) Y_{t_j - t_j^*} \]

where \( X_t \) is a Lévy process and \( X(0) = 0 \). Define a process \( Y_t \) as follows:

\[ Y_t = \exp[X_t] \]

then \( Y_t \) is an exponential Lévy process.

We assume there are times \( t_k = kh, k = 1, 2, 3, ..., \) and at each time \( t_j \) the dividend paid is \( D_j = a_j S(t_j) \); where \( S(t_j) \) denotes the stock price just before the moment of the dividend payment. We assume that each \( a_j, j=1, 2, .. \) is a positive constant number between 0 and 1. We set \( t_0 = 0 \), but it is not a Dividend-paying time.

3 Stock Pricing Formula

3.1 Stock Pricing Formula in Case of Dividends Announced and Paid at the Same Time

In this subsection, we consider the stock price process where dividends will be announced at time \( kh, k = 1, 2, 3, ... \) and be paid at the same time. Specific ways about paying dividends can be seen in section 2. The stock price \( S(t_j^*) \) after dividend payment is the stock price just before the moment of dividend payment less corresponding dividend payment:

\[ S(t_j^*) = S(t_j) - a_j S(t_j) \]

Lemma 3.1

\[ Y_t \cdot Y_{t-s} = Y_s, \ \forall s \leq t \] \hspace{1cm} (1)

Proof: By the definition of \( Y(t) \), we have

\[ Y_t \cdot Y_{t-s} = \exp[X_t] \cdot \exp[X_{t-s}] \]

\[ = \exp[X_t + X_{t-s}] \] \hspace{1cm} (2)
According to the property of stationary independent increments of Lévy process \([7]\), we have
\[ X_t + X_{t-s} = X_s. \]

Substituting above formula into (2), we get
\[ Y_t \cdot Y_{t-s} = \exp[X_t] = Y_t \]

**Theorem 3.1** Given \(d_0 = 0\), for \(\forall t_k \leq t < t_{k+1}\), \(k = 0, 1, 2, \ldots\), the stock pricing formula is
\[ S(t) = (1 - a_1)(1 - a_2) \cdots (1 - a_n)S_0 \cdot Y_t \]
which implies,
\[ S(t_{k+1}) = (1 - a_1)(1 - a_2) \cdots (1 - a_k)S_0 \cdot Y_{t_{k+1}} \]
and
\[ S(t_{k+1}) = (1 - a_1)(1 - a_2) \cdots (1 - a_{k+1})S_0 \cdot Y_{t_{k+1}} \]

**Proof:** (1) When \(k = 0\), for \(0 \leq t < t_1\),
\[ S(t) = S_0Y_t = S_0 \cdot \exp(X_t) \]
Because \(X\) is a continuous process, so \(X(t_{k+1}) = X(t_{k+1})\). Then we have
\[ S(t_{k+1}) = (1 - a_1)(1 - a_2) \cdots (1 - a_n)S_0 \cdot Y_{t_{k+1}} \]

(2) Suppose the theorem holds for \(X(t_n) = X(t_n)\). Then we have
\[ S(t_{n+1}) = (1 - a_1)(1 - a_2) \cdots (1 - a_n)S_0 \cdot Y_{t_{n+1}} \]

The proof of theorem is completed.

**Remarks:** In fact, about the development of stock price process, the intuitive understanding is very important. We need deriving stock pricing formula recursively at first and then prove it rigorously. By theorem 3.1, we know that the stock price process follows the path of an exponential Lévy process with jumps at dividend-paying time. For \(t_k \leq t < t_{k+1}\), \(k = 0, 1, 2, \ldots\), \(k\) times dividends have been paid, so the parameters \(a_1, a_2, \ldots, a_n\) are all known. We also know the value of exponential Lévy process \(Y_k\), so the stock price at time \(t\) can be calculated immediately by (3).

### 3.2 Stock Pricing Formula in Case of Dividends Announced in Advance

In this subsection, we assume that the dividends paid at time \(t_k = kh\) are announced at time \((k - 1)h + \alpha h\) and are equal to
\[ D_k = a_kS((k - 1 + \alpha)h) \]
where \(\alpha\) is a given constant in \((0, 1)\), \(a_k\) is a positive constant in \((0, 1)\) and \(k = 1, 2, \ldots\). Let \(b_k = e^{\alpha (1 - \epsilon)h} a_k\), the present value of future dividends at the announcement moment is
\[ D_k^* = e^{-(1 - \epsilon)h}D_k \]

For \(kh \leq t < (k + 1)h\), no dividend is announced or paid, then the stock pricing process follows the path of an exponential Lévy process with \(S(kh)\) as initial value. For \(kh \leq t < (k + 1)h\), namely from the announcement time until the moment of the dividend payment, the stock price \(S(t)\) includes ex-dividend stock price process \(S^{ex}(t)\) and the present value of the next known dividend payment at the time \(t\),
\[ S(t) = S^{ex}(t) + D_k^*e^{-\alpha((k+1)h-t)} \]
\[ = S^{ex}(t) + D_k^*e^{-(1 - \epsilon)h} \]
Theorem 3.2 Set \( b_0 = 0 \), for \( kh \leq t < kh + \epsilon \) , \( k = 0, 1, 2, \ldots \) the stock pricing formula is

\[
S(t) = (1-b_1)(1-b_2)\cdots(1-b_k)S_0Y_t
\]  

for \( kh + \epsilon \leq t < (k + 1)h \), the stock pricing formula is

\[
S(t) = (1-b_1)(1-b_2)\cdots(1-b_k)S_0 \times [(1-b_{k+1})Y_{t,h} + b_{k+1}Y_{t,h}e^{(t-kh-h)}]
\]  

(12)

Proof: (1) When \( k = 0 \), for \( 0 \leq t < \epsilon \) because \( X \) is a continuous process, from the definition of \( Y(t) \), we have

\[
S(t) = S_0Y_t, \quad S_{ah} = S_0Y_ah, \quad D_1 = a_1S_{ah}.
\]

For \( \epsilon \leq t < h \), let \( S^a(t) \) denote the ex-dividend stock price process, then \( S^a(t) \) follows the path of an exponential Lévy process with \( S_{ah} - D^*_1 \) as initial value. Using (9) and (10), we have

\[
S^a(t) = [S_0 ah - D^*_1]Y_{t,ah} = (1-b_1)S_0Y_t
\]

Then by (10) and (11), we have

\[
S(t) = Sex(t) + D^*_1e^{(t-h)}
\]  

(14)

From (14), we have

\[
S(h^+) = (1-b_1)S_0Y_{ah} + b_1S_0Ya\bar{e}^{(h-k\epsilon)}
\]

According to (10), we have

\[
b_1S_0Ya\bar{e}^{(h-k\epsilon)} = D^*_1e^{(h-k\epsilon)} = D_1
\]

Because \( Y_1 \) is a continuous process, we have

\[
S(h) = S(h^+) - D_1
\]  

\[
= (1-b_1)S_0Y_h
\]  

\[
= (1-b_1)S_0Y_k
\]

Thus, when \( k = 0 \), the theorem holds.

(2) Suppose the theorem holds when \( k = n \), namely, for \( nh \leq t < nh + \epsilon \),

\[
S(t) = (1-b_1)(1-b_2)\cdots(1-b_n)S_0Y_t
\]

for \( nh + \epsilon \leq t < (n + 1)h \),

\[
S(t) = (1-b_1)(1-b_2)\cdots(1-b_n)S_0 \times [(1-b_{n+1})Y_{t,h} + b_{n+1}Y_{t,h}e^{(t-kh-h)}]
\]  

(15)

From (15), we have

\[
S((n+1)h^+) = (1-b_1)(1-b_2)\cdots(1-b_n)S_0 \times [(1-b_{n+1})Y_{(n+1)h} + b_{n+1}Y_{(n+1)h}e^{(h-k\epsilon)}]
\]

According to (10), we have

\[
b_{n+1}Y_{(n+1)h}e^{(h-k\epsilon)} = D^*_{n+1}e^{(h-k\epsilon)} = D_{n+1}.
\]

Because \( Y_1 \) is a continuous process, we have

\[
S((n+1)h) = S((n+1)h^+) - D_{n+1}
\]  

\[
= (1-b_1)(1-b_2)\cdots(1-b_n)S_0Y_{(n+1)h} - D_{n+1}
\]

When \( k = n + 1 \), for \( (n + 1)h \leq t < (n + 1)h + \epsilon \), the stock pricing process follows the path of an exponential Lévy process with \( S((n + 1)h) \) as initial value, namely

\[
S(t) = S((n + 1)h)Y_{(n+1)h} - D_{n+1} \times (1-b_1)(1-b_2)\cdots(1-b_n)S_0Y_{(n+1)h}
\]

for \( (n + 1)h + \epsilon \leq t < (n + 2)h \), using (10) and (11), this leads to

\[
S(t) = S(t) + D_{n+2}e^{(t-(n+1)h)}
\]  

\[
= [(1-b_1)(1-b_2)\cdots(1-b_n)(1-b_{n+1})S_0Y_{(n+1)h} - D_{n+1}Y_{(n+1)h} + D_{n+2}e^{(t-(n+1)h)}]
\]

(13)

The proof of theorem is completed.

Remarks: In case of dividends announced in advance, for \( kh \leq t < kh + \epsilon \), \( k = 0, 1, 2, \ldots \) times dividends have been paid, so parameters \( a_1, a_2, \ldots, a_t \) are all known, and the values of \( b_1, b_2, \ldots, b_k \) can be deduced. The value of exponential Lévy process \( Y_t \) at time \( t \) is also known, so the stock price at time \( t \) can be calculated immediately according to (12). For \( kh + \epsilon \leq t < (k + 1)h \), previous dividends have been paid, while the \( (k + 1) \)-th dividends have already been announced in advance, so parameters \( a(1), a(2), \ldots, a(k + 1) \) are all known. Then the values of \( b_1, b_2, \ldots, b_{k+1} \) can be deduced. The value of exponential Lévy process \( Y_t \) at time \( t \) is also known, so the stock price at time \( t \) can be calculated immediately according to (13).

3.3 Stock Pricing Formula in Case of Changing Dividend Policy

Theorem 3.1 and Theorem 3.2 give the stock pricing formulas in the case of dynamic changes of the proportion of dividend payment. The results can be generalized to the more general case-dividend policy modulated by an external Markov chain.

We extend the model in section II to incorporate the dependence of dividend payment on the external economic environment, where the latter is modeled by a finite-state Markov chain \( \{I(t)\} \). Suppose that the stock pays dividends \( D_j \) at discrete times \( tj = jh \), the dividends have the form

\[
D_j = a(I(t_j))S(t_j)
\]

where \( S(t_j) \) denotes the stock price just before the moment of the dividend payment. We assume that \( a(I(t_j)) \), \( j = 1, 2, \ldots \) is a function depending on the state of \( I(t) \) at time \( t \). We set \( t_0 = 0 \), but it is not a dividend payment
time. Suppose \( \{I(t); t \geq 0\} \) is a homogeneous, irreducible and recurrent markov process with values in the set of \( I = \{1, 2, 3, \ldots, m\} \) and its intensity matrix is \( \Lambda = (\alpha_{ij}) \), where \( \alpha_{ii} = -\alpha_i \). The transition probability matrix of the corresponding Embedded chain of \( \{I(t); t \geq 0\} \) is

\[
P = (\rho_{ij}), \quad \rho_{ij} = \begin{cases} 0, & i = j \\ \frac{\alpha_{ij}}{\alpha_i}, & i \neq j, i, j \in J \end{cases}
\]

Using theorem 3.1, we have

**Theorem 3.3** Assume that the dividends are paid and announced at the same time, let \( a_0 = 0 \), for \( t_k \leq t < t_{k+1}, k = 0, 1, 2, \ldots \), the stock pricing formula is

\[
S(t) = (1 - a(I(t^-)))(1 - a(I(t^-))) \cdots (1 - a(I(t^-)))S_0Y_t
\]

**proof:** Completely similar to the proof of Theorem 3.1.

For Theorem 3.2, it has a similar extension. We will not repeat them here.

**4 Conclusions**

Based on the reality of stock paying discrete stochastic dividends in actual financial market, this article, considering the impact of discrete stochastic dividends on stock price dynamics, proposes an exponential Lévy model with dividend jump, and deduces the concrete stock pricing formulas in three cases. The method in this article has several advantages: there is no difficulty in coping with dividends; the stock price can be easily calculated at any time. In the future, it is deserved to study the formulas of European option and American option of exponential Lévy model with dividend jump.

**Acknowledgement**

This work was supported by the National Natural Science Foundation of China (No.10671212), Shenzhen University PhD. fund (No.000048). The author would like to thank the editor and the anonymous reviewers for their constructive comments and suggestions for improving the quality of the paper.

**References**